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CHARACTER FORMULA FOR WREATH PRODUCTS OF COMPACT GROUPS WITH THE INFINITE SYMMETRIC GROUP

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Abstract. Let $G = \mathfrak{S}_\infty(T)$ be the wreath product of a compact group T with the infinite symmetric group $\mathfrak{S}_\infty$. We study the characters of factor representations of finite type of G, and give a formula which expresses all the characters explicitly.

1. Characters of factor representations of finite type

1.1. *Characters.* Let G be a Hausdorff topological group, $\mathcal{P}(G)$ the set of continuous positive definite functions on G, $K(G)$ the set of $f \in \mathcal{P}(G)$ which are invariant under inner automorphisms, $K_1(G)$ the set of $f \in K(G)$ normalized as $f(e) = 1$ at the identity element $e \in G$, and $E(G)$ the set of extremal points in the convex set $K_1(G)$. Let π_1 and π_2 be two continuous unitary representations (= URs) of G, and $\mathfrak{U}_i = \pi_i(G)''$ $(i = 1, 2)$ the von Neumann algebra generated by $\pi_i(G)$. We say that π_1 and π_2 are *quasi-equivalent* if there exists an isomorphism Φ from $\mathfrak{U}_1$ onto $\mathfrak{U}_2$ as $*$-algebras such that $\Phi(\pi_1(g)) = \pi_2(g)$ for $g \in G$.

THEOREM 1.1 ([HH3, Theorems 1.5.4 and 1.6.1]). *Let π be a continuous unitary representation (= UR) of G such that the von Neumann algebra $\mathfrak{U} = \pi(G)''$ has a faithful normal finite trace t on the set $\mathfrak{U}^+$ of positive elements in $\mathfrak{U}$. Normalize t as $t(I) = 1$*

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at the identity operator I and put $f = \phi \circ \pi$ with the unique linear extension ϕ of t onto $\mathfrak{U}$. Then $f \in K_1(G)$, and π is quasi-equivalent to the UR π_f associated to f in [GR]. *An isomorphism Φ from $\mathfrak{U}$ to $\mathfrak{U}_f = \pi_f(G)''$ can be given explicitly.*

A UR π is called *factorial* if $\mathfrak{U}$ is a factor. If the factor is of finite type, there exists a unique faithful finite normal trace t on $\mathfrak{U}^+$ normalized as $t(I) = 1$. Then, with the unique extension ϕ of t to a linear form on $\mathfrak{U}$, the function

$$f(g) = \phi(\pi(g)) \qquad (g \in G) \tag{1}$$

belongs to $K_1(G)$ and is called a *character* of π.

Theorem 1.2 ([HH3, Theorem 1.6.2]). *For a Hausdorff topological group G, let* $\mathrm{URff}(G)$ *be the set of all quasi-equivalence classes of URs of G, factorial of finite type. Then there exists a canonical bijective correspondence between* $\mathrm{URff}(G)$ *and $E(G)$ through* (1) *above.*

In this connection, every element f in $E(G)$ is called a *character* of G of finite type. In [Dix, 17.3], the above canonical bijection is asserted under the condition that G is locally compact and unimodular.

1.2. *Case of a limit of LCG inductive system.* Now let $K_{\leq 1}(G) \supset K_1(G)$ be the set of $f \in K(G)$ such that $f(e) \leq 1$. Then the set of extremal points of $K_{\leq 1}(G)$ is the union of $E(G)$ and $\{0\}$. In the case where G is locally compact, it is known that the convex set $K_{\leq 1}(G)$ is compact in the weak topology $\sigma(L^\infty(G), L^1(G))$ (cf. [Dix, 17.3]). We extend this result to the case where $G = \lim_{n\to\infty} G_n$ is the inductive limit of a countable inductive system $G_1 \to G_2 \to \cdots \to G_n \to \cdots$ of locally compact groups, where each homomorphism from G_n into G_{n+1} is assumed to be homeomorphic. In [TSH], this kind of inductive system is called a *countable LCG inductive system* and it is proved that G with the inductive limit topology τ_{ind} becomes a topological group and that G has sufficiently many continuous positive definite functions and accordingly sufficiently many URs.

In the following we treat a certain case where all G_n's are compact.

For a topological group G, let $\mathfrak{F}(G)$ be the space of functions ψ on G such that $\psi(g) = 0$ except for a finite number of $g \in G$, with the convolution $\psi_1 * \psi_2(g) := \sum_{h\in G} \psi_1(gh^{-1})\psi_2(h)$ and the conjugation $\psi^*(g) := \overline{\psi(g^{-1})}$. Put $f(\psi) = \sum_{g\in G} f(g)\psi(g)$ for $f \in K(G)$. For two elements $f_1, f_2 \in K(G)$, we introduce a partial order $f_1 \geq f_2$ by $f_1(\psi^* * \psi) \geq f_2(\psi^* * \psi)$ $(\psi \in \mathfrak{F}(G))$, and put

$$K(G; f) := \{f' \in K_{\leq 1}(G);\ f' \leq f\}. \tag{2}$$

Then we see that functions $f' \in K(G; f)$ are uniformly equicontinuous.

Lemma 1.3. *For an $f \in K_1(G)$, take an $f' \in K(G; f)$. Then f' is extremal in $K(G)$ or $f' \in E(G)$ if and only if $f' \in \bigcap_{m\in\boldsymbol{N}} \mathrm{Extr}(K(G; mf))$, where $\mathrm{Extr}(A)$ denotes the set of extremal points of a convex set A. This means that*

$$E(G) \cap K(G; f) = \bigcap_{m\in\boldsymbol{N}} \mathrm{Extr}(K(G; mf)).$$

THEOREM 1.4. *Assume that G is a union of countable compact subsets $C_n \nearrow G$ and that the topology on G is the inductive limit of topologies τ^{C_n} on C_n. Then, for an $f \in K_1(G)$, the convex subset $K(G; f)$ of the space $C(G)$ of continuous functions on G is compact in the topology of uniform convergence on every C_n.*

In the situation of Theorem 1.4, take $f' \in E(G) \cap K(G; f)$. Then f' is extremal in the compact convex subset $K(G; mf)$ for each $m \in \boldsymbol{N}$, and it has an expression as an integral on the set of extremal points $\mathrm{Extr}(K(G; mf))$, which is due to Choquet and Bishop-K. de Leeuw [BL]. We apply this fact to prove Theorem 11.1, where we can take $C_n = G_n$.

2. Wreath products of compact groups with the infinite symmetric group. For a set I, we denote by $\mathfrak{S}_I$ the group of all finite permutations on A. A permutation σ on I is called *finite* if its support $\mathrm{supp}(\sigma) := \{ i \in I \ ; \ \sigma(i) \neq i \}$ is finite. We call the *infinite symmetric group* the permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$. The index $\boldsymbol{N}$ is frequently replaced by ∞. The symmetric group $\mathfrak{S}_n$ is naturally imbedded in $\mathfrak{S}_\infty$ as the permutation group of the set $I_n := \{ 1, 2, \ldots, n \} \subset \boldsymbol{N}$.

Let T be a compact group. We consider a wreath product group $\mathfrak{S}_I(T)$ of T with a permutation group $\mathfrak{S}_I$ as follows:

$$\mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I \, , \ \ D_I(T) = \prod_{i \in I}{}' \, T_i \, , \ \ T_i = T \ (i \in I), \tag{3}$$

where the symbol $\prod'$ means the restricted direct product, and $\sigma \in \mathfrak{S}_I$ acts on $D_I(T)$ as

$$D_I(T) \ni d = (t_i)_{i \in I} \stackrel{\sigma}{\longmapsto} \sigma(d) = (t'_i)_{i \in I} \in D_I(T), \quad t'_i = t_{\sigma^{-1}(i)} \ (i \in I). \tag{4}$$

Identifying groups $D_I(T)$ and $\mathfrak{S}_I$ with their images in the semidirect product $\mathfrak{S}_I(T)$, we have $\sigma \, d \, \sigma^{-1} = \sigma(d)$. The groups $D_{I_n}(T)$ and $\mathfrak{S}_{I_n}(T)$ are denoted by $D_n(T)$ and $\mathfrak{S}_n(T)$ respectively, then $G := \mathfrak{S}_\infty(T)$ is an inductive limit of $G_n := \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$. Since T is compact, G_n is also compact, and the inductive system $(G_n)_{n \geq 1}$ is an example of countable LCG inductive systems in [TSH]. We introduce in G its inductive limit topology τ_{ind}. Then G with τ_{ind} becomes a topological groups (cf. 2.7 in [TSH]). By definition, a subset $B \subset G$ is τ_{ind}-open if and only if $B \cap G_n$ is open in G_n for any $n \geq 1$. A general theory of unitary representations of the inductive limit group G of a countable LCG inductive system is carried out in [TSH, §5] using continuous positive definite functions on the group.

Denote by τ_{ind}^D the inductive limit topology on $D_\infty(T)$ of the topologies on $D_n(T)$, then the topology τ_{ind} on $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ is given as the product of τ_{ind}^D and the discrete topology $\tau_{disc}^{\mathfrak{S}}$ on $\mathfrak{S}_\infty$. When T is a finite group, the topology τ_{ind} in $G = \mathfrak{S}_\infty(T)$ is discrete. When T is infinite, τ_{ind} is neither discrete nor locally compact, and a subset $\{(d, \mathbf{1}) \ ; \ d \in D_\infty(T)\} \cong D_\infty(T)$ is an open neighbourhood of the identity element e of G, where $\mathbf{1}$ denotes the trivial permutation on $\boldsymbol{N}$. In the case where T is abelian, we put $P_I(d) = \prod_{i \in I} t_i$ for $d = (t_i)_{i \in I} \in D_I(T)$, and define a subgroup of $\mathfrak{S}_I(T)$ as

$$\mathfrak{S}_I^e(T) = D_I^e(T) \rtimes \mathfrak{S}_I \quad \text{with} \quad D_I^e(T) := \{ d = (t_i)_{i \in I} \ ; \ P_I(d) = e_T \}. \tag{5}$$

This kind of groups $\mathfrak{S}_\infty(T)$ and $\mathfrak{S}^e_\infty(T)$ with T abelian, contain the infinite Weyl groups of classical types, $W_{\mathbf{A}_\infty} = \mathfrak{S}_\infty$ of type $\mathbf{A}_\infty$, $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{B}_\infty/\mathbf{C}_\infty$, and $W_{\mathbf{D}_\infty} = \mathfrak{S}^e_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{D}_\infty$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r,1,n)$ of complex reflexion groups $G(r,1,n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [Sho]).

In general, by Theorem 1.2, for a topological group G, the set $E(G)$ of all extremal elements in the convex set $K_1(G)$ is equal to the set of all characters of factor represetations of G of finite type, type I_n ($n < \infty$) or II_1. When G is discrete, $K_1(G)$ is compact in the weak topology and $E(G)$ is closed in it. This is the case of $G = \mathfrak{S}_\infty(T)$ with T finite, and this case has been treated in [HH1]–[HH2], and the case of $\mathfrak{S}_\infty$ itself in [Hir3].

3. Structure of wreath product groups $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$

3.1. *Standard decomposition of elements.* Introduce the following notation: for $d = (t_i)_{i\in I} \in D_I(T)$, $I \subset \boldsymbol{N}$, we put $\mathrm{supp}_I(d) := \{\, i \in I \;;\; t_i \neq e_T \,\}$ and we omit the suffix I if $I = \boldsymbol{N}$ or I is specified from the context.

An element $g = (d,\sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

Case 1: σ is cyclic and $\mathrm{supp}(d) \subset \mathrm{supp}(\sigma)$;

Case 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i\in\boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

The element $(d,\mathbf{1})$ in Case 2 is denoted by $\xi_q = (t_q,(q))$, where (q) denotes superfluously a trivial cyclic permutation of length 1 indicating the place $q \in \boldsymbol{N}$ of $t_q \in T$, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{\, q \,\}$. For a cyclic permutation $\sigma = (i_1 \;\; i_2 \;\; \cdots \;\; i_\ell)$ of ℓ integers, we define its *length* as $\ell(\sigma) = \ell \geq 2$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d,\sigma)$, and $\ell(\xi_q) = 1$.

An arbitrary element $g = (d,\sigma) \in G$ is expressed as a product of basic elements as

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m \tag{6}$$

with $g_j = (d_j,\sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \leq j \leq m)$, are mutually disjoint. This expression of g is unique up to the orders of ξ_{q_k}'s and g_j's, and is called *standard decomposition* of g. Note that, for $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ gives the cycle decomposition of σ.

To write down the conjugacy class of $g = (d,\sigma)$, there appear products of components t_i of $d = (t_i)$, where the orders of taking products are crucial when T is not abelian. We denote by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of T, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in T. For a basic component $g_j = (d_j,\sigma_j)$ of g, let $\sigma_j = (i_{j,1} \;\; i_{j,2} \;\; \ldots \;\; i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{\, i_{j,1}, i_{j,2}, \ldots, i_{j,\ell_j} \,\}$ with $\ell_j = \ell(\sigma_j)$. For $d_j = (t_i)_{i\in K_j}$, we put

$$P_{\sigma_j}(d_j) := \left[t'_{\ell_j}t'_{\ell_j-1}\cdots t'_2t'_1\right] \in T/\!\sim \qquad \text{with} \quad t'_k = t_{i_{j,k}} \;\; (1 \leq k \leq \ell_j). \tag{7}$$

Note that the product $P_{\sigma_j}(d_j)$ is well-defined, because, for $t_1, t_2, \ldots, t_\ell \in T$, we have $t_1t_2\cdots t_\ell \sim t_kt_{k+1}\cdots t_\ell t_1\cdots t_{k-1}$ for any k, that is, the conjugacy class does not depend on any cyclic permutation of $(t_1, t_2, \ldots, t_\ell)$.

Lemma 3.1. **(i)** *Let $\sigma \in \mathfrak{S}_\infty$ be a cycle, and put $K = \mathrm{supp}(\sigma)$. Then, an element $g = (d,\sigma) \in \mathfrak{S}_K(T)$ $(=: G_K)$ is conjugate in it to $g' = (d',\sigma) \in G_K$ with $d' = (t'_i)_{i\in K}, t'_i = e_T$ $(i \neq i_0)$, $[t'_{i_0}] = P_\sigma(d)$ for some $i_0 \in K$.*

(ii) *Identify* $\tau \in \mathfrak{S}_\infty$ *with its image in* $G = \mathfrak{S}_\infty(T)$. *Then we have, for* $g = (d, \sigma)$,

$$\tau\, g\tau^{-1} = (\tau(d), \tau\sigma\tau^{-1})\ \ (=: (d', \sigma')), \quad \textit{and} \quad P_{\sigma'}(d') = P_\sigma(d).$$

THEOREM 3.2. *Let* T *be a compact group. Take an element* $g \in G = \mathfrak{S}_\infty(T)$ *and let its standard decomposition into basic elements be* $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ *in* (6) , *with* $\xi_{q_k} = (t_{q_k}, (q_k)\,)$, *and* $g_j = (d_j, \sigma_j)$, σ_j *cyclic,* $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. *Then the conjugacy class of* g *is determined by the set*

$$[t_{q_k}] \in T/\!\sim \quad (1 \le k \le r) \quad \textit{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)\,) \quad (1 \le j \le m), \tag{8}$$

where $P_{\sigma_j}(d_j) \in T/\!\sim$ *and* $\ell(\sigma_j) \ge 2$.

3.2. *The case where* T *is abelian.* In the case where T is abelian, the set $T/\!\sim$ of conjugacy classes is equal to T itself. Take $g \in G$, and take its standard decomposition (6). For $g_j = (d_j, \sigma_j)$, put $g'_j := (d'_j, \sigma_j)$, where $d'_j = (t'_i)_{i\in\boldsymbol{N}}$ with $t'_{i_0} = P(d_j) := \prod_{i\in K_j} t_i$ for some $i_0 \in K_j = \mathrm{supp}(\sigma_j)$, and $t'_i = e_T$ elsewhere.

LEMMA 3.3. *Let* T *be abelian. For a* $g = (d, \sigma) \in \mathfrak{S}_\infty(T)$, *define* g'_j $(1 \le j \le m)$ *as above and put* $g' = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g'_1 g'_2 \cdots g'_m$. *Then,* g *and* g' *are mutually conjugate in* $\mathfrak{S}_\infty(T)$. *A complete set of parameters of the conjugacy classes of non-trivial elements* $g \in \mathfrak{S}_\infty(T)$ *is given by the set*

$$\{\, t'_1, t'_2, \ldots, t'_r \,\} \quad \textit{and} \quad \{\, (u_j, \ell_j)\ ;\ 1 \le j \le m\,\}, \tag{9}$$

where $t'_k = t_{q_k} \in T^* := T \setminus \{\, e_T\,\}$, $u_j = P(d_j) \in T$, $\ell_j \ge 2$, *and* $r + m > 0$.

3.3. *Finite-dimensional representations.* Among factor representations of finite type of $G = \mathfrak{S}_\infty(T)$, those of type I are one-dimensional characters given below.

LEMMA 3.4. *A finite-dimensional continuous irreducible unitary representation (= IUR)* π *of* G *is a one-dimensional character, and is given in the form* $\pi = \pi_{\zeta,\varepsilon}$ *with*

$$\pi_{\zeta,\varepsilon}(g) = \zeta(P(d))\ (\mathrm{sgn}_{\mathfrak{S}})^{\varepsilon}\,(\sigma) \quad \textit{for}\ \ g = (d, \sigma) \in \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty,$$

where ζ *is a one-dimensional continuous character of* T, $P(d)$ *is a product of components* t_i *of* $d = (t_i)$, *and* $\mathrm{sgn}_{\mathfrak{S}}(\sigma)$ *denotes the usual sign of* σ *and* $\varepsilon = 0, 1$.

4. Characters of $\mathfrak{S}_\infty(T)$ with T any compact group

4.1. *Character formula for factor representations of finite type.* Let $\widehat{T}$ be the dual of T consisting of all equivalence classes of continuous irreducible unitary representations (= IURs). We identify every equivalence class with one of its representatives. Thus $\zeta \in \widehat{T}$ is an IUR and denote by χ_ζ its character: $\chi_\zeta(t) = \mathrm{tr}(\zeta(t))$ $(t \in T)$, then $\dim\zeta = \chi_\zeta(e_T)$.

For one-dimensional characters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$\chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^{\varepsilon} \quad (\sigma \in \mathfrak{S}_\infty\ ;\ \varepsilon = 0, 1). \tag{10}$$

As a parameter for characters of $G = \mathfrak{S}_\infty(T)$, we consider a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{\, 0, 1\,\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \tag{11}$$

of decreasing sequences of non-negative real numbers

$$\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}\,, \quad \alpha_{\zeta,\varepsilon,1} \ge \alpha_{\zeta,\varepsilon,2} \ge \alpha_{\zeta,\varepsilon,3} \ge\ \cdots\ \ge 0\,;$$

and a set of non-negative $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which altogether satisfy the condition

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1, \tag{12}$$

with

$$\|\alpha_{\zeta,\varepsilon}\| = \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i}\,, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta\,.$$

Note that, under the condition (12), there exists a countable subset $\widehat{T}_0 \subset \widehat{T}$ such that $\alpha_{\zeta,\varepsilon} = \mathbf{0}$ and $\mu_\zeta = 0$ except for $\zeta \in \widehat{T}_0$.

The following is our main theorem in this paper.

THEOREM 4.1. *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact group T with $\mathfrak{S}_\infty$. Then, for a parameter*

$$A := ((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu), \tag{13}$$

in (11)–(12)*, the following formula determines a character f_A of G: for an element $g \in G$, let* (6) *be its standard decomposition, then*

$$f_A(g) = \prod_{1\leq k\leq r}\Big\{\sum_{\zeta\in\widehat{T}}\Big(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\Big)\chi_\zeta(t_{q_k})\Big\} \tag{14}$$
$$\times\prod_{1\leq j\leq m}\Big\{\sum_{\zeta\in\widehat{T}}\Big(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\Big(\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta}\Big)^{\ell(\sigma_j)}\chi_\varepsilon(\sigma_j)\Big)\chi_\zeta(P_{\sigma_j}(d_j))\Big\},$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$. Conversely any character of factor representation of finite type of G is given in the form of f_A.

REMARK 4.1. Let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}$ without the components $g_j = (d_j, \sigma_j)$ with $\ell(\sigma_j) \geq 2$, then the formula gives

$$f_A(g) = \prod_{1\leq k\leq r}\Big\{\sum_{\zeta\in\widehat{T}}\Big(\sum_{\varepsilon\in\{0,1\}}\sum_{i\in\boldsymbol{N}}\frac{\alpha_{\zeta,\varepsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta}\Big)\chi_\zeta(t_{q_k})\Big\}$$

Put $t_{q_k} = e_T$ in the right hand side, then we get $\sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1$, by the condition (12), and the character formula in Theorem 4.1 is valid even for $g = e$ with $f_A(e) = 1$. In the case where T is not discrete or equivalently not finite, the continuity at $g = e$ is thus guaranteed by (12).

4.2. *Case where T is finite.* The case of $\mathfrak{S}_\infty$ itself is a special case of $\mathfrak{S}_\infty(T)$ with the trivial $T = \{e_T\}$. In this case, in the parameter $A = ((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu)$, the μ part does not appear, and so the equality condition (12): $\sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1$, should be replaced by the inequality $\|\alpha\| + \|\beta\| \leq 1$ for $\alpha = (\alpha_p)_{p\in\boldsymbol{N}}, \beta = (\beta_p)_{p\in\boldsymbol{N}}$ in [Tho]. This inconsistency can be explained as follows.

Assume T is finite. Let $\mathbf{1}_T$ be the trivial representation of T, and put $\widehat{T}^* := \widehat{T}\setminus\{\mathbf{1}_T\}$, $T^* = T\setminus\{e_T\}$. Then, $\sum_{\zeta\in\widehat{T}}(\dim\zeta)\,\chi_\zeta$ as functions on T, we have

$$0 = \sum_{\zeta\in\widehat{T}}(\dim\zeta)\,\chi_\zeta\,, \quad 1 = \chi_{\mathbf{1}_T} = -\sum_{\zeta\in\widehat{T}^*}(\dim\zeta)\,\chi_\zeta\,, \quad \text{on } T^*. \tag{15}$$

Because of this linear dependence, we can accept the parameter A for f_A not necessarily under the equality condition (12) but under the weaker inequality condition

$$\sum_{\zeta\in\widehat{T}}\sum_{\varepsilon\in\{0,1\}}\|\alpha_{\zeta,\varepsilon}\|+\|\mu\|\le 1\,, \tag{16}$$

losing the validity of the formula of f_A for $t_{q_k}=e_T$ and accordingly for $g=e$. We understand that this is the case of $\mathfrak{S}_\infty$ itself (without the μ part). Under this condition (16), the uniqueness of the part $\mu=(\mu_\zeta)_{\zeta\in\widehat{T}}, \mu_\zeta\in\boldsymbol{R}$, is lost.

REMARK 4.2. The character formula (14) in Theorem 4.1 is valid for Thoma's case of the infinite symmetric group $\mathfrak{S}_\infty$ itself. We consider it as an extreme case of wreath product groups $\mathfrak{S}_\infty(T)$ with a trivial group $T=\{e_T\}$. For the parameter A of character f_A, we consider in addition to $\alpha_{0,\mathbf{1}_T}, \alpha_{1,\mathbf{1}_T}$, a fake parameter $\mu=(\mu_{\mathbf{1}_T})$ for the trivial $T=\{e_T\}$ by putting $\mu_{\mathbf{1}_T}:=1-(\|\alpha_{0,\mathbf{1}_T}\|+\|\alpha_{1,\mathbf{1}_T}\|)$. Then the condition (12) holds. The relation between the parameter A and the *Thoma parameter* $(\{\alpha_n\},\{\beta_n\},\gamma)$ in [Tho] is given by

$$\{\alpha_n\}=\alpha_{0,\mathbf{1}_T},\quad \{\beta_n\}=\alpha_{1,\mathbf{1}_T},\quad \gamma=\mu_{\mathbf{1}_T}. \tag{17}$$

REMARK 4.3. In the case where $\mu_\zeta=1$ for some $\zeta\in\widehat{T}$, we have

$$f_A(g)=\prod_{1\le k\le r}\frac{1}{\dim\zeta}\,\chi_\zeta(t_{q_k})\qquad \text{for}\ \ g=\xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}, \tag{18}$$

and $f_A(g)=0$ if g has components $g_j=(d_j,\sigma_j)$ with $\ell(\sigma_j)\ge 2$. This case is related to a kind of 'ζ-twisted' regular representation of the group $\mathfrak{S}_\infty\cong G/D$, $D=D_\infty(T)$. This means that, taking an IUR ρ_ζ of D given above, we obtain the induced representation $R_\zeta:=\mathrm{Ind}_D^G\rho_\zeta$. In the case where $\zeta=\mathbf{1}_T$ the trivial representation of T, R_ζ is essentially the regular representation of $\mathfrak{S}_\infty\cong G/D$. Taking appropriately a positive definite matrix element F of R_ζ, and an increasing sequence G_N, we get f_A as a limit of centralizations F^{G_N} of F.

5. Characters of wreath product group $\mathfrak{S}_\infty(T)$ with T abelian. When T is abelian, the general character formula (14) for $G=\mathfrak{S}_\infty(T)$ with a compact group T has a simplified form. In this case, $\widehat{T}$ is nothing but the dual group consisting of all one-dimensional continuous characters of T, and for each $\zeta\in\widehat{T}$, its character χ_ζ is identified with ζ itself. For a $g\in G$, let its standard decomposition be as in (6). Put $K_j=\mathrm{supp}(\sigma_j)$, and for $d_j=(t_i)_{i\in K_j}\in D_{K_j}(T)\hookrightarrow D_\infty(T)$, put

$$P_{K_j}(d_j)=\prod_{i\in K_j}t_i\,,\quad \zeta(d_j):=\zeta(P_{K_j}(d_j))=\prod_{i\in K_j}\zeta(t_i)\,. \tag{19}$$

As a parameter for characters of $G=\mathfrak{S}_\infty(T)$, we consider a set $\alpha_{\zeta,\varepsilon}$'s and μ in (11) satisfying the condition (12).

THEOREM 5.1. *Let $G=\mathfrak{S}_\infty(T)$ be a wreath product group of a compact abelian group T with $\mathfrak{S}_\infty$. Then, for a parameter $A=((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}};\mu)$ in* (11)–(12), *the following formula determines a character f_A of G: for an element $g\in G$, let its standard* *decomposition be as in (6), then*

$$f_A(g) = \prod_{1\le k\le r} \Big\{ \sum_{\zeta\in\widehat{T}} \Big(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,i} + \mu_\zeta \Big) \zeta(t_{q_k}) \Big\} \times \prod_{1\le j\le m} \Big\{ \sum_{\zeta\in\widehat{T}} \Big(\sum_{\varepsilon\in\{0,1\}} \sum_{i\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,i})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \Big) \zeta(d_j) \Big\}, \tag{20}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$*, and* $\zeta(d_j)$ *as in* (19).

Conversely any character of G *is given in the form of* f_A*.*

REMARK 5.1. For the natural subgroup $G^e = \mathfrak{S}^e_\infty(T) = D^e_\infty(T) \rtimes \mathfrak{S}_\infty$ of $G = \mathfrak{S}_\infty(T)$, it is proved that a general character formula can be deduced from the one for G, by restriction from G to G^e.

6. Method of proving Theorem 4.1. Our proof of Theorem 4.1 can be carried out just as in the case of finite groups T in [HH2]. It consists of two parts. The first part is to prepare seemingly sufficiently big family of factorizable (hence extremal by the criterion in the second part) continuous positive definite class functions on $G = \mathfrak{S}_\infty(T)$. The second part is to guarantee that actually all extremal continuous positive definite class functions or characters have been already obtained in the first part.

6.1. *The first part of the proof.* The first part has two important ingredients. One is a method of *taking limits of centralizations* of positive definite functions. This method has been applied in [Hir3] to the case of $\mathfrak{S}_\infty$ and reestablished the results in [Tho].

The other is *inducing up positive definite functions* from subgroups. After choosing appropriate subgroups H and their representations π, we use their matrix elements f_π as positive definite functions on H to be induced up to G, and then to be centralized. We have constructed in [Hir1] a huge family of IURs of a wreath product group $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with any finite group T, by taking so-called wreath product type subgroups H in a 'saturated fashion', and their IURs π of a certain form to get IURs of G as induced representations $\rho = \mathrm{Ind}_H^G \pi$. For our present purpose, actually it is sufficient to choose simpler subgroups of degenerate wreath product type and their IURs π.

6.2. *The second part of the proof.* The second part contains also two ingredients. The first one is to establish the criterion that *a positive definite class function on* $G = \mathfrak{S}_\infty(T)$ *is extremal or indecomposable if and only if it is factorizable* (Theorem 11.1).

The second one is to determine the range of parameters for extremal positive definite class functions f. Since f is factorizable, $f(g)$ is written as

$$f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \prod_{1\le j\le m} f(g_j)$$

for $g = \xi_{q_1}\cdots\xi_{q_r}\, g_1\cdots g_m$. Then, we take a kind of 'Fourier transform' of f on $G = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with respect to the subgroup $D_\infty(T)$, and get a positive definite class function on $\mathfrak{S}_\infty$. Then for this we can appeal to Korollar 1 to Satz 2 in [Tho].

7. Subgroups and their representations for $\mathfrak{S}_\infty(T)$**.** In place of the purpose in [Hir1]-[Hir2] of getting IURs, our present purpose is to get all the characters of $G =$

$\mathfrak{S}_\infty(T)$. In the papers [Hir3], [HH1] and [HH2] respectively in the case of $\mathfrak{S}_\infty$, of $\mathfrak{S}_\infty(T)$ with T finite abelian, and of $\mathfrak{S}_\infty(T)$ with T finite in general, we have applied the method of taking limits of centralizations of the trivial inducing up $F = \mathrm{Ind}_H^G f_\pi$ of a matrix element f_π of a UR π of H. The limits turn out to be characters. Here, by definition, $\mathrm{Ind}_H^G f_\pi = f_\pi$ on H and is equal to zero outside H, and $f_\pi(h) := \langle \pi(h)w_0, w_0\rangle$ for a unit vector $w_0 \in V(\pi)$, the representation space of π. Now take a partition of $\boldsymbol{N}$ as

$$\boldsymbol{N} = \Big(\bigsqcup_{(\zeta,\varepsilon)\in \widehat{T}_0\times\{0,1\}} \Big(\bigsqcup_{p\in P_{\zeta,\varepsilon}} I_p\Big)\Big) \sqcup \Big(\bigsqcup_{\zeta\in\widehat{T}_0} I_\zeta\Big), \tag{21}$$

where $\widehat{T}_0$ is a countable subset of $\widehat{T}$, and each $P_{\zeta,\varepsilon}$ is an infinite index set, and the subsets I_p, I_ζ are all infinite. Corresponding to this partition, we define a subgroup

$$H = \Big({\prod}'_{(\zeta,\varepsilon)\in \widehat{T}_0\times\{0,1\}} \Big({\prod}'_{p\in P_{\zeta,\varepsilon}} H_p\Big)\Big) \times \Big({\prod}'_{\zeta\in\widehat{T}_0} H_\zeta\Big), \tag{22}$$

with

$$H_p = \mathfrak{S}_{I_p}(T), \quad H_\zeta = D_{I_\zeta}(T) \subset \mathfrak{S}_{I_\zeta}(T)\,,$$

where $\prod'$ denotes the restricted direct product. Note that, for the original case of $\mathfrak{S}_\infty$, we take the trivial group $T = \{e_T\}$, and for $\zeta = \boldsymbol{1}_T \in \widehat{T}$ we take $H_\zeta = \{\boldsymbol{1}_{I_\zeta}\} \subset \mathfrak{S}_{I_\zeta}$ with the identity element $\boldsymbol{1}_{I_\zeta}$ of $\mathfrak{S}_{I_\zeta}$. As a UR π of H, we take

$$\pi = \Big(\otimes^{\,b}_{(\zeta,\varepsilon)\in \widehat{T}_0\times\{0,1\}} \Big(\otimes^{\,b_{\zeta,\varepsilon}}_{p\in P_{\zeta,\varepsilon}} \pi_p\Big)\Big) \otimes \Big(\otimes^{\,b'}_{\zeta\in\widehat{T}_0} \pi_\zeta\Big). \tag{23}$$

Here $b, b_{\zeta,\varepsilon}, b'$ are reference vectors. Furthermore $b_{\zeta,\varepsilon} = (b_p)_{p\in P_{\zeta,\varepsilon}}$ with $b_p \in V(\pi_p), \|b_p\| = 1$ $(p \in P_{\zeta,\varepsilon})$, and π_p for $H_p = \mathfrak{S}_{I_p}(T)$ is given as

$$\pi_p((d,\sigma)) = \Big(\otimes^{\,a_p}_{i\in I_p} \zeta_i(t_i)\Big) I(\sigma)\,\mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad \text{for} \quad d = (t_i)_{i\in I_p},\ \sigma \in \mathfrak{S}_{I_p}, \tag{24}$$

where $a_p = (a_i)_{i\in I_p}$ is a reference vector with $a_i \in V(\zeta_i), \|a_i\| = 1$, and $\zeta_i = \zeta$ as a representation of $T_i = T$ $(i \in I_p)$, and $I(\sigma)$ is defined as

$$I(\sigma):\ v = \otimes_{i\in I_p} v_i \mapsto \otimes_{i\in I_p} v'_i\,,\ \ v'_i = v_{\sigma^{-1}(i)} \qquad (v_i \in V(\zeta_i), i \in I_p)\,.$$

Moreover $b' = (b_\zeta)_{\zeta\in\widehat{T}_0}$ with $b_\zeta \in V(\zeta), \|b_\zeta\| = 1$, and for $\zeta \in \widehat{T}_0$, π_ζ of H_ζ is given as

$$\pi_\zeta(d) = \otimes^{a_\zeta}_{i\in I_\zeta} \zeta_i(t_i) \quad \text{for} \ \ d = (t_i)_{i\in I_\zeta} \in H_\zeta = D_{I_\zeta}(T), \tag{25}$$

where $a_\zeta = (a_i)_{i\in I_\zeta}$ with $a_i \in V(\zeta_i), \|a_i\| = 1$, and $\zeta_i = \zeta$ for $T_i = T$ $(i \in I_\zeta)$.

We put $\widehat{b} := \otimes_{(\zeta,\varepsilon)\in\widehat{T}_0\times\{0,1\}} b_{\zeta,\varepsilon}$, $b_{\zeta,\varepsilon} := \otimes_{p\in P_{\zeta,\varepsilon}} b_p$, and $\widehat{b'} := \otimes_{\zeta\in\widehat{T}_0} b_\zeta$, $b_\zeta = \otimes_{i\in I_\zeta} a_i$, and further take a matrix element $f_\pi(h) := \langle \pi(h)w_0, w_0\rangle$ for $w_0 := \widehat{b} \otimes \widehat{b'} \in V(\pi)$.

8. Increasing sequences of subgroups $G_N \nearrow G = \mathfrak{S}_\infty(T)$. Depending on the choice of increasing series $G_N \nearrow G$ of subgroups, we get various positive definite class functions of G as limits of centralizations F^{G_N} for the trivial inducing up $F = \mathrm{Ind}_H^G f_\pi$ of a matrix element f_π, which turn out to be characters. We choose a series G_N as $G_N = \mathfrak{S}_{J_N}(T), J_N \nearrow \boldsymbol{N}$, and demand an asymptotic condition as

$$\frac{|I_p \cap J_N|}{|J_N|} \to \lambda_p \ \ (p \in P), \quad \frac{|I_\zeta \cap J_N|}{|J_N|} \to \mu_\zeta \ \ (\zeta \in \widehat{T}), \tag{26}$$

where $P := \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} P_{\zeta,\varepsilon}$ is the union of index sets. We have

$$\sum_{p\in P}\lambda_p + \sum_{\zeta\in\widehat{T}}\mu_\zeta \le 1, \tag{27}$$

where the inequality may happen since $|P| = \infty$.

For each $(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}$, let reorder the numbers $\{\lambda_p\,;\,p\in P_{\zeta,\varepsilon}\}$ in the decreasing order and put it as $\alpha_{\zeta,\varepsilon} := (\alpha_{\zeta,\varepsilon,i})_{i\in\boldsymbol{N}}$: $\alpha_{\zeta,\varepsilon,1}\ge\alpha_{\zeta,\varepsilon,2}\ge\cdots$, and also put $\mu := (\mu_\zeta)_{\zeta\in\widehat{T}}$. Then,

$$\sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \le 1,$$

which is nothing but the condition (16) if it is in the case of a finite group T. In the case of infinite T, if the inequality <1 holds, the continuity at $g = e$ of $f'(g) = \lim_{N\to\infty}F^{G_N}(g)$ is not guaranteed since $f'(e) = 1$ (cf. Remark 4.1). So we only pick up the case where the equality holds in (27) or the equality (12) holds (cf. Remark 4.2). As a pointwise limit of the series of centralizations F^{G_N}, we obtain the character f_A with $A = ((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu)$ in Theorem 4.1 under the asymptotic condition (26).

9. Partial centralization with respect to $D_{J_N}(T)$. As an increasing sequence $G_N \nearrow G = \mathfrak{S}_\infty(T)$ of subgroups, we have chosen $G_N = \mathfrak{S}_{J_N}(T) = D_{J_N}(T)\rtimes\mathfrak{S}_{J_N}$ with $J_N \nearrow \boldsymbol{N}$. Put $D_N = D_{J_N}(T)$ and $S_N = \mathfrak{S}_{J_N}$ for simplicity, then $G_N = D_N\rtimes S_N$, and we identify $d'\in D_N$ and $\sigma'\in S_N$ with their images in G_N respectively. Our task is to calculate centralizations F^{G_N} of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$, and to determine their limits. We see that for $h\in H$

$$F^{G_N}(h) = \int_{G_N} f_\pi(g'hg'^{-1})\,d\mu_{G_N}(g') = \frac{1}{|S_N|}\sum_{\sigma'\in S_N\,:\,\sigma'h\sigma'^{-1}\in H}\widetilde{f_\pi}(\sigma'h\sigma'^{-1}), \tag{28}$$

where $\widetilde{f_\pi}$ is a *partial centralization* of f_π with respect to $D_N\cong T^{J_N}$ defined as

$$\widetilde{f_\pi}(h') = \int_{D_N} f_\pi(d'h'd'^{-1})\,d\mu_{D_N}(d') \qquad (h'\in H), \tag{29}$$

with the normalized Haar measure $d\mu_{D_N}$ on D_N.

Put $K = \{1,2,\ldots,\ell\}$, and let $\sigma = (1\ 2\ \ldots\ \ell)$ be a cycle with $\mathrm{supp}(\sigma) = K$ and $g = (d,\sigma)$ a basic element in $\mathfrak{S}_K(T)$ with $d = (t_i)_{i\in K}$. Then, for $d' = (s_i)_{i\in K}\in D_K(T)$,

$$d'gd'^{-1} = (d'',\sigma) \quad\text{with}\quad d'' = d'd\cdot\sigma(d'^{-1}) = (s_it_is_{i-1}^{-1})_{i\in K} \quad (s_0 = s_\ell). \tag{30}$$

On the other hand, for a decomposable vector $v = \otimes_{i\in K}v_i\in V(\otimes_{i\in K}\zeta_i)$ with $v_i\in V(\zeta_i), \zeta_i = \zeta$, the restriction Π of π_p onto $\mathfrak{S}_K(T)\subset\mathfrak{S}_{I_p}(T)$ is given as

$$\Pi(g)v = \otimes_{i\in K}(\zeta(t_i)v_{\sigma^{-1}(i)}) = \otimes_{i\in K}(\zeta(t_i)v_{i-1}) \quad (v_0 = v_\ell).$$

LEMMA 9.1. *Let $\otimes_{i\in K}\zeta_i$ be a tensor product representation of $D_K(T)\cong T^K$ of $\zeta_i = \zeta$ of $T_i = T$ $(i\in K)$, and take decomposable vectors $v = \otimes_{i\in K}v_i$ from $V(\otimes_{i\in K}\zeta_i)$ with $v_i\in V(\zeta_i), \|v_i\| = 1$. Then, as an integration with respect to the normalized Haar measure*

$d\mu_{D_K(T)}(s) = \prod_{i\in K} d\mu_T(s_i), s = (s_i)_{i\in K} \in T^K \cong D_K(T),$

$$\int_{D_K(T)} \langle \Pi(sgs^{-1})v, w\rangle \, d\mu_{D_K(T)}(s) \;=\; \frac{\chi_\zeta(t_\ell t_{\ell-1}\cdots t_2 t_1)}{(\dim\zeta)^\ell} \;=\; \frac{\chi_\zeta(P_\sigma(d))}{(\dim\zeta)^\ell}\,.$$

Let H be a subgroup of G given in (21)–(22), and π its UR given in (23)–(25). Taking a unit vector $w_0 \in V(\pi)$ as in the last part of §7, we put $f_\pi(h) = \langle \pi(h)w_0, w_0\rangle$ $(h \in H)$.

PROPOSITION 9.2. *Take a $g = (d,\sigma)$ from H and let $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$, $\xi_q = (t_q,(q))$, $g_j = (d_j,\sigma_j)$, be its standard decomposition. Then, the partial centralization $\widetilde{f_\pi}(g)$ of matrix element f_π is given as follows. Let $K(\zeta)$ be the set of $k\,(1 \le k \le r)$ such that $\xi_{q_k} \in H_p$ with $p \in \bigsqcup_{\varepsilon\in\{0,1\}} P_{\zeta,\varepsilon}$ or $\xi_{q_k} \in H_\zeta$, and $J(\zeta,\varepsilon)$ be the set of $j\,(1 \le j \le m)$ such that $g_j = (d_j,\sigma_j) \in H_p$ with $p \in P_{\zeta,\varepsilon}$. Then,*

$$\widetilde{f_\pi}(g) = \Bigg(\prod_{\zeta\in\widehat{T}}\ \prod_{k\in K(\zeta)} \frac{\chi_\zeta(t_{q_k})}{\dim\zeta}\Bigg)\Bigg(\prod_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ \prod_{j\in J(\zeta,\varepsilon)} \frac{\chi_\zeta(P_{\sigma_j}(d_j))}{(\dim\zeta)^{\ell(\sigma_j)}}\,\mathrm{sgn}(\sigma_j)^\varepsilon\Bigg). \tag{31}$$

10. Limits of centralizations of matrix elements.

For any element in G, there exists an element in H conjugate to it. Therefore, from (28), it is enough for us to determine the value of F^{G_N} on H. Take $g = (d,\sigma) \in H$ and take its its standard decomposition as in Proposition 9.2. Since $H = (\prod'_{p\in P} H_p) \times (\prod'_{\zeta\in\widehat{T}_0} H_\zeta)$, the condition $g \in H$ means that each ξ_{q_k} belongs to one of H_p and H_ζ, and each g_j belongs to one of H_p. Furthermore, the latter condition can be expressed by means of supports as

$$\mathrm{supp}(\xi_{q_k}) = \{\,q_k\,\} \subset I_p \ \text{ or } \ \subset I_\zeta\,, \quad \text{and} \quad K_j = \mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j) \subset I_p. \tag{32}$$

Hence, using Proposition 9.2, we can calculate as in [Hir3] and [HH1], and get the first half of Theorem 4.1:

THEOREM 10.1. *Let f_A be the class function on $G = \mathfrak{S}_\infty(T)$ given in* (15) *with parameter A in* (13) *satisfying the conditions* (11)–(12). *Then f_A is obtained as a limit of centralizations F^{G_N} of a positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with (H,π) given above. The limit is taken according to an increasing sequence of subgroups $G_N = \mathfrak{S}_{J_N}(T)$ with $J_N \nearrow \boldsymbol{N}$ obeying the asymptotic condition* (26).

11. Factorizability for extremal positive definite class functions

THEOREM 11.1. *Let T be a compact group, and f a continuous positive definite class function on $G = \mathfrak{S}_\infty(T)$ normalized as $f(e) = 1$. Then f is extremal or $f \in E(G)$, if and only if it has one of the following properties which are mutually equivelent:*

(FTP) [Factorizability Property] *For any $g = (d,\sigma) \in G$, let*

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m, \quad \xi_q = (t_q,(q)), \quad g_j = (d_j,\sigma_j),$$

be its standard decomposition. Then,

$$f(g) = \prod_{1\le k\le r} f(\xi_{q_k}) \times \prod_{1\le j\le m} f(g_j). \tag{33}$$

(FTP′) *For any two elements g, g' with disjoint supports, $f(gg') = f(g)f(g')$.*

We rewrite these conditions in another form. As in Theorem 3.2, conjugacy classes of *basic elements* in G is given by the set Ω of the following objects ω :

$$\omega = ([t], \ell\,) \in (T/\!\sim) \times \boldsymbol{N}, \tag{34}$$

and the conjugacy class of $g \in G, \neq e$, with the above standard decomposition is determined by the collection $\{\,([t_{q_k}], \ell = 1)\ (1 \le k \le r),\ (P_{\sigma_j}(d_j), \ell(\sigma_j)\,)\ (1 \le j \le m)\,\}$, and the conjugate class of $g = e$ by $\omega_0 = ([e_T], \ell = 1)$. For $g \neq e$, denote by $n_\omega(g)$ the multiplicity of $\omega \in \Omega$ in this collection for g. We put $n_{\omega_0}(e) = 1$ and $n_{\omega_0}(g) = 0$ $(g \in G, \neq e)$ by definition.

Put $\boldsymbol{Z}_{\ge 0} := \{\, n \in \boldsymbol{Z}\ ;\ n \ge 0\,\}$ and denote by $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$ the set of all $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega}$, $n_\omega \in \boldsymbol{Z}_{\ge 0}$, with $n_\omega = 0$ for almost all ω, and $n_{\omega_0} = 1$ if $n_\omega = 0$ $(\forall \omega \neq \omega_0)$ and $n_{\omega_0} = 0$ otherwise. Then, $\boldsymbol{n}(g) := (n_\omega(g))_{\omega\in\Omega}$ is an element of $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$, and the correspondence

$$\Phi :\ [g] \ \mapsto\ \boldsymbol{n}(g) \in (\boldsymbol{Z}_{\ge 0})^{(\Omega)} \tag{35}$$

gives a bijective map from the set $G/\!\sim$ of all conjugacy classes of $g \in G$ onto $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$. We introduce in the latter the topology in $G/\!\sim$ through the map Φ.

For $\omega = ([t], \ell) \in \Omega$, put $\omega^{-1} := ([t^{-1}], \ell)$. Then, if ω is the conjugacy class of $\xi_q = (t_q, (q))$ or of $g_j = (d_j, \sigma_j)$, then ω^{-1} is that of ξ_q^{-1} or of g_j^{-1} respectively. Hence, $n_\omega(g^{-1}) = n_{\omega^{-1}}(g)$, and the transformation $[g] \mapsto [g^{-1}]$ in the set $G/\sim$ of conjugacy classes of elements in G induces an involutive transformation ι on $(\boldsymbol{Z}_{\ge 0})^{(\Omega)}$ given as

$$\iota\ :\ (\boldsymbol{Z}_{\ge 0})^{(\Omega)} \ni \boldsymbol{n} = (n_\omega)_{\omega\in\Omega} \mapsto \boldsymbol{n'} = (n'_\omega)_{\omega\in\Omega}\ \text{ with }\ n'_\omega = n_{\omega^{-1}}\ (\omega \in \Omega). \tag{36}$$

We put $\Omega_{re} := \{\,\omega \in \Omega\ ;\ \omega^{-1} = \omega\,\}$, $\Omega_c := \{\,\omega \in \Omega\,;\, \omega^{-1} \neq \omega\,\}$, then $\Omega = \Omega_{re} \sqcup \Omega_c$. Furthermore put $D_\omega := D = \{\, z \in \boldsymbol{C}; |z| \le 1\,\} \subset \boldsymbol{C}$ for $\omega \in \Omega$, and $I_\omega := [-1, 1] \subset \boldsymbol{R}$ for $\omega \in \Omega_{re}$, and $S := \prod_{\omega\in\Omega} D_\omega$. With the product topology S is compact, and on it we have two commuting involutions as $\jmath(s) := (s'_\omega)_{\omega\in\Omega}$ with $s'_\omega := s_{\omega^{-1}}$, and $\overline{s} := (\overline{s_\omega})_{\omega\in\Omega}$ (conjugate numbers), for $s = (s_\omega)_{\omega\in\Omega}$. Then we put

$$S' := \{\, s \in S = \textstyle\prod_{\omega\in\Omega} D_\omega\ ;\ \jmath(s) = \overline{s}\}, \tag{37}$$

then for $s \in S'$, $s_{\omega^{-1}} = \overline{s_\omega}$ and so $s_\omega \in I_\omega$ for $\omega \in \Omega_{re}$.

For a continuous positive definite class function f on G, put $s(f) = (s_\omega)_{\omega\in\Omega}$ with $s_\omega = f(g_\omega)$, where g_ω denotes a basic element in the class ω (put $g_{\omega_0} = e$ and $s_{\omega_0} = f(g_{\omega_0}) = f(e)$). Since $f \in K_{\le 1}(G)$ has the symmetry

$$f(g^{-1})\ =\ \overline{f(g)} \qquad (g \in G), \tag{SYM1}$$

and since ω^{-1} is represented by g_ω^{-1}, there holds a symmetry condition for $s = s(f)$

$$\jmath(s) = \overline{s} \qquad (\text{or } s \in S'\,). \tag{SYM2}$$

Define a positive definite class function $\overline{f}$ by $\overline{f}(g) = \overline{f(g)}$ $(g \in G)$, then $s(\overline{f}) = \overline{s(f)}$.

On the product space $S' \times (\boldsymbol{Z}_{\ge 0})^{(\Omega)}$, we define a function

$$P(\boldsymbol{n}, s)\ =\ \prod_{\omega\in\Omega} {s_\omega}^{n_\omega} \qquad \text{with}\ \ {s_\omega}^0 = 1, \tag{38}$$

for $\boldsymbol{n} = (n_\omega)_{\omega\in\Omega}, s = (s_\omega)_{\omega\in\Omega}$. Then, $P(\iota(\boldsymbol{n}), s) = P(\boldsymbol{n}, \jmath(s)) = P(\boldsymbol{n}, \overline{s})$. Fixing an $s = (s_\omega) \in S'$, we get a function $\Psi_s(\boldsymbol{n}) := P(\boldsymbol{n}, s)$ on $(\boldsymbol{Z}_{\ge 0})^{(\Omega)} \cong G/\!\sim$. Similarly, fixing an $\boldsymbol{n}$, we get a function on S' by $P_{\boldsymbol{n}}(s) := P(\boldsymbol{n}, s)\,(s \in S')$.

For every $s \in S'$, we get a factorizable class function on G as

(39) $$f_s := \Psi_s \circ \Phi \quad \text{or} \quad f_s(g) \;=\; \Psi_s(\boldsymbol{n}(g)) = P(\boldsymbol{n}(g), s) = P_{\boldsymbol{n}(g)}(s).$$

Then $P_{\boldsymbol{n}(g)}(s)$ satisfies the next symmetry condition, equivalent to (SYM1), for $f = f_s$:

(SYM3) $$P_{\iota(\boldsymbol{n})}(s) = \overline{P_{\boldsymbol{n}}(s)} \quad \text{for} \quad \boldsymbol{n} = \boldsymbol{n}(g) \in (\boldsymbol{Z}_{\geq 0})^{(\Omega)}.$$

Thus the condition (FTP) above is rewritten as follows:

(FTP$''$) *There exists an* $s = (s_\omega)_{\omega\in\Omega}$ *in* S' *such that* $f = f_s$ *in* (39).

12. Final step of the proof of Theorem 4.1. The functions f_A in Theorem 4.1 are given as limits of centralizations F^{G_N} of positive definite function $F = \mathrm{Ind}_H^G f_\pi$ with a matrix element f_π of a UR π of a subgroup H of $G = \mathfrak{S}_\infty(T)$. We see that $f = f_A$ is factorizable and the corresponding $s(f) = s = (s_\omega)_{\omega\in\Omega}$ is easily given from the formula (14).

Conversely here we prove that *any extremal positive definite class function (or a character) f on G, normalized as $f(e) = 1$, is given in the form of f_A in* (15) *with parameter* $A = ((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu)$ *in* (11)–(12).

By Theorem 11.1, we should examine when a factorizable class function $f = f_s = \Psi_s \circ \Phi$ with $s = (s_\omega)_{\omega\in\Omega}$ is positive definite. For $\omega = ([t], \ell) \in \Omega$, $\ell = 1$, we have a class function X on T as $X(t) := s_{([t],1)}$ $(t \in T)$. Then, X is a continuous positive definite class function on T. So X is expressed as a linear combination of $\chi_\zeta, \zeta \in \widehat{T}$, as $X(t) = \sum_{\zeta\in\widehat{T}} a_\zeta\, \chi_\zeta(t)$ $(t \in T)$ with $a_\zeta = \int_T X(t)\,\overline{\chi_\zeta(t)}\, d\mu_T(t) \geq 0$, $\sum_{\zeta\in\widehat{T}}(\dim\zeta)\, a_\zeta = 1$. The sum for X is absolutely convergent. For $\ell \geq 2$, we have also a continuous class function $Y_\ell(t)$ on T by $Y_\ell(t) := s_{([t],\ell)}$, where $s_{([t],\ell)} = s_\omega$ for $\omega = ([t], \ell) \in \Omega$. Then, similarly as for X, it is expressed as $Y_\ell(t) = \sum_{\zeta\in\widehat{T}} b_{\zeta,\ell}\, \chi_\zeta(t)\, (t \in T)$ with $b_{\zeta,\ell} = \int_T Y_\ell(t)\, \overline{\chi_\zeta(t)}\, d\mu_T(t)$.

For $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$, we have from (33) and (39)

(40) $$f(g) = \prod_{i\leq k\leq r} \Big(\sum_{\zeta\in\widehat{T}} a_\zeta\; \chi_\zeta(t_{q_k})\Big) \times \prod_{1\leq j\leq m} \Big(\sum_{\zeta\in\widehat{T}} b_{\zeta,\ell(\sigma_j)}\; \chi_\zeta(P_{\sigma_j}(d_j))\Big).$$

The following function $F_{\zeta_0,\varepsilon}$ for $(\zeta_0, \varepsilon) \in \widehat{T} \times \{0, 1\}$ is a special kind of functions f_A in Theorem 4.1, which has been proved to be positive definite and invariant: for $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m,\ \xi_q = (t_q, (q)),\ g_j = (d_j, \sigma_j)$,

$$F_{\zeta_0,\varepsilon}(g) = \prod_{1\leq k\leq r} \frac{\chi_{\zeta_0}(t_{q_k})}{\dim\zeta_0} \times \prod_{1\leq j\leq m} \frac{\chi_{\zeta_0}(P_{\sigma_j}(d_j))}{(\dim\zeta_0)^{\ell(\sigma_j)}}\, \mathrm{sgn}(\sigma_j)^\varepsilon.$$

Then the product $f'(g) := (f\,\overline{F_{\zeta_0,\varepsilon}})(g) = f(g)\,\overline{F_{\zeta_0,\varepsilon}}(g)$ is positive definite. Take a subgroup $D_n := D_{I_n}(T)$ with n sufficiently large so that $\mathrm{supp}(g) \subset I_n$. *A partial Fourier transform* $\mathcal{F}_{\zeta_0,\varepsilon;n}(f)$ of f with respect to $F_{\zeta_0,\varepsilon}$ is by definition the integral of f' with respect to D_n:

$$\mathcal{F}_{\zeta_0,\varepsilon;n}(f)(g) := \int_{D_n} f(d'g)\, \overline{F_{\zeta_0,\varepsilon}}(d'g)\; d\mu_{D_n}(d').$$

Considering it on $\mathfrak{S}_n$, we get a positive definite class function on $\mathfrak{S}_n$, for any n. For $\sigma \in \mathfrak{S}_n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be its decomposition into mutually disjoint cycles, then

$$\mathcal{F}_{\zeta_0,0;n}(f)(\sigma) = \left(\frac{a_{\zeta_0}}{\dim\zeta_0}\right)^{n-|\mathrm{supp}(\sigma)|} \times \prod_{1\le j\le m} \frac{b_{\zeta_0,\ell(\sigma_j)}}{(\dim\zeta_0)^{\ell(\sigma_j)}}\,.$$

Here we can apply Korollar 1 of Satz 2 in [Tho], and obtain the desired expression of f, through detailed calculations.

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