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THE $\mathbb{Z}_2$-GRADED STICKY SHUFFLE PRODUCT HOPF ALGEBRA

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Abstract. By abstracting the multiplication rule for $\mathbb{Z}_2$-graded quantum stochastic integrals, we construct a $\mathbb{Z}_2$-graded version of the Itô Hopf algebra, based on the space of tensors over a $\mathbb{Z}_2$-graded associative algebra. Grouplike elements of the corresponding algebra of formal power series are characterised.

1. Introduction. This paper concerns $\mathbb{Z}_2$-graded algebras. An associative algebra $\mathcal{A}$, not necessarily unital, is $\mathbb{Z}_2$-*graded* if, as a vector space, it is the internal direct sum $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ of *even* and *odd* subspaces which satisfy

$$\mathcal{A}_0\mathcal{A}_0, \mathcal{A}_1\mathcal{A}_1 \subset \mathcal{A}_0, \qquad \mathcal{A}_0\mathcal{A}_1, \mathcal{A}_1\mathcal{A}_0 \subset \mathcal{A}_1.$$

The *parity* δ is the function on the set $\mathcal{A}_0 \cup \mathcal{A}_1 - \{0\}$ of which is 0 on $\mathcal{A}_0$ and 1 on $\mathcal{A}_1$. The *Chevalley tensor product* [1] of two such algebras $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1$ is the $\mathbb{Z}_2$-graded associative algebra got by equipping the vector space tensor product $\mathcal{A} \otimes \mathcal{B}$ with the product defined by bilinear extension of the rule that, for homogeneous $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $(a \otimes b)(a' \otimes b') = (-1)^{\delta(b)\delta(b')} aa' \otimes bb'$ and the $\mathbb{Z}_2$-grading defined by $\delta(a \otimes b) = \delta(a) + \delta(b) \pmod 2$. In this paper the tensor product algebra of two $\mathbb{Z}_2$-graded associative algebras will always be understood as the Chevalley tensor product. Thus, for example, a $\mathbb{Z}_2$-*graded Hopf algebra* is a unital $\mathbb{Z}_2$-graded associative algebra $\mathcal{H}$ equipped with a counital coassociative coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ which is multiplicative and $\mathbb{Z}_2$-graded (in the sense that even subspaces map to even and odd to odd) when $\mathcal{H} \otimes \mathcal{H}$ is equipped with the Chevalley structure, together with an antipode S which is $\mathbb{Z}_2$-graded-antimultiplicative, that is satisfies $S(hh') = (-1)^{\delta(h)\delta(h')} S(h')S(h)$.

In [8], generalising the shuffle product Hopf algebra used in [2], a Hopf algebra structure was introduced in the space of tensors over an associative algebra which can be used [6] to quantise Lie bialgebras in which the Lie bracket is formed by taking commutators

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in the associative algebra in a comparatively [2] straightforward way. The purpose of the present work is to initiate a $\mathbb{Z}_2$-graded version of this circle of ideas. It is shown that, given a not necessarily unital $\mathbb{Z}_2$-graded associative algebra $\mathcal{L}$, the space of tensors $\mathcal{T}(\mathcal{L})$ can be equipped with the structure of a $\mathbb{Z}_2$-graded Hopf algebra which contains a sub-Hopf algebra isomorphic to the universal enveloping superalgebra of the Lie superalgebra got by taking supercommutators in $\mathcal{L}$.

2. The $\mathbb{Z}_2$-graded sticky shuffle product. Let $\mathcal{L}$ be a $\mathbb{Z}_2$-*graded vector space*, that is, a vector-space internal direct sum $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ of *even* and *odd* subspaces $\mathcal{L}_0$ and $\mathcal{L}_1$. The vector space $\mathcal{T}(\mathcal{L}) = \bigoplus_{n=0}^{\infty}(\bigotimes^n \mathcal{L})$ (where $\bigotimes^0 \mathcal{L}$ is the underlying field $\mathbb{F}$) of all tensors over $\mathcal{L}$ becomes a unital associative algebra when equipped with the product defined by bilinear extension of the rule that, for $L_1, L_2, ..., L_{m+n} \in \mathcal{L}$ of definite parity,

$$\begin{aligned}&(0,0,...,0,L_1\otimes L_2\otimes\cdots\otimes L_m,0,0,...)(0,0,...,0,L_{m+1}\otimes L_{m+2}\otimes\cdots\otimes L_{m+n},0,0,...)\\ &= \sum_{\pi\in\mathcal{S}_{m,n}} (-1)^{\sigma(\pi;L_1,L_2,...,L_{m+n})}(0,0,...,0,L_{\pi(1)}\otimes L_{\pi(2)}\otimes\cdots\otimes L_{\pi(m+n)}0,0,...). \qquad (1)\end{aligned}$$

Here $\mathcal{S}_{m,n}$ is the set of permutations π of $\{1,2,...,m+m\}$ such that $\pi(1) < \pi(2) < \cdots < \pi(m)$ and $\pi(m+1) < \pi(m+2) < \cdots \pi(m+n)$ and $\sigma(\pi; L_1, L_2, ..., L_{m+n})$ counts the number of transpositions of adjacent pairs of odd elements needed to effect the reordering $(L_1, L_2, ..., L_{m+n}) \to (L_{\pi(1)}, L_{\pi(2)}, \cdots, L_{\pi(m+n)})$. Thus, for example

$$(0,L_1,0,0,...)(0,L_2,0,0,...) = (0,0,L_1\otimes L_2 + (-1)^{\delta(L_1)\delta(L_2)}L_2\otimes L_1,0,0,...),$$

$$\begin{aligned}&(0,0,L_1\otimes L_2,0,0,...)(0,L_3,0,0,...)\\ &= (0,0,0,L_1\otimes L_2\otimes L_3 + (-1)^{\delta(L_2)\delta(L_3)}L_1\otimes L_3\otimes L_2\\ &\quad + (-1)^{\delta(L_2)\delta(L_3)+\delta(L_2)\delta(L_3)}L_3\otimes L_1\otimes L_2,0,0,...).\end{aligned}$$

The unit element is $(1_{\mathbb{F}}, 0, 0, ...)$. In the totally even case $\mathcal{L}_1 = \{0\}$ this is just the shuffle product, so called because of the analogy with shuffling packs of cards, so we call it the $\mathbb{Z}_2$-*graded shuffle product* corresponding to the $\mathbb{Z}_2$-grading $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$.

Now suppose $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ is a not necessarily unital $\mathbb{Z}_2$-*graded associative algebra*. Thus it is equipped with an associative multiplication with the properties $\mathcal{L}_0\mathcal{L}_0$, $\mathcal{L}_1\mathcal{L}_1 \subset \mathcal{L}_0$, $\mathcal{L}_0\mathcal{L}_1$, $\mathcal{L}_1\mathcal{L}_0 \subset \mathcal{L}_1$. We define a corresponding *sticky* $\mathbb{Z}_2$-*graded shuffle product* by adding extra terms to the shuffle product (1) based on this multiplication rule:

$$\begin{aligned}&(0,0,...,0,L_1\otimes L_2\otimes\cdots\otimes L_m,0,0,...)(0,0,...,0,L_{m+1}\otimes L_{m+2}\otimes\cdots\otimes L_{m+n},0,0,...)\\ &= \sum_{\pi\in\mathcal{S}_{m,n}} (-1)^{\sigma(\pi;L_1,L_2,...,L_{m+n})}\Big\{(0,0,...,0,L_{\pi(1)}\otimes L_{\pi(2)}\otimes\cdots\otimes L_{\pi(m+n)}0,0,...)\\ &\quad + \sum_{k=1}^{m\wedge n}\Big(0,0,..., \sum_{\tau\in\mathcal{C}_{\pi;k}}(L_{\tau(1)}\otimes L_{\tau(2)}\otimes\cdots\otimes L_{\tau(m+n-k)}),0,0,...\Big)\Big\}. \qquad (2)\end{aligned}$$

Here for each $k \in \{1,2,...,m\wedge n\}$, $\mathcal{C}_{\pi;k}$ is the class of ordered partitions $\tau = (\tau(1), \tau(2), ..., \tau(m+n-k))$ of the ordered set $(\pi(1), \pi(2), ..., \pi(m+n))$, consisting of k pairs and $m+n-2k$ singletons, obtained by bracketing some adjacent pairs $\{\pi(r), \pi(s)\}$ with $r \in \{1,2,...,m\}$, $s \in \{m+1, m+2, ..., m+n\}$ and $\pi(s) = \pi(r)+1$. For a singleton $\tau(j)$

$= \pi(r)$, $L_{\tau(j)}$ is defined to be L_r while for a pair $\tau(j) = \{\pi(r), \pi(s)\}$, $L_{\tau(j)} = L_r L_s$. Thus, in a sticky shuffle, after the initial shuffle, a card from the first pack may stick to an adjacent card from the second pack and form a single card. For example

$$(0, L_1, 0, 0, ...)(0, L_2, 0, 0, ...) = (0, L_1L_2, L_1 \otimes L_2 + (-1)^{\delta(L_1)\delta(L_2)} L_2 \otimes L_1, 0, 0, ...)$$

$$(0, 0, L_1 \otimes L_2 0, 0, ...)(0, L_3, 0, 0, ...)$$

$$\begin{aligned} &= (0, 0, L_1 \otimes L_2L_3, L_1 \otimes L_2 \otimes L_3, 0, 0, ...) \\ &\quad + (-1)^{\delta(L_2)\delta(L_3)}(0, 0, L_1L_3 \otimes L_2, L_1 \otimes L_3 \otimes L_2, 0, 0, ...) \\ &\quad +(-1)^{\delta(L_2)\delta(L_3)+\delta(L_2)\delta(L_3)}(0, 0, 0, L_3 \otimes L_1 \otimes L_2, 0, 0, ...). \end{aligned} \tag{3}$$

This procedure defines a unital associative multiplication in $\mathcal{T}(\mathcal{L})$ with unit $1_{\mathbb{F}}$.

If $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ is the complex $\mathbb{Z}_2$-graded algebra of quantum stochastic Itô differentials of [4],[5] then, for $a < b \in \mathbb{R}^+$, the linear iterated stochastic integral map I_a^b from $\mathcal{T}(\mathcal{L})$ to operators on the restricted exponential domain in the appropriate Fock space $\mathcal{F}_a^b$, for which

$$I_a^b(0, 0, ..., 0, L_1 \otimes L_2 \otimes \cdots \otimes L_m, 0, 0, ...) = \int_{a<t_1<t_2<\cdots<t_m<b} dL_1(t_1)dL_2(t_2)...dL_m(t_m),$$

is multiplicative, in the weak sense that for arbitrary restricted exponential vectors φ, ψ and α, β in $\mathcal{T}(\mathcal{L})$

$$\langle \varphi, I_a^b(\alpha\beta)\psi \rangle = \langle I_a^b(\alpha)^\dagger \varphi, I_a^b(\beta)\psi \rangle.$$

The $\mathbb{Z}_2$-graded sticky shuffle product algebra $\mathcal{T}(\mathcal{L})$ is itself $\mathbb{Z}_2$-graded by the linear extension of the rule that, for homogeneous $L_1, L_2, ..., L_m$,

$$\delta(L_1 \otimes L_2 \otimes \cdots \otimes L_m) = \sum_{j=1}^{m} \delta(L_j) \text{ (addition } \bmod 2).$$

It reduces to the corresponding unsticky product when the associative multiplication in $\mathcal{L}$ is the trivial one in which all products vanish.

From (3) it follows that the map $\phi : \mathcal{L} \ni L \mapsto (0, L, 0, 0, ...) \in \mathcal{T}(\mathcal{L})$ is a homomorphism of Lie superalgebras [9] when $\mathcal{L}$, and similarly $\mathcal{T}(\mathcal{L})$, is equipped with the *supercommutator* formed by bilinear extension of the rule for homogeneous L, K that $[L, K] = LK - (-1)^{\delta(L)\delta(K)} KL$. Thus ϕ has a unique extension Φ to a homomorphism of $\mathbb{Z}_2$-graded unital associative algebras from the universal enveloping superalgebra of the supercommutator Lie superalgebra $\mathcal{L}$ into$\mathcal{T}(\mathcal{L})$. It follows from the $\mathbb{Z}_2$-graded Poincaré-Birkhoff-Witt theorem [9] that the map Φ is injective.

3. The $\mathbb{Z}_2$-graded sticky shuffle product Hopf algebra. The $\mathbb{Z}_2$-graded shuffle product algebra $\mathcal{T}(\mathcal{L})$ becomes a $\mathbb{Z}_2$-graded Hopf algebra under the coproduct Δ defined by linear extension of the rule

$$\begin{aligned} &\Delta(0, 0, ..., 0, L_1 \otimes L_2 \otimes \cdots \otimes L_m, 0, 0, ...) \\ &= \sum_{j=0}^{m} (0, 0, ..., 0, L_1 \otimes L_2 \otimes \cdots \otimes L_j, 0, 0, ...) \\ &\otimes(0, 0, ..., 0, L_{j+1} \otimes L_{j+2} \otimes \cdots \otimes L_m, 0, 0, ...). \end{aligned} \tag{4}$$

The counit is the map ε taking each tensor into its zero-rank component and the antipode is the map S given by linear extension of

$$S(0,0,...,0,L_1\otimes L_2\otimes\cdots\otimes L_m,0,0,...)=(-1)^m\Delta(0,0,...,0,L_m\otimes L_{m-1}\otimes\cdots\otimes L_1,0,0,...). \tag{5}$$

In fact Δ remains multiplicative if shuffles are replaced by sticky shuffles.

THEOREM 1. *The coproduct defined by* (4) *is multiplicative for the product defined by* (2).

Proof. We abbreviate $(0,0,...,0,L_1\otimes L_2\otimes\cdots\otimes L_m,0,0,...)$ as $L_1\otimes L_2\otimes\cdots\otimes L_m$. Thus we have to show that, for $L_1,L_2,...,L_{m+n}\in\mathcal{L}$ of definite parity,

$$\begin{aligned}&\Delta((L_1\otimes L_2\otimes\cdots\otimes L_m)(L_{m+1}\otimes L_{m+2}\otimes\cdots\otimes L_{m+n}))\\ &=\Delta(L_1\otimes L_2\otimes\cdots\otimes L_m)\Delta(L_{m+1}\otimes L_{m+2}\otimes\cdots\otimes L_{m+n}).\end{aligned}$$

We give the proof in the case $m=2$, $n=1$. Then, replacing the symbol $\otimes$ by $\boxtimes$ for the tensor product between elements of different copies of $\mathcal{T}(\mathcal{L})$ for clarity, and denoting the grades of homogeneous elements L, L' and L'' of $\mathcal{L}$ by δ, δ' and δ'' respectively, we have

$$\begin{aligned}&\Delta((L\otimes L')L'')\\ &=\Delta(L\otimes L'\otimes L''+L\otimes L'L''+(-1)^{\delta'\delta''}(L\otimes L''\otimes L'+LL''\otimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}L''\otimes L\otimes L')\\ &=1_{\mathcal{T}(\mathcal{L})}\boxtimes(L\otimes L'\otimes L''+L\otimes L'L''+(-1)^{\delta'\delta''}(L\otimes L''\otimes L'+LL''\otimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}L''\otimes L\otimes L')\\ &\quad+L\boxtimes(L'\otimes L'')+(L\otimes L')\boxtimes L''+L\boxtimes L'L''\\ &\quad+(-1)^{\delta'\delta''}(L\boxtimes(L''\otimes L')+(L\otimes L'')\boxtimes L'+LL''\boxtimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}(L''\boxtimes(L\otimes L')+(L''\otimes L)\boxtimes L')\\ &\quad+(L\otimes L'\otimes L''+L\otimes L'L''+(-1)^{\delta'\delta''}(L\otimes L''\otimes L'+LL''\otimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}L''\otimes L\otimes L')\boxtimes 1_{\mathcal{T}(\mathcal{L})}.\end{aligned}$$

On the other hand

$$\begin{aligned}\Delta(L\otimes L')\Delta(L'')&=\{1_{\mathcal{T}(\mathcal{L})}\boxtimes(L\otimes L')+L\boxtimes L'+(L\otimes L')\boxtimes 1_{\mathcal{T}(\mathcal{L})}\}\\ &\quad\{1_{\mathcal{T}(\mathcal{L})}\boxtimes L''+L''\boxtimes 1_{\mathcal{T}(\mathcal{L})}\}\\ &=1_{\mathcal{T}(\mathcal{L})}\boxtimes(L\otimes L'\otimes L''+L\otimes L'L''+(-1)^{\delta'\delta''}(L\otimes L''\otimes L'+LL''\otimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}L''\otimes L\otimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}L''\boxtimes(L\otimes L')+L\boxtimes(L'\otimes L''+(-1)^{\delta'\delta''}L''\otimes L'+L'L'')\\ &\quad+(-1)^{\delta'\delta''}(L\otimes L''+(-1)^{\delta\delta''}L''\otimes L+LL'')\boxtimes L'+(L\otimes L')\boxtimes L''\\ &\quad+(L\otimes L'\otimes L''+L\otimes L'L''+(-1)^{\delta'\delta''}(L\otimes L''\otimes L'+LL''\otimes L')\\ &\quad+(-1)^{(\delta+\delta')\delta''}L''\otimes L\otimes L')\boxtimes 1_{\mathcal{T}(\mathcal{L})}\end{aligned}$$

as required. ∎

Thus $\mathcal{T}(\mathcal{L})$ becomes a $\mathbb{Z}_2$-graded bialgebra. By regarding it as a deformation of the $\mathbb{Z}_2$-graded unsticky shuffle product algebra (for example by inserting a formal deformation parameter h into the product in $\mathcal{L}$ so that the extra sticky terms are prefixed by positive powers of h; see [8] for the corresponding ungraded argument) we may invoke the $\mathbb{Z}_2$-graded version of the theorem [3] that a deformation bialgebra of a Hopf algebra is itself a Hopf algebra, to conclude that $\mathcal{T}(\mathcal{L})$ is a $\mathbb{Z}_2$-graded Hopf algebra, in which the antipode got by adding correction terms of lower rank to the right hand side of (5). We call it the *$\mathbb{Z}_2$-graded sticky shuffle product Hopf algebra.*

In the case when $\mathcal{L}$ is the $\mathbb{Z}_2$-graded algebra of quantum stochastic Itô differentials [4], [5], the coproduct Δ is related to the splitting or continuous tensor product structure of Fock space as follows. For $a < b < c \in \mathbb{R}^+$, making the identification $\mathcal{F}_a^c = \mathcal{F}_a^b \otimes \mathcal{F}_b^c$, we have

$$I_a^c = (I_a^b \otimes I_b^c)\Delta.$$

Here the algebras of processes on $\mathcal{F}_a^c$, $\mathcal{F}_a^b$, $\mathcal{F}_b^c$ must be $\mathbb{Z}_2$-graded using the corresponding grading operators [5] Γ_a^c, Γ_a^b, Γ_b^c. For example for homogeneous $L \in \mathcal{L}$ then

$$I_a^c(0, L, 0, 0, ...) = I_a^b(0, L, 0, 0, ...) \otimes \text{id}_{\mathcal{F}_b^c} + (\Gamma_a^b)^{\delta(L)} \otimes I_b^c(0, L, 0, 0, ...)$$

where $\otimes$ means the usual operator tensor product operator.

We define the iterated coproducts $\Delta^{(n)} : \mathcal{T}(\mathcal{L}) \to \bigotimes^n(\mathcal{T}(\mathcal{L}))$, n=0, 1, 2, ..., by $\Delta^{(0)} = \varepsilon$, $\Delta^{(1)} = \text{id}_{\mathcal{T}(\mathcal{L})}$, $\Delta^{(n)} = (\Delta \otimes \text{id}_{\bigotimes^{n-1}\mathcal{T}(\mathcal{L})})\Delta^{(n-1)}$, $n > 1$. Thus $\Delta^{(2)} = \Delta$. The following useful theorem also holds in the ungraded case [6].

Theorem 2. *For n=0, 1, 2, ...denote by α_n the component of rank n of $\alpha \in \mathcal{T}(\mathcal{L})$. Then, in the decomposition*

$$\bigotimes^n \mathcal{T}(\mathcal{L}) = \bigoplus_{m_1, m_2, ... m_n = 0}^{\infty} \left(\left(\bigotimes^{m_1}\right) \otimes \left(\bigotimes^{m_2}\right) \otimes \cdots \otimes \left(\bigotimes^{m_n}\right) \right),$$

the component of $\Delta^{(n)}(\alpha)$ of joint rank $(1, 1, ..., \overset{(n)}{1})$ is α_n.

Proof. The Theorem holds when $n = 0, 1$ by the definitions of $\Delta^{(0)}, \Delta^{(1)}$. From (4) it holds when $n = 2$ for tensors whose second rank components are product tensors and hence generally by linearity. For an nth rank product tensor then by iteration of (4) we have

$$\begin{aligned} &\Delta^{(n)}(0, 0, ..., 0, L_1 \otimes L_2 \otimes \cdots \otimes L_m, 0, 0, ...) \\ = &\sum_{0 \le j_1 \le j_2 \le \cdots \le j_n \le m} \{(0, 0, ..., 0, L_1 \otimes L_2 \otimes \cdots \otimes L_{j_1}, 0, 0, ...) \\ &\otimes (0, 0, ..., 0, L_{j_1+1} \otimes L_{j_1+2} \otimes \cdots \otimes L_{j_1+j_2}, 0, 0, ...) \otimes \cdots \\ &\otimes (0, 0, ..., 0, L_{j_1+j_2+\cdots j_n+1} \otimes L_{j_1+j_2+\cdots j_n+2} \otimes \cdots \otimes L_m, 0, 0, ...)\}. \end{aligned}$$

From this it follows that the component of joint rank $(1, 1, ..., \overset{(n)}{1})$ of $\Delta^{(n)}(0, 0, ..., 0, L_1 \otimes L_2 \otimes \cdots \otimes L_n, 0, 0, ...)$ is just $L_1 \otimes L_2 \otimes \cdots \otimes L_n$ and that product tensors of other ranks cannot contribute to this component. The result now follows by linearity. ∎

4. Grouplike elements of $\mathcal{T}(\mathcal{L})[[h]]$. We equip the vector space $\mathcal{T}(\mathcal{L})[[h]]$ of formal power series with coefficients in $\mathcal{T}(\mathcal{L})$ with the convolution multiplication

$$\sum_{N=0}^{\infty} h^N \alpha^{(N)} \sum_{N=0}^{\infty} h^N \beta^{(N)} = \sum_{N=0}^{\infty} h^N \sum_{j=0}^{N} \alpha^{(N-j)} \beta^{(j)}.$$

The coproduct Δ extends to a map from $\mathcal{T}(\mathcal{L})[[h]]$ to $(\mathcal{T}(\mathcal{L}) \otimes \mathcal{T}(\mathcal{L}))[[h]]$ by action on coefficients; $\Delta(\sum_{N=0}^{\infty} h^N \alpha^{(N)}) = \sum_{N=0}^{\infty} h^N \Delta(\alpha^{(N)})$. As an illustration of the use of Theorem 2 let us characterise elements $\alpha[h]$ of $\mathcal{T}(\mathcal{L})[[h]]$ which are *group-like,* meaning that $\Delta\alpha[h] = \alpha[h] \otimes \alpha[h]$. Here the tensor product is rearranged into a formal power series with coefficients in $\mathcal{T}(\mathcal{L}) \otimes \mathcal{T}(\mathcal{L})$ by convolution; thus

$$\sum_{N=0}^{\infty} h^N \alpha^{(N)} \otimes \sum_{N=0}^{\infty} h^N \beta^{(N)} = \sum_{N=0}^{\infty} h^N \sum_{j=0}^{N} \alpha^{(N-j)} \otimes \beta^{(j)}.$$

Theorem 3 may be compared with the proof based on calculus of a corresponding result in the ungraded case [7].

THEOREM 3. *Let $\alpha[h]$ be a nonzero grouplike element of $\mathcal{T}(\mathcal{L})[[h]]$. Then there exists a formal power series $L[h]$ with coefficients in $\mathcal{L}$ and vanishing zero-order coefficient such that*

$$\alpha[h] = \Big(1, L[h], L[h] \otimes L[h], ..., \bigotimes^{n} L[h], ...\Big). \tag{6}$$

Conversely every element of this form is grouplike.

Proof. Note first that for $L[h] = hL^{(1)} + h^2 L^{(2)} + \cdots \in h\mathcal{L}[[h]]$ the right hand side of (6) is a well defined element of $\mathcal{T}(\mathcal{L})[[h]]$;

$$\Big(1, L[h], L[h] \otimes L[h], ..., \bigotimes^{n} L[h], ...\Big).$$
$$= (1, 0, 0, ...) + h(0, L^{(1)}, 0, 0, ...) + h^2(0, L^{(2)}, L^{(1)} \otimes L^{(1)}, 0, 0, ...) + \cdots.$$

Suppose $\alpha[h] \in \mathcal{T}(\mathcal{L})[[h]]$ is nonzero and satisfies $\Delta\alpha[h] = \alpha[h] \otimes \alpha[h]$. Then in particular $\alpha_0[h] = (\alpha_0[h])^2$ whence either $\alpha_0[h] = 0$ or $\alpha_0[h] = 1$. In the former case the grouplike property implies that $\alpha[h] = 0$. Thus $\alpha_0[h] = 1$. Iterating the grouplike property gives

$$\Delta^{(n)}(\alpha[h]) = \bigotimes^{n}(\alpha[h]).$$

Hence by Theorem 2 the component of rank n is given by

$$\alpha_n[h] = \Big(\bigotimes^{n}(\alpha[h])\Big)_{(1,1,...,\overset{(n)}{1})} = \Big(\bigotimes^{n}(1, L[h], ...)\Big)_{(1,1,...,\overset{(n)}{1})} = \bigotimes^{n} L[h]$$

where $L[h] = \alpha_1[h] \in \mathcal{L}[[h]]$. But for $\alpha[h] = (1, L[h], L[h] \otimes L[h], ...)$ to be well defined as an element of $\mathcal{T}(\mathcal{L})[[h]]$ it is necessary that the zero-order coefficient $L_0 = 0$ other wise the zero-order coefficient of $\alpha[h]$ will be nonterminating. Hence $\alpha[h]$ is as claimed. The converse follows directly from (4). ∎

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