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q-WHITE NOISE AND NON-ADAPTED STOCHASTIC INTEGRAL

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Abstract. The q-white noise is studied as the time derivative of the q-Brownian motion. As an application of the q-white noise, a non-adapted (non-commutative) stochastic integral with respect to the q-Brownian motion is constructed.

1. Introduction. A Fock representation of q-commutation relation (introduced by Greenberg [7], and Bożejko and Speicher [3]) was first studied in [5] by constructing a q-Fock space as the space of representation, see also [2]. A representation of the q-commutation relation ($-1 \le q \le 1$) is given as the form:

$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle \cdot \mathbf{1}, \qquad \zeta, \eta \in H.$$

The q-commutation relation ($-1 < q < 1$) provides an interpolation between the fermionic and bosonic commutation relations which correspond to $q = -1$ and $q = 1$, respectively. The spaces of the representation of the fermionic and bosonic commutation relations are called the Fermion and Boson Fock spaces, respectively. Also, the full Fock space corresponds to $q = 0$. Recently, in [10], we constructed a q-Fock space as the space of the representation of the q-commutation relation such that for $0 < q < 1$, the q-Fock space is interpolated between the full Fock space and the Boson Fock space.

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On the other hand, stochastic calculus with respect to the q-annihilation process, q-creation process and q-gauge process has been developed in [15]. Also, in [6], a stochastic integral of adapted biprocess with respect to q-Brownian motion was developed by using the method used for the free case in [1].

Main purpose of this paper is to study the q-white noise as like as the (standard Gaussian) white noise [8, 11, 12, 13]. Then we construct a non-adapted stochastic integral with respect to the q-Brownian motion, more generally, with respect to the q-annihilation process and the q-creation process.

The paper is organized as follows. In Section 2 we briefly recall the notions in q-Fock space [10]. In Section 3 we study the q-white noise within a rigged q-Fock space. In Section 4 we construct a non-adapted stochastic integrals with respect to the q-Brownian motion.

2. q-Fock space. Let $\Gamma_0(H)$ be the full Fock space (with the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_0$) over a complex Hilbert space H. Let $\Gamma_0^{\mathrm{finite}}(H)$ be the linear span of vectors of the forms $\xi_1 \otimes \cdots \otimes \xi_n \in H^{\otimes n}$, $n = 0, 1, 2, \ldots$, where $H^{\otimes 0} = \mathbb{C}\Omega$ for the vacuum vector $\Omega \in \Gamma_0(H)$.

Let $q \in (-1, 1)$ be fixed. For each $n = 0, 1, 2, \ldots$, we put

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [0]_q = 0.$$

The q-factorial is defined as

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \qquad [0]_q! = 1.$$

Let S_n denote the symmetric group of all permutations on $\{1, \ldots, n\}$ and $I(\sigma)$ denote the number of inversions of the permutation $\sigma \in S_n$ defined by

$$I(\sigma) = \#\{(i, j) \mid 1 \le i < j \le n, \, \sigma(i) > \sigma(j)\}.$$

The operator P_q is defined on $\Gamma_0^{\mathrm{finite}}(H)$ by a linear extension of

$$P_q \Omega = \Omega;$$
$$P_q(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{\sigma \in S_n} q^{I(\sigma)} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$

Put

$$\xi_1 \otimes_q \cdots \otimes_q \xi_n := P_q(\xi_1 \otimes \cdots \otimes \xi_n), \qquad \xi_i \in H, \qquad i = 1, \ldots, n.$$

and then

$$\xi_1 \otimes_q \cdots \otimes_q \xi_n = \sum_{i=1}^{n} q^{i-1} \xi_i \otimes (\xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n). \tag{2.1}$$

Let $\Gamma_q^{\mathrm{finite}}(H)$ be the linear span of vectors of the forms $\xi_1 \otimes_q \cdots \otimes_q \xi_n \in H^{\otimes n}$, $n = 0, 1, 2, \ldots$. Define a sesquilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_q$ on $\Gamma_q^{\mathrm{finite}}(H)$ by a sesquilinear extension of

$$\langle\!\langle \xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes_q \cdots \otimes_q \eta_m \rangle\!\rangle_q := \delta_{nm} [n]_q! \langle\!\langle \xi_1 \otimes_q \cdots \otimes_q \xi_n, \eta_1 \otimes \cdots \otimes \; eta_m \rangle\!\rangle_0.$$

Then by applying Theorem 2.2 in [5], we see that the sesquilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_q$ is the strictly positive, i.e., $\langle\!\langle \xi, \xi \rangle\!\rangle_q > 0$ for $0 \neq \xi \in \Gamma_q^{\mathrm{finite}}(H)$. The completion of $\Gamma_q^{\mathrm{finite}}(H)$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_q$ is called the *q-Fock space* and denoted by $\Gamma_q(H)$.

For each $\zeta \in H$, we define the *q-creation operator* $a^*(\zeta)$ and the *q-annihilation operator* $a(\zeta)$ on the dense subspace $\Gamma_q^{\text{finite}}(H)$ of the q-Fock space $\Gamma_q(H)$ as follows:

$$a^*(\zeta)\Omega = \zeta;$$
$$a^*(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n = \frac{1}{\sqrt{[n+1]_q}}\zeta \otimes_q \xi_1 \otimes_q \cdots \otimes_q \xi_n$$

and

$$a(\zeta)\Omega = 0;$$
$$a(\zeta)\xi_1 \otimes_q \cdots \otimes_q \xi_n = \sqrt{[n]_q}\zeta \otimes^1 (\xi_1 \otimes_q \cdots \otimes_q \xi_n),$$

where $f \otimes^1 g$ is the left 1-contraction of $f \in H$ and $g \in H^{\otimes m}$, see [13]. From (2.1), we have

$$\zeta \otimes^1 (\xi_1 \otimes_q \cdots \otimes_q \xi_n) = \sum_{i=1}^{n} q^{i-1}\langle \zeta, \xi_i\rangle \xi_1 \otimes_q \cdots \otimes_q \check{\xi}_i \otimes_q \cdots \otimes_q \xi_n,$$

where the symbol $\check{\xi}_i$ means that ξ_i has to be deleted in the tensor product and $\langle \cdot, \cdot \rangle$ denotes the inner product on H.

THEOREM 2.1 ([10]). *Let $\zeta \in H$.*

(1) *The operators $a^*(\zeta)$ and $a(\zeta)$ are bounded on $\Gamma_q(H)$. Moreover,*
$$\|a(\zeta)\|_{\mathrm{OP}} = \|a^*(\zeta)\|_{\mathrm{OP}} \le 1/\sqrt{1-q}\,|\zeta|_H. \tag{2.2}$$
(2) *The operators $a^*(\zeta)$ and $a(\zeta)$ are adjoints of each other on $\Gamma_q^{\text{finite}}(H)$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_q$.*
(3) *The q-creation and q-annihilation operators fulfill the q-commutation relation, i.e.,*
$$a(\zeta)a^*(\eta) - qa^*(\eta)a(\zeta) = \langle \zeta, \eta\rangle \cdot \mathbf{1}, \qquad \zeta, \eta \in H.$$

The *Boson Fock space* is defined by

$$\Gamma_1(H) = \bigoplus_{n=0}^{\infty} H^{\widehat{\otimes} n} = \{\phi = (f_n)_{n=0}^{\infty} | f_n \in H^{\widehat{\otimes} n}, n = 0, 1, \ldots \text{ and } \|\phi\|_1 < \infty\},$$

where $H^{\widehat{\otimes} n}$ is the symmetric n-tensor product and $\|\phi\|_1^2 = \sum_{n=0}^{\infty} |f_n|^2$. Then we have the following

THEOREM 2.2 ([10]). *For any $0 \le q \le 1$ we have the following continuous inclusions:*

$$\Gamma_1(H) \subset \Gamma_q(H) \subset \Gamma_0(H).$$

In particular, $\Gamma_1(H)$ is isometrically embedded into $\Gamma_q(H)$ and the second inclusion is contraction.

3. q-White noise. Let $H = L^2(\mathbf{R}, dt)$ be the (complex) Hilbert space of L^2-functions on $\mathbf{R}$ with respect to the Lebesgue measure dt and the norm is denoted by $|\cdot|_0$. Let A be the harmonic oscillator given by

$$A = 1 + t^2 - \frac{d^2}{dt^2} = \left(t + \frac{d}{dt}\right)^*\left(t + \frac{d}{dt}\right) + 2.$$

Recall that

$$e_j(t) = (\sqrt{\pi}\, 2^j j!)^{-1/2} H_j(t) e^{-t^2/2}, \qquad j = 0, 1, 2, \ldots,$$

where H_j is the Hermite polynomial of degree j, constitute an orthonormal basis of $L^2(\mathbf{R})$ and $Ae_j = (2j+2)e_j$, and so

$$\|A^{-1}\|_{\mathrm{OP}} = 1/2 < 1, \qquad \|A^{-r}\|_{\mathrm{HS}}^2 = \sum_{n=0}^{\infty} \frac{1}{(2n+2)^{2r}} < \infty, \qquad r > 1/2.$$

For $p \in \mathbf{R}$ we define

$$|\xi|_p^2 = |A^p \xi|_0^2 = \sum_{j=0}^{\infty} (2j+2)^{2p} |\langle \xi, e_j \rangle|^2, \qquad \xi \in H.$$

Now, for $p \geq 0$, setting $E_p = \{\xi \in H\,;\, |\xi|_p < \infty\}$ and defining E_{-p} to be the completion of H with respect to $|\cdot|_{-p}$, we obtain a chain of Hilbert spaces $\{E_p\,;\, p \in \mathbf{R}\}$. Define their limit spaces:

$$E = \mathcal{S}(\mathbf{R}) = \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* = \mathcal{S}(\mathbf{R})^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p},$$

where E^* is the dual space of E which is well-known as the Schwartz space. Identifying H with its dual space, we have

$$E \subset E_p \subset H = L^2(\mathbf{R}, dt) \subset E_{-p} \subset E^*. \tag{3.1}$$

By taking q-Fock space from (3.1), we have the following natural inclusions:

$$\Gamma_q(E_p) \subset \Gamma_q(H) \subset \Gamma_q(E_{-p}), \qquad p \geq 0.$$

By the general duality theory, $\Gamma_q(E_{-p})$ is the strong dual space of $\Gamma_q(E_p)$. The norm generated by the sesquilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{q;r}$ on $\Gamma_q(E_r)$ is denoted by $\|\cdot\|_{q;r}$.

Let $T \in \mathcal{L}(H, H)$ and $\Gamma(T)$ be the second quantization of T on $\Gamma_0^{\mathrm{finite}}(H)$, i.e.,

$$\Gamma(T)(\xi_1 \otimes \cdots \otimes \xi_n) = T^{\otimes n}(\xi_1 \otimes \cdots \otimes \xi_n), \qquad \xi_1, \ldots, \xi_n \in H, \qquad n = 1, 2, \ldots.$$

Then since $T^{\otimes n}$ and P_q commute, for any $\xi_1, \ldots, \xi_n \in H$ we have

$$\begin{aligned}\|\Gamma(T)(\xi_1 \otimes_q \cdots \otimes_q \xi_n)\|_{q;0} &= \|T^{\otimes n} P_q^{1/2}(\xi_1 \otimes \cdots \otimes \xi_n)\|_0^2 \\ &\leq \|T\|_{\mathrm{OP}}^{2n} \|P_q^{1/2}(\xi_1 \otimes \cdots \otimes \xi_n)\|_0^2 \\ &= \|T\|_{\mathrm{OP}}^{2n} \|\xi_1 \otimes_q \cdots \otimes_q \xi_n\|_{q;0},\end{aligned}$$

see Lemma 1.4 in [2]. Therefore, for any $q \in (-1, 1)$ and $T \in \mathcal{L}(H, H)$ with $\|T\|_{\mathrm{OP}} \leq 1$, the second quantization $\Gamma(T)$ of T can be extended to $\Gamma_q(H)$ as a bounded operator.

LEMMA 3.1. *For any $n \geq 1$ and $r > 1/2$, $(A^{-r})^{\otimes n}$ is of Hilbert–Schmidt type on $H^{\otimes_q n}$, where $H^{\otimes_q n}$ is the completion of $\{\xi_1 \otimes_q \cdots \otimes_q \xi_n \,|\, \xi_i \in H,\ i = 1, 2, \ldots, n\}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_q$.*

Proof. Let $\{\varphi_{n,i}\}_{i=1}^{\infty}$ be a complete orthonormal basis for $\Gamma_q(H)$. Since $P_q^{[n]}$ is invertible (see [4]), $\{\sqrt{[n]_q!}\, {P_q^{[n]}}^{-1/2} \varphi_{n,i}\}_{i=1}^{\infty}$ is an orthonormal sequence in $\Gamma_0(H)$, where $P_q^{[n]}$ is the restriction of P_q to $H^{\otimes n}$. Let $\{\phi_{n,k}\}_{k=1}^{\infty}$ be a complete orthonormal basis for $\Gamma_0(H)$

containing $\{\sqrt{[n]_q!}\, P_q^{[n]^{-1/2}} \varphi_{n,i}\}_{i=1}^{\infty}$. Then we have

$$\begin{aligned}\|(A^{-r})^{\otimes n}\|_{\mathrm{HS;q}}^2 &= \sum_{i=1}^{\infty} \langle\!\langle (A^{-r})^{\otimes n}\varphi_{n,i},\, \varphi_{n,i}\rangle\!\rangle_q^2 \\ &= \sum_{i=1}^{\infty} [n]_q! \langle\!\langle (A^{-r})^{\otimes n} P_q^{[n]^{-1/2}}\varphi_{n,i},\, P_q^{[n]^{-1/2}}\varphi_{n,i}\rangle\!\rangle_0^2 \\ &\le \sum_{k=1}^{\infty} \langle\!\langle (A^{-r})^{\otimes n}\phi_{n,k},\, \phi_{n,k}\rangle\!\rangle_0^2 = \|(A^{-r})^{\otimes n}\|_{\mathrm{HS}}^2,\end{aligned}$$

where $\|(A^{-r})^{\otimes n}\|_{\mathrm{HS;q}}$ is the Hilbert–Schmidt norm of $(A^{-r})^{\otimes n}$ on $\Gamma_q(H)$, which completes the proof. ■

THEOREM 3.2. *For any $r, s \in \mathbf{R}$ with $\|A^{-(s-r)}\|_{\mathrm{HS}} < 1$, the natural inclusion*

$$i_{s,r} : \Gamma_q(E_s) \to \Gamma_q(E_r)$$

is of Hilbert–Schmidt type. In particular, for any $r > 1$, $\Gamma(A^{-r})$ is of Hilbert–Schmidt type on $\Gamma_q(H)$.

Proof. Let $\{\varphi_{s;n,k}\}_{n,k=0}^{\infty}$ be a complete orthonormal basis for $\Gamma_q(E_s)$, where for each $n \ge 1$, $\{\varphi_{s;n,k}\}_{k=0}^{\infty}$ is a complete orthonormal basis for $E_s^{\otimes_q n}$ which is the completion of $\Gamma_q^{\mathrm{finite}}(H) \cap E_s^{\otimes n}$ with respect to $\|\cdot\|_{q;s}$. First, we note that for any n, k

$$\|\varphi_{s;n,k}\|_{q;r} = \|(A^{-(s-r)})^{\otimes n}\varphi_{s;n,k}\|_{q;s}.$$

Therefore, by Lemma 3.1 we have

$$\|i_{s,r}\|_{\mathrm{HS}}^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \|(A^{-(s-r)})^{\otimes n}\varphi_{s;n,k}\|_{q;s}^2 \le \sum_{n=0}^{\infty} \|A^{-(s-r)}\|_{\mathrm{HS}}^{2n}$$

which is finite for any $r, s \in \mathbf{R}$ with $\|A^{-(s-r)}\|_{\mathrm{HS}} < 1$. ■

We put

$$\Gamma_q(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_q(E_p), \qquad \Gamma_q(E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma_q(E_{-p}).$$

Then we obtain a complex nuclear triple:

$$\Gamma_q(E) \subset \Gamma_q(H) \subset \Gamma_q(E)^*$$

which can be considered as a *q-white noise triplet* from the following:

DEFINITION 3.3. Let $G : H \to \mathcal{L}(\Gamma_q(H), \Gamma_q(H))$ be defined by

$$G_f = a(f) + a^*(f), \qquad f \in H.$$

For notational convenience, for any $t \in [0, \infty)$ we write $B_t = G_{\mathbf{1}_{[0,t]}}$. Then $\{B_t\}_{t\ge 0}$ is called the *q-Brownian motion.*

Note that for any $p > 5/12$ the map $\mathbf{R} \ni t \mapsto \delta_t \in E_{-p}$ is continuous, where δ_t is the delta function. Moreover, for any $0 \le \alpha \le 1$ with $p > 5/12 + \alpha/2$ there exists a constant $C \ge 0$ such that

$$|\delta_s - \delta_t|_{-p} \le C|s-t|^{\alpha}, \qquad s, t \in \mathbf{R},$$

see Theorem B.1 in [14].

REMARK. Let $p \geq 0$. For any $f \in H$, the operators $a(f)$ and $a^*(f)$ can be considered as operators in $\mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$ since $\mathcal{L}(\Gamma_q(H), \Gamma_q(H)) \subset \mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$. Also, for each $t \geq 0$, $a_t^* \equiv a^*(\delta_t)$ is a bounded operator in $\mathcal{L}(\Gamma_q(E_{-p}), \Gamma_q(E_{-p}))$ for any $p > 5/12$ of which the proof is similar to the proof of (2) in Theorem 2.1, see [10], and so $a_t \equiv a(\delta_t) \in \mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_p))$.

THEOREM 3.4. *For any $p > 5/12$, the map $t \mapsto B_t \in \mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$ is differentiable.*

Proof. By linearity of the map $E_{-p} \ni x \mapsto a(x) \in \mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$ and (2.2), we have

$$\left\| \frac{a(\mathbf{1}_{[0,t+\Delta_t]}) - a(\mathbf{1}_{[0,t]})}{\Delta_t} - a_t \right\|_{\mathrm{OP}} = \frac{1}{|\Delta_t|} \left\| a(\mathbf{1}_{[0,t+\Delta_t]} - \mathbf{1}_{[0,t]} - \Delta_t \delta_t) \right\|_{\mathrm{OP}}$$
$$\leq \frac{1}{\sqrt{1-q}} \left| \frac{\mathbf{1}_{[0,t+\Delta_t]} - \mathbf{1}_{[0,t]}}{\Delta_t} - \delta_t \right|_{-p} \to 0$$

as $\Delta_t \to 0$. Similarly, we have

$$\lim_{\Delta_t \to 0} \left\| \frac{a^*(\mathbf{1}_{[0,t+\Delta_t]}) - a^*(\mathbf{1}_{[0,t]})}{\Delta_t} - a_t^* \right\|_{\mathrm{OP}} = 0.$$

Hence B_t is differentiable in t and

$$\frac{dB_t}{dt} = a_t + a_t^*, \qquad t \geq 0,$$

in $\mathcal{L}(\Gamma(E_p), \Gamma(E_{-p}))$. ■

To simplify notation, we write $W_t = a_t + a_t^*$ for any $t \geq 0$. The process $\{W_t\}_{t \geq 0}$ is called the *q-white noise.*

4. Non-adapted stochastic integral. From now on we fix a positive real number p with $p > 5/12$. A family $\{\Xi_t\}_{t \geq 0}$ of operators in $\mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$ is called a quantum stochastic process.

A quantum stochastic process $\{\Xi_t\}_{t \geq 0}$ is said to be *uniformly measurable* if there exists a sequence $\{\Xi_{n,t}\}_{t \geq 0}$ of countable-valued quantum stochastic processes such that $\Xi_{n,t}$ converges to Ξ_t in $\mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$ for almost all $t \geq 0$. It is well known by N. Dunford that the uniform measurability of a quantum stochastic process $\{\Xi_t\}_{t \geq 0}$ is equivalent to the following conditions:

(1) $\{\Xi_t\}_{t \geq 0}$ is weakly measurable, i.e., for any $\phi, \varphi \in \Gamma_q(E_p)$ the map

$$\mathbf{R}_+ \ni t \mapsto \langle\langle \Xi_t \phi, \varphi \rangle\rangle \in \mathbf{C}$$

is measurable.

(2) $\{\Xi_t\}_{t \geq 0}$ is almost separable-valued in $\mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_{-p}))$, see [9].

From now on, for notational convenience we denote by $\| \cdot \|_{\mathrm{OP};r,s}$ the operator norm on $\mathcal{L}(\Gamma_q(E_r), \Gamma_q(E_s))$.

DEFINITION 4.1. Let $\{\Xi_t\}_{t \geq 0} \subset \mathcal{L}(\Gamma_q(E_r), \Gamma_q(E_s))$ be a quantum stochastic process.

(1) A countable-valued process $\{\Xi_t\}_{t \geq 0}$ is said to be *(Bochner) integrable* on $[0, T]$ if $\|\Xi_t\|_{\mathrm{OP};r,s}$ is integrable on $[0, T]$.

(2) The process $\{\Xi_t\}_{t\ge 0}$ is said to be *integrable* on $[0,T]$ if $\{\Xi_t\}_{t\ge 0}$ is uniformly measurable, i.e., there exists a sequence $\{\Xi_{n,t}\}_{t\ge 0}$ of countable-valued integrable processes such that $\Xi_{n,t}$ converges to Ξ_t in $\mathcal{L}(\Gamma_q(E_r),\Gamma_q(E_s))$ for almost all $t\in[0,T]$, and

$$\lim_{n\to\infty}\int_0^T \|\Xi_{n,t}-\Xi_t\|_{\mathrm{OP};r,s}\,dt=0. \tag{4.1}$$

In this case, we write

$$\int_0^T \Xi_t\,dt=\lim_{n\to\infty}\int_0^T \Xi_{n,t}\,dt.$$

THEOREM 4.2. *Let $\{\Xi_t\}_{t\ge 0}\subset\mathcal{L}(\Gamma_q(E_{-p}),\Gamma_q(E_{-p}))$ be an integrable process on $[0,T]$. Then $\{\Xi_t a_t\}_{t\ge 0}$ and $\{\Xi_t a_t^*\}_{t\ge 0}$ are integrable on $[0,T]$. Moreover, $\{\Xi_t W_t\}$ is integrable on $[0,T]$.*

Proof. Since the maps

$$\mathbf{R}_+\ni t\mapsto a_t\in\mathcal{L}(\Gamma_q(E_p),\Gamma_q(E_p)),\qquad \mathbf{R}_+\ni t\mapsto a_t^*\in\mathcal{L}(\Gamma_q(E_{-p}),\Gamma_q(E_{-p}))$$

are continuous, by assumption $\{\Xi_t a_t\}_{t\ge 0}$ and $\{\Xi_t a_t^*\}_{t\ge 0}$ are uniformly measurable on $[0,T]$. On the other hand, there exists a sequence $\{\Xi_{n,t}\}_{t\ge 0}$ of countable-valued integrable processes such that $\Xi_{n,t}$ converges to Ξ_t for almost all $t\in[0,T]$, and (4.1) holds with $r=s=-p$. For any $n=1,2,\ldots$, we put

$$a_{n,t}=a_{t_i},\qquad t\in[(i-1)T/n,iT/n),\qquad i=1,2,\ldots,n.$$

Then $a_{n,t}$ converges to a_t for almost all $t\in[0,T]$. Therefore, by the dominated convergence theorem we have

$$\begin{aligned}
&\int_0^T \|\Xi_{n,t}a_{n,t}-\Xi_t a_t\|_{\mathrm{OP};p,-p}\,dt\\
&\le\int_0^T \|\Xi_{n,t}-\Xi_t\|_{\mathrm{OP};-p,-p}\|a_{n,t}-a_t\|_{\mathrm{OP};p,p}\,dt+\int_0^T \|\Xi_t\|_{\mathrm{OP};-p,-p}\|a_{n,t}-a_t\|_{\mathrm{OP};p,p}\,dt\\
&+\int_0^T \|\Xi_{n,t}-\Xi_t\|_{\mathrm{OP};-p,-p}\|a_t\|_{\mathrm{OP};p,p}\,dt\\
&\le 3K\int_0^T \|\Xi_{n,t}-\Xi_t\|_{\mathrm{OP};-p,-p}\,dt+\int_0^T \|\Xi_t\|_{\mathrm{OP};-p,-p}\|a_{n,t}-a_t\|_{\mathrm{OP};p,p}\,dt\\
&\to 0
\end{aligned}$$

as $n\to\infty$, where $K=\sup\{\|a_t\|_{\mathrm{OP};p,p}\,|\,t\in[0,T]\}$ which is finite by the continuity. Hence $\{\Xi_t a_t\}_{t\ge 0}$ is integrable. The rest of the proof is similar. ■

THEOREM 4.3. *Let $\{\Xi_t\}_{t\ge 0}\subset\mathcal{L}(\Gamma_q(E_p),\Gamma_q(E_p))$ be an integrable process on $[0,T]$. Then $\{a_t\Xi_t\}_{t\ge 0}$ and $\{a_t^*\Xi_t\}_{t\ge 0}$ are integrable on $[0,T]$. Moreover, $\{W_t\Xi_t\}$ is integrable on $[0,T]$.*

Proof. The proof is similar to the proof of Theorem 4.2. ■

REMARK. By Theorems 4.2 and 4.3, the quantum stochastic processes $\{a_t\}$, $\{a_t^*\}$ and $\{W_t\}$ are integrable on $[0,T]$ and we have

$$a(\mathbf{1}_{[0,t]})=\int_0^t a_s ds,\qquad a^*(\mathbf{1}_{[0,t]})=\int_0^t a_s^* ds,\qquad B_t=\int_0^t W_s ds.$$

Therefore, we write

$$\int_0^t W_s \Xi_s ds = \int_0^t dB_s \Xi_s, \qquad \int_0^t \Xi_s W_s ds = \int_0^t \Xi_s dB_s$$

and call them the stochastic integrals with respect to the q-Brownian motion.

DEFINITION 4.4. Let $\{\Xi_t\}_{t\geq 0} \subset \mathcal{L}(\Gamma_q(E_r), \Gamma_q(E_s))$ be a quantum stochastic process. The process $\{\Xi_t\}_{t\geq 0}$ is said to be *square integrable* on $[0,T]$ if $\{\Xi_t\}_{t\geq 0}$ is uniformly measurable, i.e., there exists a sequence $\{\Xi_{n,t}\}_{t\geq 0}$ of countable-valued integrable processes such that $\Xi_{n,t}$ converges to Ξ_t in $\mathcal{L}(\Gamma_q(E_r), \Gamma_q(E_s))$ for almost all $t \in [0,T]$, and

$$\lim_{n\to\infty} \int_0^T \|\Xi_{n,t} - \Xi_t\|_{r;s}^2 \, dt = 0.$$

THEOREM 4.5. *Let* $\{\Xi_t^{(1)}\}_{t\geq 0} \subset \mathcal{L}(\Gamma_q(E_{-p}), \Gamma_q(E_{-p}))$, $\{\Xi_t^{(2)}\}_{t\geq 0} \subset \mathcal{L}(\Gamma_q(E_p), \Gamma_q(E_p))$ *be square integrable processes on* $[0,T]$. *Then* $\{\Xi_t^{(1)} a_t \Xi_t^{(2)}\}_{t\geq 0}$ *and* $\{\Xi_t^{(1)} a_t^* \Xi_t^{(2)}\}_{t\geq 0}$ *are integrable on* $[0,T]$. *Moreover,* $\{\Xi_t^{(1)} W_t \Xi_t^{(2)}\}_{t\geq 0}$ *is integrable on* $[0,T]$.

Proof. The proof is a simple modification of the proof of Theorem 4.2. ■

Now, we consider the stochastic integrals with respect to the q-Brownian motion which are usual operators in $\mathcal{L}(\Gamma_q(H), \Gamma_q(H))$.

Let $T \geq 0$ be fixed. The algebraic tensor product $L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H))$ is identified with a linear subspace of $\mathcal{L}(\Gamma_q(H), \Gamma_q(H))$-valued square integrable functions on $[0,T]$ by the identification:

$$f = \sum_{i=1}^n f_i \otimes \Xi_i \;\leftrightarrow\; \sum_{i=1}^n f_i(\cdot) \otimes \Xi_i.$$

Define seminorms $\|\cdot\|_l$ and $\|\cdot\|_r$ on $L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H))$ by

$$\|f\|_l = \left\|\sum_{i=1}^n (a^*(f_i) + a(f_i))\Xi_i\right\|_{\mathrm{OP}}, \qquad \|f\|_r = \left\|\sum_{i=1}^n \Xi_i (a^*(f_i) + a(f_i))\right\|_{\mathrm{OP}}$$

for $f = \sum_{i=1}^n f_i \otimes \Xi_i \in L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H))$. Put

$$\mathcal{N}_\epsilon = \{f \in L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H)) \mid \|f\|_\epsilon = 0\}, \qquad \epsilon = l, r.$$

The completion of $L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H))/\mathcal{N}_\epsilon$ with respect to the norm $\|\cdot\|_\epsilon$ is denoted by $\mathfrak{B}_\epsilon^2([0,T])$, where $\epsilon = l, r$.

Note that for any $f = \sum_{i=1}^n f_i \otimes \Xi_i \in L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H))$ it is (Bochner) integrable on $[0,T]$ and

$$\int_0^T W_s f(s) ds = \sum_{i=1}^n (a^*(f_i) + a(f_i))\, \Xi_i, \qquad \int_0^T f(s) W_s ds = \sum_{i=1}^n \Xi_i\, (a^*(f_i) + a(f_i)).$$

Therefore, for any $f \in \mathfrak{B}_l^2([0,T])$ and $g \in \mathfrak{B}_r^2([0,T])$ there exist sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ in $L^2[0,T] \otimes_{\mathrm{a}} \mathcal{L}(\Gamma_q(H), \Gamma_q(H))$ such that

$$\lim_{n\to\infty} \|f_n - f\|_l = 0, \qquad \lim_{n\to\infty} \|g_n - g\|_r = 0$$

which implies that

$$\left\{\int_0^T W_s f_n(s)ds\right\}_{n=1}^{\infty}, \quad \left\{\int_0^T g_n(s)W_s ds\right\}_{n=1}^{\infty}$$

are Cauchy sequences in $\mathcal{L}(\Gamma_q(H), \Gamma_q(H))$ and the limits are denoted by

$$\int_0^T W_s f(s)ds = \lim_{n\to\infty}\int_0^T W_s f_n(s)ds, \quad \int_0^T g(s)W_s ds = \lim_{n\to\infty}\int_0^T g_n(s)W_s ds.$$

REMARK. For the stochastic integrals with respect to the q-Brownian motion, we used $\mathfrak{B}^2_\epsilon([0,T])$ as the space of integrands which is obtained by taking completion with respect to uniform operator norms. But we can consider a bigger space as a space of integrands in the stochastic integrals by taking completion with respect to weaker topologies, e.g., strong operator topology, weak operator topology or L^2 space of operators with certain state, and then we can consider some connections with the results in [6] and [15] which are now in progress.

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