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FACTORIALITY OF VON NEUMANN ALGEBRAS CONNECTED WITH GENERAL COMMUTATION RELATIONS—FINITE DIMENSIONAL CASE

ILONA KRÓLAK
Institute of Mathematics, Wrocław University
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: ikrol@math.uni.wroc.pl

Abstract. We study a certain class of von Neumann algebras generated by selfadjoint elements $\omega_i = a_i + a_i^+$, where a_i, a_i^+ satisfy the general commutation relations:

$$a_i a_j^+ = \sum_{r,s} t_{j\,s}^{i\,r}\; a_r^+ a_s + \delta_{ij} Id.$$

We assume that the operator T for which the constants $t_{j\,s}^{i\,r}$ are matrix coefficients satisfies the braid relation. Such algebras were investigated in [BSp] and [K] where the positivity of the Fock representation and factoriality in the case of infinite dimensional underlying space were shown. In this paper we prove that under certain conditions on the number of generators our algebra is a factor. The result was obtained for q-commutation relations by P. Śniady [Snia] and recently by E. Ricard [R]. The latter proved factoriality without restriction on the dimension, but it cannot be easily generalized to the general commutation relation case. We generalize the result of Śniady and present a simpler proof. Our estimate for the number of generators in case $q > 0$ is better than in [Snia].

0. Introduction. Let $\mathcal{H}_R$ be a real Hilbert space and $\mathcal{H} = \mathcal{H}_R + \mathrm{i}\mathcal{H}_R$ be its complexification. Denote by $\overline{\mathcal{H}}$ the complex conjugate of $\mathcal{H}$. Let $\overline{T} : \overline{\mathcal{H}} \otimes \mathcal{H} \to \mathcal{H} \otimes \overline{\mathcal{H}}$ be linear.

Consider the algebra of all tensors over $\mathcal{H}$ and $\overline{\mathcal{H}}$ plus Id with tensor multiplication as a product denoted by $\Upsilon(\mathcal{H}, \overline{\mathcal{H}})$. Define the Wick algebra as the following quotient algebra:

$$(\star) \qquad W(T) \simeq \Upsilon(\mathcal{H}, \overline{\mathcal{H}}) / \langle \overline{f} \otimes g - \overline{T}(\overline{f} \otimes g) - \langle g|f\rangle Id \rangle.$$

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For $f, g \in \mathcal{H}$ we set $a(f) = \overline{f}$, $a^+(g) = g$, where the action of the operators $a(f)$ and $a^+(g)$ is given by multiplication by $\overline{f}$ and g, respectively. We are interested in positive representations of $W(T)$, that is, the representations of the $a(f)$'s as operators on a Hilbert space such that $a^+(f)$ is the restriction of the operator adjoint to $a(f)$.

In this paper we will deal with the Fock representation. It is constructed from a cyclic vector Ω with the property $a(f)\Omega = 0$ for all $f \in \mathcal{H}$. Such a representation is unique (up to isomorphism) and carries a unique hermitian form. There are various criteria implying that this form is positive definite. Most of them are written in the language of the properties of the operator $T \in L(\mathcal{H}\otimes\mathcal{H})$ defined as $\langle T(f\otimes g)|h\otimes w\rangle = \langle \overline{T}(\overline{h}\otimes f)|w\otimes\overline{g}\rangle$. It was proved (see [BSp]) that if T is a selfadjoint contraction and satisfies the braid relation (or Yang-Baxter relation)

(BR) $$[(1\otimes T)(T\otimes 1)(1\otimes T) = (T\otimes 1)(1\otimes T)(T\otimes 1)] \quad \text{on } \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$$

then the form is positive definite. In such a case the completion of the Fock space

$$\mathcal{F}^{fin}(\mathcal{H}) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$$

with scalar product denoted by $\langle\cdot|\cdot\rangle_T$ is a Hilbert space.

We study the von Neumann algebra $\Gamma_T(\mathcal{H})$ generated by the operators

$$\omega(f) = a(f) + a^+(f), f \in \mathcal{H}_R.$$

The choice of the relations $(\star)$ was made since several examples of such structures were investigated in the literature. The case $T(f\otimes g) = (g\otimes f)$ is known as the canonical commutation relations (CCR). The case $T(f\otimes g) = -(g\otimes f)$ is known as the canonical anti-commutation relations (CAR). More generally for $T(f\otimes g) = q(g\otimes f)$, where $q \in [-1,1]$ we use the name q-canonical commutation relations (q-CCR). The case $T = 0$ plays a fundamental role. In [Vo] Voiculescu proved that $\Gamma_0(\mathcal{H}) = VN(\mathbf{F}_N)$, where $\mathbf{F}_N$ denotes the free group with $N = \dim\mathcal{H}$ generators. Other examples as well as those mentioned above were studied in a series of papers (see e.g. [JSW], [BSp], [BKSp], [PW] and references therein).

For our needs we will use the following assumptions:

I. $T = T^*$, $\|T\| = q < 1$,
II. T satisfies the braid relation,
III. Under the identification $\overline{\mathcal{H}}_R \simeq \mathcal{H}_R$ we have $T = \overline{T}$ on $\mathcal{H}_R$.

If we choose $\{e_i\}$ to be an orthonormal basis of $\mathcal{H}_R$ and define matrix coefficients of T by the formula $\langle T(e_a\otimes e_b)|\, e_c\otimes e_d\rangle = t_{ab}^{cd}$ then the relations $(\star)$ have the form

$$a_i a_j^+ = \sum_{r,s} t_{js}^{ir}\; a_r^+ a_s + \delta_{ij} Id$$

and the equation $T = \overline{T}$ on $\mathcal{H}_R$ can be rewritten as $t_{js}^{ir} = t_{ij}^{rs}$.

Under assumptions I, II, III the algebra $\Gamma_T(\mathcal{H})$ can be continuously embedded into the representation space. The map $D\colon \Gamma_T(\mathcal{H}) \to \mathcal{F}_T(\mathcal{H})$ defined as $D(X) = X\Omega$ gives the embedding. The operator $\psi(f_1\otimes\ldots\otimes f_n)$ which corresponds to a vector $f_1\otimes\ldots\otimes f_n$ is called the Wick product of the operators $\omega(f_1),\ldots,\omega(f_n)$. Such an element of $\Gamma_T(\mathcal{H})$

exists and is unique. In [K] an explicit formula for the Wick product was given. It provides a useful tool for studying $\Gamma_T(\mathcal{H})$ (see [K] and [Nu]).

In [K] we proved that if the number of generators is infinite then for T such that $\|T\| = q < 1$ the algebra $\Gamma_T(\mathcal{H})$ is a type II factor. In this paper we show that the result remains true if $\dim \mathcal{H}$ is greater than some finite natural number $N(q)$. Recently E. Ricard [R] proved factoriality without restriction on the dimension. We generalize and present another method of the proof of the result of Śniady.

1. Notations and known theorems. This section is based on [BSp] and [Bo]. Unless otherwise stated we assume that $T \in B(\mathcal{H}_R \otimes \mathcal{H}_R)$ is a selfadjoint, strict contraction ($\|T\| = q < 1$) which satisfies the braid relation.

Definition 1. For fixed m define a function $\varphi : S_m \to B(\mathcal{F}(\mathcal{H}))$ by quasi multiplicative extension of

$$\varphi(e) = 1, \quad \varphi(\pi_i) = T_i, \quad i \le m-1,$$

where

$$T_i(f_1 \otimes \ldots \otimes f_s) = f_1 \otimes \ldots \otimes f_{i-1} \otimes T(f_i \otimes f_{i+1}) \otimes \ldots \otimes f_s.$$

This means that for a reduced word $\sigma = \pi_{i_1} \cdots \pi_{i_k}$ we put $\varphi(\sigma) = T_{i_1} \cdots T_{i_k}$.

Definition 2. Fix the operator T and define $P^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma)$.

According to the main theorem in [BSp] $P^{(n)}$ is nonnegative for every n (even strictly positive in case $\|T\| < 1$) and therefore we can introduce a new scalar product on $\mathcal{H}^{\otimes n}$

$$\langle \xi \mid \eta \rangle_T = \langle P^{(n)} \xi \mid \eta \rangle.$$

The T-Fock space $\mathcal{F}_T(\mathcal{H})$ is the completion of the Fock space $\mathcal{F}^{fin}(\mathcal{H}) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to the scalar product mentioned above.

Definition 3. For $f \in \mathcal{H}_R$ we define the *left* and *right creation operators* $a_l^+(f)$, $a_r^+(f)$ as

$$a_l^+(f) f_1 \otimes \ldots \otimes f_n = f \otimes f_1 \otimes \ldots \otimes f_n \quad \text{and} \quad a_r^+(f) f_1 \otimes \ldots \otimes f_n = f_1 \otimes \ldots \otimes f_n \otimes f$$

and the *right* and *left annihilation operators* $a_l(f)$, $a_r(f)$ as their adjoints on the T-Fock space.

By [K] the operators $a_l^+(f)$ and $a_r^+(f)$ are bounded and their adjoints are well defined. Explicit formulas for annihilation operators on $\mathcal{H}^{\otimes n}$ are the following

$$a_l(f) = l_l(f)(1 + T_1 + T_1 T_2 + \ldots + T_1 T_2 T_3 \cdots T_{n-1})$$

and

$$a_r(f) = l_r(f)(1 + T_{n-1} + T_{n-1} T_{n-2} + \ldots + T_{n-1} T_{n-2} \cdots T_1),$$

where $l_{l,r}(f)\Omega = 0$,

$$l_l(f)(f_1 \otimes \ldots \otimes f_n) = \langle f_1 \mid f \rangle f_2 \otimes \ldots \otimes f_n,$$
$$l_r(f)(f_1 \otimes \ldots \otimes f_n) = \langle f_n \mid f \rangle f_1 \otimes \ldots \otimes f_{n-1}.$$

Definition 4. Let $\Gamma_T(\mathcal{H})$ be the von Neumann algebra generated by the selfadjoint operators $\omega_l(f) = a_l(f) + a_l^+(f)$, $f \in \mathcal{H}_R$ on $\mathcal{F}_T(\mathcal{H})$.

The vector Ω is separating and cyclic for $\Gamma_T(\mathcal{H})$. Moreover the functional $\tau_T(X) = \langle X\Omega\,|\Omega\rangle_T$ is a faithful normal trace. The map $D\colon \Gamma_T(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ defined as $D(X) = X(\Omega)$ is a continuous embedding of $\Gamma_T(\mathcal{H})$ into $\mathcal{F}_T(\mathcal{H})$.

NOTATION 5. For $\xi \in \mathcal{F}(\mathcal{H})$ we denote by $\psi(\xi)$ the unique element of $\Gamma_T(\mathcal{H})$ such that $\psi(\xi)\Omega = \xi$.

2. The result. Let $e_1, e_2, \ldots$ be an orthonormal basis of $\mathcal{H}_R$. Let

$$\omega_l(f) = a_l(f) + a_l^+(f) \text{ and } \omega_r(f) = a_r(f) + a_r^+(f).$$

Notice that $\omega_r(e_i) \in (\Gamma_T(\mathcal{H}))'$. Take

$$M = \sum_{i=1}^{d} (\omega_l(e_i) - \omega_r(e_i))^2,$$

where d is a natural number with $d \leq \dim \mathcal{H}$. This operator is positive and selfadjoint. We will show that

$$\mathrm{Ker}(M) = \mathbf{C}\Omega.$$

THEOREM 6. *There exists d_0 such that for $d \geq d_0$ operator M is strictly positive on $\mathcal{F}_T(\mathcal{H})\backslash \mathbf{C}\Omega$.*

To prove the theorem we need some estimates.

LEMMA 7. *Fix $f \in \mathcal{H}_R^{\otimes n}$.*

a) $$\|L_{n+1}(T_n \cdots T_2 T_1) D_j(f)\|_T \leq q^n \|f\|_T,$$

b) $$\|L_1(T_2 T_3 \cdots T_{n+1}) D_j(f)\|_T \leq q^n \|f\|_T,$$

where $D_j(f) = e_j \otimes f \otimes e_j$, $L_{n+1}(f_1 \otimes \cdots \otimes f_{n+2}) = \langle f_{n+1}|f_{n+2}\rangle f_1 \otimes \ldots \otimes f_n$ and $L_1(f_1 \otimes \cdots \otimes f_{n+2}) = \langle f_1|f_2\rangle f_3 \otimes f_4 \otimes \cdots \otimes f_{n+2}$.

Proof. Notice that for arbitrary $\sigma \in S_n$ we have

$$\begin{aligned} L_{n+1}(T_n \cdots T_2 T_1) D_j \varphi(\sigma) &= L_{n+1} \varphi(\rho \times Id) \varphi(Id \times \sigma \times Id) D_j \\ &= L_{n+1}(\varphi(\rho) \otimes Id)(\varphi(Id \times \sigma) \otimes Id) D_j, \end{aligned}$$

where $\rho = \pi_n \cdots \pi_1$. Further one can verify that $Inv(\rho) + Inv(Id \times \sigma) = Inv(\rho(Id \times \sigma))$. Furthermore $\rho(Id \times \sigma) = (\sigma \times Id)\rho$ and also $Inv(\sigma \times Id) + Inv(\rho) = Inv((\sigma \times Id)\rho)$. This implies that $\varphi(\rho)\varphi(Id \times \sigma) = \varphi(\rho(Id \times \sigma)) = \varphi((\sigma \times Id)\rho) = \varphi(\sigma \times Id)\varphi(\rho)$ and finally gives

$$\begin{aligned} L_{n+1}(T_n \cdots T_1) D_j \varphi(\sigma) &= L_{n+1}(\varphi(\sigma \times Id) \otimes Id)(\varphi(\rho) \otimes Id) D_j \\ &= \varphi(\sigma) L_{n+1}(\varphi(\rho) \otimes Id) D_j \\ &= \varphi(\sigma) L_{n+1}(T_n \cdots T_2 T_1) D_j. \end{aligned}$$

This implies that operator $L_{n+1}(T_n \cdots T_2 T_1) D_j$ commutes with $P^{(n)} = \sum \varphi(\sigma)$ and therefore its norm is the same as on $\mathcal{H}^{\otimes n}$ with the usual scalar product.

The operator $L_{n+1}\colon \mathcal{H}^{\otimes(n+1)} \otimes e_j \to \mathcal{H}^{\otimes n}$ is a contraction, D_j is an isometry and

$$\|(T_n \cdots T_2 T_1)\| \leq q^n. \ \blacksquare$$

LEMMA 8. *The following estimations hold for $f \in \mathcal{F}_T(\mathcal{H})$:*

(a) $\Big\| \sum_{i=1}^{d} e_i \otimes f \otimes e_i \Big\|_T^2 \le dC_q^2 \|f\|_T^2$,

(b) $\Big\| \sum_{i=1}^{d} a_l(e_i) a_r(e_i) f \Big\|_T^2 = \Big\| \sum_{i=1}^{d} a_r(e_i) a_l(e_i) f \Big\|_T^2 \le dC_q^2 \|f\|_T^2$,

(c) $\Big\| \sum_{i=1}^{d} e_i \otimes a_r(e_i) f \Big\|_T^2 \le dC_q^2 \|f\|_T^2$,

(d) $\Big\| \sum_{i=1}^{d} a_l(e_i) f \otimes e_i \|_T^2 \le dC_q^2 \|f\|_T^2$,

(e) $\Big\| \sum_{i=1}^{d} a_l(e_i) a_l(e_i) f \Big\|_T^2 \le dC_q^2 \|f\|_T^2$,

(f) $\Big\| \sum_{i=1}^{d} a_r(e_i) a_r(e_i) f \Big\|_T^2 \le dC_q^2 \|f\|_T^2$,

(g) $\Big\| \sum_{i=1}^{d} a_{l,r}^{+}(e_i) a_{l,r}(e_i) f \Big\|_T^2 \le dC_q^2 \|f\|_T^2$,

(h) $\Big\| \sum_{i=1}^{d} \sum_{k,s} t_{is}^{ik}\, e_k \otimes a_l(e_s) f \Big\|_T^2 \le dq^2 C_q^2 \|f\|_T^2$.

Proof. Let $f = \sum f_n$, where $f_n \in \mathcal{H}^{\otimes n}$. Observe that $e_i \otimes f_n \otimes e_i \in \mathcal{H}^{\otimes n+2}$. Therefore for fixed i we have

$$\langle e_i \otimes f_n \otimes e_i \,|\, e_i \otimes f_m \otimes e_i \rangle_T = 0 \text{ if } n \ne m.$$

It is known from [BSp] that there exists some constant C_q (it can be explicitly calculated) such that for arbitrary $k < m$

$$P_T^{(m)} \le C_q (P_T^{(k)} \otimes P_T^{(m-k)}).$$

This yields

$$\begin{aligned} \Big\| \sum_{i=1}^{d} e_i \otimes f \otimes e_i \Big\|_T^2 &= \sum_{n=0}^{\infty} \Big\| \sum_{i=1}^{d} e_i \otimes f_n \otimes e_i \Big\|_T^2 \\ &= \sum_{n=0}^{\infty} \Big\langle \sum_{i=1}^{d} P_T^{(n+2)} (e_i \otimes f_n \otimes e_i) \,\Big|\, \sum_{i=1}^{d} e_i \otimes f_n \otimes e_i \Big\rangle \\ &\le \sum_{n=0}^{\infty} C_q^2 \Big\langle \sum_{i=1}^{d} (I \otimes P_T^{(n)} \otimes I)(e_i \otimes f_n \otimes e_i) \,\Big|\, \sum_{i=1}^{d} e_i \otimes f_n \otimes e_i \Big\rangle. \end{aligned}$$

Since $\langle (I \otimes P_T^{(n)} \otimes I)(e_i \otimes f_n \otimes e_i) \,|\, e_j \otimes f_n \otimes e_j \rangle = 0$ for $i \ne j$ the last expression is equal to

$$\sum_{n=0}^{\infty}\sum_{i=1}^{d} C_q^2\langle (I\otimes P_T^{(n)}\otimes I)(e_i\otimes f_n\otimes e_i)\,|\,e_i\otimes f_n\otimes e_i\rangle$$
$$= C_q^2\sum_{n=0}^{\infty} d\langle P_T^{(n)} f_n\,|\,f_n\rangle = dC_q^2\sum_{n=0}^{\infty}\|f_n\|_T^2 = dC_q^2\|f\|_T^2.$$

This gives (a).

For (b) define $B_i f = a_r(e_i)a_l(e_i)f$. By taking the adjoints on both sides we obtain $B_i^* f = a_l^+(e_i)a_r^+(e_i)f$. Further

$$\Big(\sum_{i=1}^{d} a_r(e_i)a_l(e_i)\Big)^* f = \sum_{i=1}^{d} a_l^+(e_i)a_r^+(e_i)f = \sum_{i=1}^{d} e_i\otimes f\otimes e_i.$$

From (a) we have that $\|\sum_{i=1}^{d} a_l^+(e_i)a_r^+(e_i)\|_\infty \le \sqrt{d}C_q$, which implies

$$\Big\|\sum_{i=1}^{d} a_l(e_i)a_r(e_i)\Big\|_\infty \le \sqrt{d}C_q.$$

To prove (c), as in (a) we write

$$\begin{aligned}
\Big\|\sum_{i=1}^{d} e_i\otimes a_r(e_i)f\Big\|_T^2 &= \sum_{n=0}^{\infty}\Big\|\sum_{i=1}^{d} e_i\otimes a_r(e_i)f_n\Big\|_T^2\\
&= \sum_{n=0}^{\infty}\Big\langle \sum_{i=1}^{d} P_T^{(n)}(e_i\otimes a_r(e_i)f_n)\,\Big|\,\sum_{i=1}^{d} e_i\otimes a_r(e_i)f_n\Big\rangle\\
&\le \sum_{n=0}^{\infty} C_q\Big\langle \sum_{i=1}^{d}(I\otimes P_T^{(n-1)})(e_i\otimes a_r(e_i)f_n)\,\Big|\,\sum_{i=1}^{d} e_i\otimes a_r(e_i)f_n\Big\rangle\\
&= \sum_{n=0}^{\infty}\sum_{i=1}^{d} C_q\langle (I\otimes P_T^{(n-1)})(e_i\otimes a_r(e_i)f_n)\,|\,e_i\otimes a_r(e_i)f_n\rangle\\
&= C_q\sum_{n=0}^{\infty}\sum_{i=1}^{d}\langle P_T^{(n-1)}(a_r(e_i)f_n)\,|\,a_r(e_i)f_n\rangle\\
&= C_q\sum_{n=0}^{\infty}\sum_{i=1}^{d}\|a_r(e_i)f_n\|_T^2 \le C_q\sum_{n=0}^{\infty}\sum_{i=1}^{d} C_q\|f_n\|_T^2 = C_q^2 d\|f\|_T^2.
\end{aligned}$$

In the last inequality we used an estimate of the norm of $a_r(e_i)$, i.e. $\|a_r(e_i)\|_\infty \le \sqrt{C_q}$.

For (d) it is obvious that $e_i\otimes a_r(e_i)f = a_l^+(e_i)a_r(e_i)f$. Also $a_l(e_i)f\otimes e_i = a_r^+(e_i)a_l(e_i)f$ $= (a_l^+(e_i)a_r(e_i))^* f$. Now (d) can be derived from (c). Analogously we can prove (e), (f), (g). Finally for (h) and for $f=\sum f_n$, where $f_n\in\mathcal{H}^{\otimes n}$ we have

$$\sup_{\|f\|=1}\Big\|\sum_{i=1}^{d}\sum_{k,s} t_{is}^{ik}\, e_k\otimes a_l(e_s)f\Big\|_T^2 = \sup_{\|f\|=1}\Big\|\sum_{k,s}\sum_{i=1}^{d} t_{is}^{ik}\, e_k\otimes a_l(e_s)f\Big\|_T^2$$
$$= \sup_{\|f\|=1}\sum_{n=0}^{\infty}\Big\|\sum_{k,s}\sum_{i=1}^{d} t_{is}^{ik}\, e_k\otimes a_l(e_s)f_n\Big\|_T^2 \le C_q\sup_{\|f\|=1}\sum_{n=0}^{\infty}\sum_{k}\Big\|\sum_{s}\sum_{i=1}^{d} t_{is}^{ik}\, a_l(e_s)f_n\Big\|_T^2$$

$$= C_q \sum_k \Big\| \sum_s \sum_{i=1}^d t_{is}^{ik}\, a_l(e_s) \Big\|_\infty^2 = C_q \sum_k \Big\| \sum_s \sum_{i=1}^d \overline{t_{is}^{ik}}\, a_l^+(e_s) \Big\|_\infty^2$$

$$= C_q \sum_k \sup_{\|f\|=1} \sum_{n=0}^\infty \Big\| \sum_s \sum_{i=1}^d \overline{t_{is}^{ik}}\, e_s \otimes f_n \Big\|_T^2 \le C_q^2 \sum_{n=0}^\infty \sum_{k,s} \Big| \sum_{i=1}^d \overline{t_{is}^{ik}} \Big|^2 \|f_n\|_T^2$$

$$= C_q^2 \sum_{k,s} \Big| \sum_{i=1}^d t_{is}^{ik} \Big|^2 = C_q^2 \Big\| T\Big(\sum_{i=1}^d e_i \otimes e_i \Big) \Big\|^2 \le dq^2 C_q^2.$$

Now we can pass to the proof of our main theorem.

Proof of Theorem 6. We have

$$\sum_{i=1}^d (\omega_l(e_i) - \omega_r(e_i))^2 = \sum_{i=1}^d a_l(e_i)a_l(e_i) + a_l^+(e_i)a_l^+(e_i) + a_l^+(e_i)a_l(e_i) + a_l(e_i)a_l^+(e_i)$$
$$+ \sum_{i=1}^d a_r(e_i)a_r(e_i) + a_r^+(e_i)a_r^+(e_i) + a_r^+(e_i)a_r(e_i) + a_r(e_i)a_r^+(e_i)$$
$$- \sum_{i=1}^d a_l(e_i)a_r(e_i) + a_l^+(e_i)a_r^+(e_i) + a_l^+(e_i)a_r(e_i) + a_l(e_i)a_r^+(e_i)$$
$$- \sum_{i=1}^d a_r(e_i)a_l(e_i) + a_r^+(e_i)a_l^+(e_i) + a_r^+(e_i)a_l(e_i) + a_r(e_i)a_l^+(e_i).$$

By the definition of $a_r(e_i)$ and $a_l(e_i)$ we have for $f \in \mathcal{H}^{\otimes n}$

$$a_l(e_i)a_r^+(e_i)f = a_l(e_i)f \otimes e_i + L_1(T_2T_3 \cdots T_{n+1})D_j(f),$$
$$a_r(e_i)a_l^+(e_i)f = e_i \otimes a_l(e_i)f + L_{n+1}(T_n \cdots T_2T_1)D_j(f),$$
$$a_l(e_i)a_l^+(e_i)f = f + \sum_{i=1}^d \sum_{k,s} t_{is}^{ik}\, e_k \otimes a_l(e_s)f,$$
$$a_r(e_i)a_r^+(e_i)f = f + \sum_{i=1}^d \sum_{k,s} t_{is}^{ik}\, a_r(e_s)f \otimes e_k.$$

From Lemma 7, for $f \in \mathcal{F}_T \backslash \mathbf{C}\Omega$

$$\|L_{n+1}(T_n \cdots T_2T_1)D_j(f)\| \le q\|f\|_T \quad \text{and} \quad \|L_1(T_2T_3 \cdots T_{n+1})D_j(f)\| \le q\|f\|_T.$$

Therefore $M = 2dId + M_1 - M_2 + M_3$, where

$$\|M_1\| \le 2\sqrt{d}C_q\, q, \quad \|M_2\| \le 2q\, d, \quad \|M_3\| \le 14C_q\sqrt{d}.$$

This implies

$$M \ge 2d(1-q) - 2\sqrt{d}qC_q - 14\sqrt{d}C_q.$$

It is easy to see that there exists d_0 such that the last expression is positive for arbitrary $d \ge d_0$.

THEOREM 9. *Let d_0 be the constant from Theorem 6. If $dim\mathcal{H} \ge d_0$ then the algebra $\Gamma_T(\mathcal{H})$ is a factor.*

Proof. Notice that if $X \in \Gamma_T(\mathcal{H}) \cap \Gamma_T(\mathcal{H})'$ then X commutes with $\omega_r(e_i)$ and $\omega_l(e_i)$ for arbitrary i. Therefore $X\Omega \in \mathrm{Ker}(M)$. From Theorem 6 we derive that $\mathrm{Ker}(M) = \mathbf{C}\Omega$, hence $X = \alpha Id$ and $\Gamma_T(\mathcal{H})$ is a factor.

REMARK 10. The result can be generalized to the case of q-Araki-Woods algebras ([H]).

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