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QUANTUM STOCHASTIC CONVOLUTION COCYCLES —ALGEBRAIC AND C^*-ALGEBRAIC

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Abstract. We summarise recent results concerning quantum stochastic convolution cocycles in two contexts—purely algebraic and C^*-algebraic. In each case the class of cocycles arising as the solution of a quantum stochastic differential equation is characterised and the form taken by the stochastic generator of a *-homomorphic cocycle is described. Throughout the paper a common viewpoint on the algebraic and C^*-algebraic situations is emphasised; the final section treats the unifying example of convolution cocycles on full compact quantum groups.

Introduction. Stochastic cocycles on operator algebras are basic objects of interest in quantum stochastic analysis ([L]) and in the study of more general noncommutative white noise ([HKK]); their importance in quantum probability was first recognised in [Acc]. There is a well developed theory of quantum Lévy processes ([ASW], [Fra], [Sch]), that is, stationary, independent increment, *-homomorphic processes on *-bialgebras. Close examination of these two areas naturally leads to the notion of *quantum stochastic convolution cocycle* on a quantum group (or, more generally, on a coalgebra), as introduced in [LS_1]. There it is shown that, as with 'standard' quantum stochastic cocycles ([LW_1]), quantum stochastic convolution cocycles arise as solutions of noncommutative stochastic differential equations. Indeed, all sufficiently regular convolution cocycles arise in this way. Although the results of [LS_1] are formulated in a purely algebraic context, they may be extended to the case of compact quantum groups, and many of them to operator space coalgebras (see [LS_3]). For this some technical results concerning quantum stochastic differential equations with nontrivial initial conditions are required ([LS_2]).

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Our aim here is to give a flavour of these recent results. No attempt is made to formulate theorems in the greatest possible generality; proofs will appear elsewhere. Quantum stochastic convolution cocycles may be considered in two different categories, namely algebraic and operator-space theoretic. Although the available techniques are different in each case, we seek to show that these convolution cocycles may fruitfully be considered from a common viewpoint. This has influenced the plan of the paper. The first section sets out our notation, and recalls some quantum stochastic lore. In the second section a flexible definition of such cocycles is given, together with their basic properties, covering both the algebraic and the C^*-algebraic cases. This section also shows how convolution cocycles may be obtained by solving coalgebraic quantum stochastic differential equations, moreover the form taken by the stochastic generator of a unital *-homomorphic cocycle (due to Schürmann) is recalled. Results for the algebraic case are given in the third section; those for the C^*-algebraic cases in the fourth. In the final section we briefly present the case where both algebraic and C^*-algebraic convolution cocycles coexist on the same underlying space, namely the case of full compact quantum groups.

In view of space limitations, we do not discuss generalisations to compact quantum hypergroups, the structure of generators of completely positive, contractive convolution cocycles, or dilations of such cocycles on a C^*-bialgebra to quantum Lévy processes ([S]). Currently the main remaining challenge in these developments is the extension of the results to the context of locally compact quantum groups ([KuV], [Kus]). This presents serious technical difficulties. At the root of these is an insufficient understanding of the interplay between operator space theory on the one hand, and multiplier algebras with their strict topology on the other.

1. Preliminaries. In this section we set out our notation and recall some quantum stochastic theory ([Hud], [Mey], [Par]; we follow [L]). "Quantum stochastic" will usually be abbreviated to QS.

General notations. All vector spaces in this note are complex and inner products are linear in their second argument. Let h be a Hilbert space. For a function $f : \mathbb{R}_+ \to \mathsf{h}$ and subinterval I of $\mathbb{R}_+$, f_I denotes the function $\mathbb{R}_+ \to \mathsf{h}$ which agrees with f on I and is zero outside I (cf. standard indicator-function notation). This convention also applies to vectors, by viewing them as constant functions—for example

$$\xi_{[s,t[}, \text{ for } \xi \in \mathsf{h} \text{ and } 0 \leq s < t.$$

Now let E be a dense subspace of h. The following notation will be employed:

$$\widehat{E} := \operatorname{Lin}\{\widehat{\xi} : \xi \in E\}, \text{ where } \widehat{\xi} := \begin{pmatrix} 1 \\ \xi \end{pmatrix} \in \widehat{\mathsf{h}} := \mathbb{C} \oplus \mathsf{h},$$

and

$$|E\rangle := \{|\xi\rangle : \xi \in E\}, \text{ where } |\xi\rangle : \mathbb{C} \to \mathsf{h} \text{ is the map } \lambda \mapsto \lambda\xi.$$

Thus $\widehat{E}$ is a dense subspace of $\widehat{\mathsf{h}}$ and $|E\rangle$ is a dense subspace of $B(\mathbb{C}; \mathsf{h})$. Next let $\mathcal{O}(E)$ denote the vector space of operators on h with domain E and define subspaces of $\mathcal{O}(E)$

as follows:

$$\begin{aligned} &\mathcal{O}^{\ddagger}(E) := \{T \in \mathcal{O}(E) : \operatorname{Dom} T^* \supset E\}, \\ &\mathcal{O}^{\mathrm{inv}}(E) := \{T \in \mathcal{O}(E) : \operatorname{Ran} T \subset E\} \text{ and} \\ &\mathcal{O}^{*}(E) := \{T \in \mathcal{O}^{\ddagger}(E) : T, T^{\dagger} \in \mathcal{O}^{\mathrm{inv}}(E)\}. \end{aligned} \tag{1}$$

Here "inv" stands for invariant, and the dagger notation is

$$T^{\dagger} := T^*|_E \tag{2}$$

for $T \in \mathcal{O}^{\ddagger}(E)$. Thus $\mathcal{O}^{\ddagger}(E)$ is an involutive vector space, with involution given by (2), and $\mathcal{O}^{*}(E)$ is a unital *-algebra; the former following from the inclusion $\operatorname{Dom}(S+\lambda T)^* \supset \operatorname{Dom} S^* \cap \operatorname{Dom} T^*$. Operator composition $\mathcal{O}^{\ddagger}(E) \times \mathcal{O}^{\mathrm{inv}}(E) \to \mathcal{O}(E)$ extends to ordered pairs (S, T) in $\mathcal{O}^{\ddagger}(E) \times \mathcal{O}(E)$ for which $\operatorname{Dom}(S^{\dagger})^* \supset \operatorname{Ran} T$, as follows:

$$S \cdot T := (S^{\dagger})^* T. \tag{3}$$

This partially defined product is bilinear in an obvious sense.

For vectors $\zeta \in \mathsf{h}, \eta \in E$ and vector space V define a map $\Omega_{\eta}^{\zeta} : V \otimes \mathcal{O}(E) \to V$ by

$$\Omega_{\eta}^{\zeta}(x \otimes T) = \langle \zeta, T\eta \rangle x \tag{4}$$

($x \in V, T \in \mathcal{O}(E)$), extended linearly. For an operator space V and vectors $\zeta, \eta \in \mathsf{h}$, the same notation is used for the completely bounded map from the spatial tensor product $\mathsf{V} \otimes B(\mathsf{H})$ to V, given by continuous linear extension of the prescription (4). Finally, ampliations are denoted

$$\iota_{\mathsf{h}} : B(\mathsf{H}) \to B(\mathsf{H} \otimes \mathsf{h}),\ T \mapsto T \otimes I_{\mathsf{h}},$$

Fock space notations. Let k be a fixed Hilbert space, called the *noise dimension space*. The Hilbert space $L^2(\mathbb{R}_+; \mathsf{k})$ is denoted K, $\mathcal{F}$ denotes the symmetric Fock space over K and her exponential vectors are written $\varepsilon(f)$ ($f \in \mathsf{K}$). Also write $\mathcal{F}_J$ when $\mathbb{R}_+$ is replaced by a subinterval J. For a subset D of k and subinterval J of $\mathbb{R}_+$, define $\mathbb{S}_{D,J} := \{f \in \mathbb{S} : f \text{ is } D\text{-valued and vanishes outside } J\}$, where

$$\mathbb{S} := \operatorname{Lin}\{d_{[0,s[} : d \in \mathsf{k}, s \in \mathbb{R}_+\}.$$

Also define a corresponding subspace of $\mathcal{F}_J$:

$$\mathcal{E}_{D,J} := \operatorname{Lin}\{\varepsilon(f) : f \in \mathbb{S}_{D,J}\},$$

dropping subscripts when $D = \mathsf{k}$, respectively $J = \mathbb{R}_+$. If D is total in k and contains 0 then the subspace $\mathcal{E}_D$ is dense in $\mathcal{F}$. For us D will be a dense subspace of k. It is often convenient to suppress ampliations by exploiting the exponential property of Fock space. Thus, for example, $\mathcal{O}(\mathcal{E}_{D,[a,b[})$ may be viewed as a subspace of $\mathcal{O}(\mathcal{E}_D)$ through the map $T \mapsto I \otimes T \otimes I$, where the identity operators are on $\mathcal{F}_{[0,a[}$ and $\mathcal{F}_{[b,\infty[}$ respectively. The CCR flow of index k, defined in terms of the second quantisation of the shift on K, is denoted $\sigma = (\sigma_t)_{t \geq 0}$. These extend to maps of unbounded operators. Thus, suppressing ampliations, $\sigma_s(\mathcal{O}(\mathcal{E}_{D,J})) = \mathcal{O}(\mathcal{E}_{D,J+s})$.

QS processes. Let D be a dense subspace of the noise dimension space k. Elements of $\mathcal{E}_D$ will play the role of test functions. By an *operator process* we understand a family $X = (X_t)_{t \geq 0}$ of operators on $\mathcal{F}$, each having the (dense) domain $\mathcal{E}_D$, being weak-operator measurable in t and adapted to the natural Fock-space operator-filtration. Thus $X : \mathbb{R}_+ \to \mathcal{O}(\mathcal{E}_D)$, $t \mapsto X_t\xi$ is weakly measurable for all $\xi \in \mathcal{E}_D$ and, for each $t \geq 0$, $X_t \in \mathcal{O}(\mathcal{E}_{D,[0,t[}) \otimes I$ where the identity is on $\mathcal{F}_{[t,\infty[}$. The linear space of all such processes is denoted $\mathbb{P}(\mathcal{E}_D)$. For a dense subspace E of a Hilbert space h, $\mathbb{P}(E \otimes \mathcal{E}_D)$ is defined similarly and we speak of h-*processes*. The collection of processes $X \in \mathbb{P}(E \otimes \mathcal{E}_D)$ having a conjugate process in $\mathbb{P}(E \otimes \mathcal{E}_D)$ (i.e. a process $X^\dagger \in \mathbb{P}(E \otimes \mathcal{E}_D)$ for which $\mathrm{Dom} X_t^* \supset X_t^\dagger$ $(t \in \mathbb{R}_+)$) is denoted $\mathbb{P}^\ddagger(E \otimes \mathcal{E}_D)$. Finally let V be a vector space. Linear maps from V to $\mathbb{P}(\mathcal{E}_D)$ are called *processes on V with domain $\mathcal{E}_D$* and the set of all such processes on V is written $\mathbb{P}(V, \mathcal{E}_D)$.

QS differential equations with functional as initial condition. For linear maps $\kappa : V \to \mathbb{C}$ and $\phi : V \to V \otimes \mathcal{O}(\widehat{D})$ (or $V \to V \otimes B(\widehat{\mathsf{k}})$ with spatial tensor product, if V is an operator space), consider the *quantum stochastic differential equation*

$$dk_t = \widehat{k}_t \circ \phi \, d\Lambda_t, \quad k_0 = \iota_{\mathcal{F}} \circ \kappa. \tag{5}$$

By a *weak solution* of this equation (with domain $\mathcal{E}_D$) we understand a process $k \in \mathbb{P}(V, \mathcal{E}_D)$ such that

$$\langle \varepsilon(f), (k_t(x) - \kappa(x)1_{\mathcal{F}})\varepsilon(g)\rangle = \int_0^t \langle \varepsilon(f), (k_s \circ \Omega_{\widehat{g}(s)}^{\widehat{f}(s)} \circ \phi)(x)\varepsilon(g)\rangle ds$$

$(t \geq 0, x \in V, f, g \in \mathbb{S}_D)$. If there is a *quantum stochastically integrable* $\widehat{\mathsf{k}}$-process K on V, with domain $\widehat{D} \otimes \mathcal{E}_D$, satisfying

$$\Omega_{\eta \otimes \varepsilon(g)}^{\zeta \otimes \varepsilon(f)} \circ K_t = \Omega_{\varepsilon(g)}^{\varepsilon(f)} \circ k_t \circ \Omega_\eta^\zeta$$

$(\zeta, \eta \in \widehat{D}, f, g \in \mathbb{S}_D, t \geq 0)$, then k is called a *strong solution.*

2. Quantum stochastic convolution cocycles. For this section 'space', 'map' and $\otimes$ mean respectively space, map and tensor product of the appropriate category. Thus in the algebraic case these are respectively vector spaces, linear maps, and algebraic tensor products, whereas in the C^*-algebraic case they are operator spaces, completely bounded maps and spatial tensor products.

**-Bialgebras.* The idea of considering quantum stochastic convolution cocycles originates in the theory of quantum Lévy processes, the noncommutative counterpart of classical Lévy processes on groups. As usual in noncommutative mathematics, in order to 'quantise' one focuses on the appropriate class of functions on the underlying set of the classical structure. Then we see that quantum stochastic convolution cocycles should 'act on' a generalisation of the algebra of complex-valued functions on a group (or, to be more precise, semigroup with identity), namely on a **-bialgebra.* By this we understand a unital *-algebra (respectively C^*-algebra) A, equipped with compatible coalgebraic structure, that is, unital *-homomorphisms

$$\Delta : \mathrm{A} \to \mathrm{A} \otimes \mathrm{A} \text{ and } \epsilon : \mathrm{A} \to \mathbb{C},$$

called *comultiplication* and *counit* respectively, satisfying

$$(\Delta \otimes \mathrm{id}_{\mathrm{A}}) \circ \Delta = (\mathrm{id}_{\mathrm{A}} \otimes \Delta) \circ \Delta \text{ and} \tag{6}$$

$$(\epsilon \otimes \mathrm{id}_{\mathrm{A}}) \circ \Delta = (\mathrm{id}_{\mathrm{A}} \otimes \epsilon) \circ \Delta = \mathrm{id}_{\mathrm{A}}. \tag{7}$$

Coalgebraic structure entails a convolution on maps, as follows. Let U, V and W be spaces for which there is a natural map $M : U \otimes V \to W$, for example $V = \mathbb{C}$ and $W = U$. Then maps $\psi : \mathrm{A} \to U$ and $\chi : \mathrm{A} \to V$ determine a map

$$\psi \star \chi := M \circ (\psi \otimes \chi) \circ \Delta : \mathrm{A} \to W$$

As an example of this notation observe that equations (6) and (7) may be expressed as follows:

$$\Delta \star \mathrm{id}_{\mathrm{A}} = \mathrm{id}_{\mathrm{A}} \star \Delta, \text{ respectively } \epsilon \star \mathrm{id}_{\mathrm{A}} = \mathrm{id}_{\mathrm{A}} \star \epsilon = \mathrm{id}_{\mathrm{A}}.$$

In these cases the natural maps are the following obvious maps: $(\mathrm{A} \otimes \mathrm{A}) \otimes \mathrm{A} \to \mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A}$, $\mathrm{A} \otimes (\mathrm{A} \otimes \mathrm{A}) \to \mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A}$, $\mathbb{C} \otimes \mathrm{A} \to \mathrm{A}$ and $\mathrm{A} \otimes \mathbb{C} \to \mathrm{A}$ respectively.

DEFINITION. A family $\{\mu_t : t \geq 0\}$ of maps $\mathrm{A} \to \mathbb{C}$ is called a *convolution semigroup of functionals* if

$$\mu_0 = \epsilon \text{ and } \mu_{s+t} = \mu_s \star \mu_t \text{ for all } s, t \geq 0.$$

The coalgebraic structure of A facilitates another algebraic operation on maps, defined as follows. For any space V and map $\psi : \mathrm{A} \to V$ define a map

$$R_\psi : \mathrm{A} \to \mathrm{A} \otimes V$$

by the formula:

$$R_\psi = (\mathrm{id}_{\mathrm{A}} \otimes \psi) \circ \Delta = \mathrm{id}_{\mathrm{A}} \star \psi.$$

Such maps are known as convolution operators. The *R-map* itself has a left inverse:

$$(\epsilon \otimes \mathrm{id}_V) \circ R_\psi = \psi.$$

Specialising again to maps $\mathrm{A} \to \mathbb{C}$, it is easily seen that the R-map intertwines convolution and composition: for maps $\lambda, \mu : \mathrm{A} \to \mathbb{C}$,

$$R_\lambda \circ R_\mu = R_{\lambda \star \mu}. \tag{8}$$

REMARK. Readers may recognise, in the above, generalisations of standard notions and operations from the theory of classical probability on algebraic structures ([Gre], [Hey]). For example R_λ corresponds to the so-called probability operator of a probability measure on a group.

QS convolution cocycles. Let $(\mathrm{A}, \Delta, \epsilon)$ be a *-bialgebra and let D be a fixed dense subspace of the noise dimension space k.

DEFINITION. A *quantum stochastic convolution cocycle* (on A with domain $\mathcal{E}_D$) is a process $l \in \mathbb{P}(\mathrm{A}, \mathcal{E}_D)$ such that, for $s, t \geq 0$,

$$l_{s+t} = l_s \star (\sigma_s \circ l_t) \text{ and } l_0 = \iota_{\mathcal{F}} \circ \epsilon.$$

The first of these conditions is referred to as the *convolution increment property.* The natural product in the definition of the convolution here is given by the identification $\mathcal{O}(\mathcal{E}_{D,[0,s[}) \otimes \mathcal{O}(\mathcal{E}_{D,[s,s+t[}) = \mathcal{O}(\mathcal{E}_{D,[0,s+t[})$ in the algebraic case, and by the inclusion $B(\mathcal{F}_{[0,s[}) \otimes B(\mathcal{F}_{[s,s+t[}) \subset B(\mathcal{F}_{[0,s+t[})$ in the C^*-algebraic case.

With each QS convolution cocycle l one may associate a family of convolution semigroups in the following way: for each $c, d \in D$ define

$$\lambda_t^{c,d}(a) = \langle \varepsilon(c_{[0,t[}), l_t(a)\varepsilon(d_{[0,t[})\rangle \mathrm{e}^{-t\langle c,d\rangle} \tag{9}$$

($a \in \mathrm{A}, t \geq 0$). It is easily checked that $\{\lambda_t^{c,d} : t \geq 0\}$ is a convolution semigroup of functionals. Moreover the convolution cocycle l is determined by this family, which we refer to as its *associated convolution semigroups* (*of functionals*). This is a consequence of the following fact, which uses the convention that step functions in $\mathbb{S}$ are right-continuous. For any $f, g \in \mathbb{S}_D$, $a \in \mathrm{A}$ and $t \geq 0$,

$$(10) \qquad \langle \varepsilon(f_{[0,t[}), l_t(a)\varepsilon(g_{[0,t[})\rangle = (\lambda_{t_1-t_0}^{c_0,d_0} \otimes \cdots \otimes \lambda_{t_n-t_{n-1}}^{c_{n-1},d_{n-1}})(\Delta_{n-1}(a))e^{\langle f_{[0,t[}, g_{[0,t[}\rangle},$$

where $0 = t_0 \leq t_1 < \ldots \leq t_n = t$ contains all the discontinuities of f and g in $[0,t]$, $c_0 = f(t_0)$, $d_0 = g(t_0), \ldots, c_{n-1} = f(t_{n-1})$ and $d_{n-1} = g(t_{n-1})$. Here $\Delta_n : \mathrm{A} \to \mathrm{A}^{\otimes(n+1)}$ is defined recursively by

$$\Delta_0 := \mathrm{id}_{\mathrm{A}}, \quad \Delta_1 := \Delta, \quad \Delta_{i+1} := \Delta_i \star \mathrm{id}_{\mathrm{A}} \text{ for } i \geq 1.$$

Exactly as was the case for convolution semigroups of functionals (see equation (8)), the convolution operator/R-map transforms QS convolution cocycles into QS cocycles. To be more precise, for a given QS convolution cocycle l, define $k_t = R_{l_t}$ for each $t \geq 0$. Then, for $s, t \geq 0$,

$$k_{s+t} = \widehat{k}_s \circ (\sigma_s \circ k_t), \quad k_0 = \iota_{\mathcal{F}},$$

where $\widehat{k}_s$ is a certain extension of k_s, defined precisely in [LW$_1$]. This correspondence remains valid at the level of associated semigroups; due to this many results of [LW$_1$] have counterparts in the convolution context. The correspondence is also crucial for the proofs of the theorems in Section 4.

The above definition of a QS convolution cocycle generalises naturally by replacing the counit in the initial condition by an *idempotent functional*. By this we mean a linear functional $\chi : \mathrm{A} \to \mathbb{C}$ satisfying

$$\chi \star \chi = \chi.$$

This is relevant, for example, in the context of Lévy processes on quantum hypergroups ([FrS]).

Coalgebraic QS differential equations, Quantum stochastic convolution cocycles on A are constructed by solving QS differential equations of the following kind

$$(11) \qquad dl_t = l_t \star_\tau d\Lambda_\varphi(t), \quad l_0 = \iota_{\mathcal{F}} \circ \epsilon$$

(τ indicating the tensor flip reversing the order of the spaces $\widehat{\mathsf{k}}$ and $\mathcal{F}$), for a map $\varphi : \mathrm{A} \to \mathcal{O}(\widehat{D})$ (or $\mathrm{A} \to B(\widehat{\mathsf{k}})$). A process $l \in \mathbb{P}(\mathrm{A}, \mathcal{E}_D)$ being a weak solution of this equation (with domain $\mathcal{E}_D$) is equivalent to l weakly satisfying the 'standard' QS differential equation (5) in which

$$\phi = R_\varphi \text{ and } \kappa = \epsilon.$$

Remark. In the algebraic case, this reads

$$\langle \varepsilon(f), (l_t(a) - \epsilon(a)I_{\mathcal{F}})\varepsilon(g)\rangle = \int_0^t \langle \varepsilon(f), l_s(a_{(1)})\varepsilon(g)\rangle \langle \widehat{f(s)}, \varphi(a_{(2)})\widehat{g(s)}\rangle \, ds,$$

in which the Sweedler notation $a_{(1)} \otimes a_{(2)}$ is used for Δa ([Swe]).

An important observation here is that if l satisfies such a QS differential equation, then the *generators* of the associated convolution semigroups of l, defined (for each $c, d \in D$) by

$$\gamma_{c,d}(a) = \lim_{t\to 0^+} \frac{1}{t}(\lambda_t^{c,d}(a) - \epsilon(a)), \quad a \in \mathrm{A},$$

satisfy

$$\gamma_{c,d}(a) = \langle \hat{c}, \varphi(a)\hat{d}\rangle. \tag{12}$$

This fact is key for determining classes of cocycles arising in this way. Specific conditions, assuring the existence of a solution, or for characterising the type of stochastically generated cocycles, will be given in the following sections—separately for the algebraic and C^*-algebraic cases.

*-*Homomorphic convolution cocycles.* It was Schürmann who observed that the stochastic generators of quantum Lévy processes acting on a Fock space (i.e. unital *-homomorphic QS convolution cocycles, in the terminology of this note) are given by triples of maps satisfying certain structure relations.

DEFINITION. Following Meyer, a triple (γ, δ, ρ) consisting of a unital *-homomorphism $\rho : \mathrm{A} \to \mathcal{O}^*(D)$ (or $\mathrm{A} \to B(\mathsf{k})$), a ρ-ϵ-derivation, that is, a map $\delta : \mathrm{A} \to |D\rangle$ (or $\mathrm{A} \to |\mathsf{k}\rangle$), satisfying

$$\delta(ab) = \delta(a)\epsilon(b) + \rho(a)\delta(b),$$

and a map $\gamma : \mathrm{A} \to \mathbb{C}$ satisfing

$$\gamma(a^*b) = \gamma(a)^*\epsilon(b) + \epsilon(a)^*\gamma(b) + \delta(a)^*\delta(b),$$

is called a *Schürmann triple* on A.

For a map $\varphi : \mathrm{A} \to \mathcal{O}^*(\widehat{D})$ (or $\mathrm{A} \to B(\widehat{\mathsf{k}})$) the following are equivalent:

(i) φ has block matrix form

$$\begin{bmatrix} \gamma & \delta^\dagger \\ \delta & \rho - \iota_{\mathsf{k}} \circ \epsilon \end{bmatrix},$$

where (γ, δ, ρ) forms a Schürmann triple;

(ii) φ satisfies

$$\varphi(1) = 0 \text{ and, for } a, b \in \mathrm{A},\ \varphi(a^*b) = \varphi(a)^*\epsilon(b) + \epsilon(a)^*\varphi(b) + \varphi(a)^*\Delta^{QS}\varphi(b). \tag{13}$$

Here $\Delta^{QS} \in B(\widehat{\mathsf{k}})$ denotes the orthogonal projection with range $\{0\} \oplus \mathsf{k}$, not a coproduct! In the algebraic case Schürmann showed that, under these conditions, the QS differential equation (11) has a unique *-homomorphic and unital solution. The functional γ is real, vanishes at 1_{A} and is *conditionally positive*, that is, positive on the positive part of the kernel of the counit. These conditions characterise *generators* γ of abstract quantum Lévy processes and, for any such functional, a GNS-type construction yields a pre-Hilbert space D' and map $\varphi' \in \mathcal{O}^*(\widehat{D'})$ satisfying (13). This leads to the Schürmann Reconstruction Theorem—every quantum Lévy process may be reconstructed (up to equivalence) from its generator ([Sch]).

The following remark connects our approach to Schürmann's (see Theorem 3.3 below).

REMARK. The invariance condition on φ and ρ may be dropped: if $\varphi : \mathrm{A} \to \mathcal{O}^{\ddagger}(\widehat{D})$, respectively $\rho : \mathrm{A} \to \mathcal{O}^{\ddagger}(D)$, then the above equivalence endures with ρ being *real and weakly multiplicative* (that is, *-homomorphic with respect to the involution and product defined in (2) and (3)).

3. Algebraic case. In this section 'spaces' and 'maps' are vector spaces and linear maps, tensor products are algebraic and $\mathcal{A}$ is a *-bialgebra. The following existence theorem is proved with the help of the Fundamental Theorem on Coalgebras.

THEOREM 3.1. *Let $\varphi \in L(\mathcal{A}; \mathcal{O}(\widehat{D}))$. Then the equation*

$$dl_t = l_t \star_\tau d\Lambda_\varphi(t), \quad l_0 = \iota_{\mathcal{F}} \circ \epsilon,$$

has a unique weak solution in $\mathbb{P}(\mathcal{A}, \mathcal{E}_D)$, it is a quantum stochastic convolution cocycle and is actually a strong solution; we denote it l^φ.

The following notation is useful for capturing necessary and sufficient conditions for an algebraic QS convolution cocycle to be 'stochastically generated'.

$\mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D) := \{X \in \mathbb{P}(\mathcal{E}_D) : \forall_{\xi \in \mathcal{E}_D}\, t \mapsto X_t \xi$ is locally Hölder-continuous with exponent $\frac{1}{2}\}$,

$\mathbb{P}^{\ddagger}_{\mathrm{Hc}}(\mathcal{E}_D) := \{X \in \mathbb{P}^{\ddagger}(\mathcal{E}_D) : X, X^\dagger \in \mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D)\}$,

$\mathbb{P}_{\mathrm{Hc}}(\mathcal{A}, \mathcal{E}_D) := L(\mathcal{A}; \mathbb{P}_{\mathrm{Hc}}(\mathcal{E}_D))$ and $\mathbb{P}^{\ddagger}_{\mathrm{Hc}}(\mathcal{A}, \mathcal{E}_D) := L(\mathcal{A}; \mathbb{P}^{\ddagger}_{\mathrm{Hc}}(\mathcal{E}_D))$.

We refer to processes in $\mathbb{P}_{\mathrm{Hc}}(\mathcal{A}, \mathcal{E}_D)$ as *Hölder-continuous processes. Continuous processes* are defined analogously. Recall the operator notation introduced in (1).

THEOREM 3.2. *Let $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_D)$. Then the following are equivalent:*

(i) *k is a quantum stochastic convolution cocycle in $\mathbb{P}^{\ddagger}_{\mathrm{Hc}}(\mathcal{A}, \mathcal{E}_D)$;*
(ii) *$k = l^\varphi$ for some $\varphi \in L(\mathcal{A}; \mathcal{O}^{\ddagger}(\widehat{D}))$.*

The stochastic generators of *-homomorphic cocycles are characterised next.

THEOREM 3.3. *Let $l = l^\varphi$ where $\varphi \in L(\mathcal{A}; \mathcal{O}^{\ddagger}(\widehat{D}))$. Then the following are equivalent:*

(i) *l is real, weakly multiplicative and unital;*
(ii) *φ satisfies the structure relations* (13).

4. C^*-Algebraic case. In this section 'spaces' and 'maps' are operator spaces and completely bounded maps; tensor products are spatial; and A is a C^*-bialgebra, that is a C^*-algebra with (topological) *-bialgebra structure. Being *-homomorphisms, the comultiplication and counit are automatically completely bounded. As the stochastic generators we consider are everywhere defined, we take $D = \mathsf{k}$. However, since solutions of QS differential equations need not be bounded, we need a further definition. Recall that $\mathcal{E}$ abbreviates $\mathcal{E}_{\mathsf{k}}$.

DEFINITION. A process $l \in \mathbb{P}(\mathrm{A}, \mathcal{E})$ is called a *weak quantum stochastic convolution cocycle* if each functional defined by (9) is continuous and the identities (10) hold.

For the technical definitions and corresponding results for 'standard' cocycles and QS differential equations see [LW$_{1-3}$], or [L]. The basic existence theorem for solutions of QS differential equations in our context is the following.

THEOREM 4.1. *Let* $\phi : \mathsf{A} \to \mathsf{A} \otimes B(\widehat{\mathsf{k}})$ *and* $\kappa : \mathsf{A} \to \mathbb{C}$ *be completely bounded linear maps. Then the QS differential equation* (5) *has a unique weakly regular weak solution on* $\mathcal{E}$*; it is actually a cb-strongly regular strong solution.*

REMARK. Complete boundedness for κ is equivalent to boundedness.

THEOREM 4.2. *Let* $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$*. The QS differential equation*

$$\mathrm{d}l_t = l_t \star_\tau \mathrm{d}\Lambda_\varphi(t), \quad l_0 = \iota_{\mathcal{F}} \circ \epsilon,$$

has a unique weakly regular weak solution, again denoted l^φ*; it is a weak QS convolution cocycle.*

As the coefficients of the QS differential equations considered above are completely bounded, identity (12) implies that the generators of each associated convolution semigroup of the cocycle l^φ must be norm bounded; in turn this implies norm continuity for these semigroups and naturally leads to the next definition.

DEFINITION. A completely bounded QS convolution cocycle is *Markov-regular* if its associated convolution semigroup of functionals $\{\lambda_t^{0,0} : t \geq 0\}$ is norm continuous.

REMARK. For a completely contractive QS convolution cocycle, all its associated convolution semigroups are norm continuous if one of them is (cf. 'standard' cocycles - [LW_1]).

THEOREM 4.3. *Let* $l \in \mathbb{P}(\mathsf{A}, \mathcal{E})$ *be a completely positive, contractive and Markov-regular QS convolution cocycle. Then there is a unique map* $\varphi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}))$ *such that* $l = l^\varphi$.

The form of 'stochastic generator' of a *-homomorphic convolution cocycle is now exactly the same as before—algebraic conditions implying analytic ones.

THEOREM 4.4. *Let* $\varphi \in L(\mathsf{A}; B(\widehat{\mathsf{k}}))$*. Then the following are equivalent:*

(i) φ *is completely bounded and* l^φ *is *-homomorphic and unital;*
(ii) φ *satisfies the structure relations* (13).

5. Cocycles on full compact quantum groups. A concept of compact quantum groups was introduced by Woronowicz, in [Wor_1]. For our purposes it is most convenient to adopt the following definition. Tensor products here are spatial/minimal.

DEFINITION ([Wor_2]). A *compact quantum group* is a pair (A, Δ), where A is a unital C^*-algebra and $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital *-homomorphic map which is coassociative and satisfies the quantum cancellation properties:

$$\overline{\mathrm{Lin}}(1_{\mathsf{A}} \otimes \mathsf{A})\Delta(\mathsf{A}) = \overline{\mathrm{Lin}}(\mathsf{A} \otimes 1_{\mathsf{A}})\Delta(\mathsf{A}) = \mathsf{A} \otimes \mathsf{A}.$$

For the concept of Hopf *-algebras and their unitary corepresentations, as well as unitary corepresentations of compact quantum groups, we refer the reader to [KlS]. For our purposes it is sufficient to note the facts contained in the following theorem.

THEOREM 5.1 ([Wor_1]). *Let* A *be a compact quantum group and let* $\mathcal{A}$ *denote the linear span of the matrix coefficients of irreducible unitary corepresentations of* A*. Then* $\mathcal{A}$ *is a dense *-subalgebra of* A*, the coproduct of* A *restricts to an algebraic coproduct* Δ_0 *on* $\mathcal{A}$ *and there is a natural counit* ϵ *and coinverse* $\mathcal{S}$ *on* $\mathcal{A}$ *which makes it a Hopf *-algebra.*

REMARK ([BMT]). In the above theorem $(\mathcal{A}, \Delta_0, \epsilon, \mathcal{S})$ is the unique dense Hopf *-subalgebra of A, in the following sense: if $(\mathcal{A}', \Delta_0', \epsilon', \mathcal{S}')$ is a Hopf *-algebra in which $\mathcal{A}'$ is a dense *-subalgebra of A and the coproduct of A restricts to an algebraic coproduct Δ_0' on $\mathcal{A}'$, then $(\mathcal{A}', \Delta_0', \epsilon', \mathcal{S}')$ equals $(\mathcal{A}, \Delta_0, \epsilon, \mathcal{S})$.

The Hopf *-algebra arising here is called the *associated Hopf *-algebra* of (A, Δ). Dijkhuizen and Koornwinder observed that the Hopf *-algebras arising in this way have intrinsic algebraic structure.

DEFINITION. A Hopf *-algebra $\mathcal{A}$ is called a *CQG algebra* if it is the linear span of all matrix elements of its finite dimensional unitary corepresentations.

THEOREM 5.2 ([DiK]). *Each Hopf *-algebra associated with a compact quantum group is a CQG algebra. Conversely, if $\mathcal{A}$ is a CQG algebra then*

$$\|a\| := \sup\{\|\pi(a)\| : \pi \textit{ is a *-representation of } \mathcal{A} \textit{ on a Hilbert space}\} \tag{14}$$

defines a C^*-norm on $\mathcal{A}$ and the completion of $\mathcal{A}$ with respect to this norm is a compact quantum group with comultiplication extending that of $\mathcal{A}$.

The compact quantum group obtained in this theorem is called the *universal compact quantum group of $\mathcal{A}$* and is denoted $\mathcal{A}_{\mathrm{u}}$.

DEFINITION. A compact quantum group (A, Δ) is called a *full compact quantum group* if the C^*-norm it induces on its associated CQG algebra $\mathcal{A}$ coincides with its canonical norm defined in (14) (equivalently, if A is *-isomorphic to $\mathcal{A}_{\mathrm{u}}$).

The notion of full compact quantum groups was introduced in [BaS] and [BMT] (in the latter they were called universal compact quantum groups). The fundamental examples of commutative and cocommutative C^*-bialgebras, namely $C(G)$ for a compact group G and universal C^*-algebra $C^*(\Gamma)$ for a discrete group Γ, are full compact quantum groups. Moreover most of the genuinely quantum (i.e. neither commutative nor cocommutative) compact quantum groups considered in the literature also fall into this category, including the queen of examples, $SU_q(2)$.

Before formulating the main results of this section we need one more proposition. The first part was proved in [BMT]; the second is a rather straightforward application of ideas of [DiK] (see also [KlS]).

PROPOSITION 5.3. *Each full compact quantum group A is a C^*-bialgebra, whose counit is the continuous extension of the counit of its associated CQG algebra $\mathcal{A}$. There is a bijective correspondence between unital *-homomorphic QS convolution cocycles on A and unital, real and weakly multiplicative QS convolution cocycles on $\mathcal{A}$.*

The above facts, together with the theorems of the previous two sections, imply the following characterisation of stochastically generated QS convolution cocycles on full compact quantum groups.

THEOREM 5.4. *Let A be a full compact quantum group and let $k \in \mathbb{P}(\mathsf{A}, \mathcal{E})$ be a completely bounded process. Then the following are equivalent:*

(i) *k and $k^\dagger$ are Hölder-continuous QS convolution cocycles;*
(ii) *$k|_{\mathcal{A}} = l^\varphi$ for some map $\varphi \in L(\mathcal{A}; B(\widehat{\mathsf{k}}))$.*

Restricting to *-homomorphic cocycles yields the following much stronger result.

THEOREM 5.5. *Let* $k \in \mathbb{P}(\mathsf{A}, \mathcal{E}_D)$, *where* A *is a full compact quantum group and* D *is a dense subspace of a Hilbert space* k. *Then the following are equivalent:*

(i) k *is a Hölder-continuous, unital and *-homomorphic QS convolution cocycle;*
(ii) k *is bounded and* $k|_{\mathcal{A}} = l^{\varphi}$ *for some map* $\varphi \in L(\mathcal{A}; \mathcal{O}^{\ddagger}(\widehat{D}))$ *satisfying the structure relations* (13).

REMARK. One of the consequences of the last two theorems is that each map φ defined on a CQG algebra $\mathcal{A}$ with values in $\mathcal{O}^{\ddagger}(\widehat{D})$ satisfying (13) must be bounded-operator-valued. However, φ need not be continuous as an operator $\mathcal{A} \to B(\widehat{\mathsf{k}})$ (see [ScS] for examples), and therefore need not extend to A. If it is continuous then it is necessarily completely bounded.

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