QUANTUM PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 73
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2006

QUANTUM STATES SATISFYING CLASSICAL PROBABILITY CONSTRAINTS

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Abstract. For linear combinations of quantum product averages in an arbitrary bipartite state, we derive new quantum Bell-form and CHSH-form inequalities with the right-hand sides expressed in terms of a bipartite state. This allows us to specify bipartite state properties sufficient for the validity of a classical CHSH-form inequality and the perfect correlation form of the original Bell inequality for any bounded quantum observables. We also introduce a new general condition on a bipartite state and quantum observables sufficient for the validity of the original Bell inequality, in its perfect correlation or anticorrelation forms. Under this general sufficient condition, a bipartite quantum state does not necessarily exhibit Bell's perfect correlations or anticorrelations.

1. Introduction. The Bell [1] and the Clauser-Horne-Shimony-Holt (*CHSH*) [2] inequalities, derived originally in the frame of the Bell local hidden variable model, describe the relations between the product expectation values under different joint measurements.

In the frame of classical probability, the product expectation values in every classical state satisfy the CHSH inequality and the perfect correlation form of the original Bell inequality ([1]) for any bounded classical observables.

In the frame of quantum probability, under joint quantum measurements on a bipartite system, the product expectation values in a bipartite quantum state do not, in general, satisfy the above inequalities. It is, however, well known [4, 5] that not only all separable quantum states but also a variety of nonseparable quantum states satisfy the CHSH inequality for any bounded quantum observables. We also proved in [3] (section 3.B.1,

2000 *Mathematics Subject Classification*: 81P15, 47A20, 47A63.

The paper is in final form and no version of it will be published elsewhere.

([1]) The original proof [1] of the perfect correlation form of the Bell inequality is true only for dichotomic classical observables admitting values $\pm\lambda$. In appendix of [3], we proved the validity of this inequality for any three bounded classical observables.

item 1) that there exist ([2]) separable quantum states that satisfy the perfect correlation form of the original Bell inequality for any bounded quantum observables ([3]).

At present, Bell-type inequalities are widely used in quantum information processing. However, from the pioneering paper of R. Werner [4] up to now a general analytical structure of bipartite quantum states satisfying a *classical* ([4]) CHSH-form inequality has not been well formalized. Moreover, a structure of bipartite quantum states satisfying the perfect correlation form of the original Bell inequality for any bounded quantum observables has been analyzed [3] in the literature only in the separable case.

The aim of this paper is to introduce *general analytical conditions* sufficient for a bipartite quantum state to satisfy a *classical* CHSH-form inequality and the perfect correlation form of the original Bell inequality for any bounded quantum observables.

In *section 2.1*, we introduce a new notion: a source-operator for a bipartite quantum state and prove (proposition 1) that, for any bipartite state, source-operators exist. We specify new notions: density source-operator (DSO) states and Bell class states, and present examples of such bipartite states. We prove (proposition 2) that the nonseparable Werner state ([5]) is a DSO state for any dimension $d \geq 2$ and represents a Bell class state if $d \geq 3$.

In *sections 2.2, 2.3*, for an arbitrary bipartite state, we derive (propositions 3, 4) new upper bounds of linear combinations of quantum product averages. These upper bounds are expressed in terms of source-operators for a bipartite state, and this allows us to specify analytically in section 3 the situations where a bipartite quantum state satisfies a classical Bell-type inequality.

In *section 3*:

(i) we prove (theorems 1, 2) that, for any bounded quantum observables ([6]), the product expectation values in a density source-operator (DSO) state satisfy a *classical* CHSH-form inequality;

(ii) we prove (theorem 3) that every Bell class state satisfies the perfect correlation form of the original Bell inequality for any bounded quantum observables and does not necessarily exhibit perfect correlations;

(iii) we introduce (theorem 4) a new general condition sufficient for a density source-operator (DSO) state and three bounded quantum observables to satisfy the original Bell inequality, in its perfect correlation or anticorrelation forms. A DSO state, satisfying this general sufficient condition, does not necessarily exhibit (proposition 5) Bell's perfect correlations or anticorrelations [1].

([2]) See Eq. (49) in [3].

([3]) These separable quantum states do not necessarily exhibit Bell's perfect correlations [1], see the discussion in [3], section 3.B.1, item 1.

([4]) Here, the term *classical* specifies the validity of some probabilistic constraint in the frame of classical probability.

([5]) This bipartite quantum state was introduced by R. Werner in [4] and is widely used in quantum information processing.

([6]) Everywhere in this paper, quantum observables may be of any spectral types.

In *section 4*, we specify (theorems 5–7) the validity of classical Bell-type inequalities under generalized quantum measurements of Alice and Bob.

2. Quantum upper bounds. General case. Let a bipartite quantum system be described in terms of a separable complex Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. In this section, for an arbitrary state (7) ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, we derive new upper bounds of linear combinations of quantum product averages:

$$\text{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)})] - \text{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_2)})], \tag{1}$$
$$\text{tr}[\rho(W_1^{(a_1)} \otimes W_2^{(b)})] - \text{tr}[\rho(W_1^{(a_2)} \otimes W_2^{(b)})],$$

$$\sum_{n,m=1,2} \gamma_{nm} \text{tr}[\rho(W_1^{(a_n)} \otimes W_2^{(b_m)})]. \tag{2}$$

Here, $W_1^{(a)}$, $W_2^{(b)}$ are any bounded quantum observables on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and γ_{nm}, $n, m = 1, 2$, are any real coefficients. For clarity, we label (8) by indices "a" quantum observables on $\mathcal{H}_1$ and by "b" on $\mathcal{H}_2$.

2.1. *Source-operators for a bipartite state.* In order to evaluate (1) and (2), we introduce in a general setting a new notion.

Denote by $\mathcal{K}_{112} := \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{K}_{122} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ the extended tensor product Hilbert spaces. Below, we use the notation $\text{tr}_{\mathcal{H}_m}^{(k)}[\cdot]$, $k = 1, 2, 3$, $m = 1, 2$, for the partial trace over the elements of a Hilbert space $\mathcal{H}_m$ standing in the k-th place of tensor products.

DEFINITION 1. For a state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, let T_{112} on $\mathcal{K}_{112}$ and T_{122} on $\mathcal{K}_{122}$ be self-adjoint trace class operators defined by the relations:

$$\text{tr}_{\mathcal{H}_1}^{(1)}[T_{112}] = \rho, \quad \text{tr}_{\mathcal{H}_1}^{(2)}[T_{112}] = \rho; \tag{3}$$

$$\text{tr}_{\mathcal{H}_2}^{(2)}[T_{122}] = \rho, \quad \text{tr}_{\mathcal{H}_2}^{(3)}[T_{122}] = \rho. \tag{4}$$

We call any of these dilations a *source-operator* for a bipartite state ρ.

PROPOSITION 1. *For a state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, there exist source-operators T_{122} and T_{112}.*

Proof. The spectral decomposition of a quantum state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ reads:

$$\rho = \sum_i \alpha_i |\Psi_i\rangle\langle\Psi_i|, \quad \langle\Psi_i, \Psi_j\rangle = \delta_{ij}, \quad \forall \alpha_i > 0, \quad \sum_i \alpha_i = 1. \tag{5}$$

Take an orthonormal basis $\{\varphi_n\}$ in $\mathcal{H}_2$ and consider the Schmidt decomposition of an eigenvector Ψ_i with respect to this basis:

$$\Psi_i = \sum_n \Phi_n^{(i)} \otimes \varphi_n, \quad \sum_n \langle\Phi_n^{(i)}, \Phi_n^{(j)}\rangle = \delta_{ij}. \tag{6}$$

Substituting (6) into (5), we derive

$$\rho = \sum_{n,m} \rho_{nm} \otimes |\varphi_n\rangle\langle\varphi_m|, \quad \text{where} \quad \rho_{nm} := \sum_i \alpha_i |\Phi_n^{(i)}\rangle\langle\Phi_m^{(i)}|, \quad \forall n, m.$$

The operators ρ_{nn} are positive with $\sum_n \text{tr}[\rho_{nn}] = 1$.

(7) We consider only normal quantum states.

(8) In the physical literature, these labels correspond to "Alice" and "Bob" names.

For any density operator σ on $\mathcal{H}_2$ and any self-adjoint trace class operator τ_{122} on $\mathcal{K}_{122}$, with $\mathrm{tr}^{(2)}_{\mathcal{H}_2}[\tau_{122}] = \mathrm{tr}^{(3)}_{\mathcal{H}_2}[\tau_{122}] = 0$, the operator

$$T_{122} = \sum_{n,m} \rho_{nm} \otimes |\varphi_n\rangle\langle\varphi_m| \otimes \sigma + \sum_{n,m} \rho_{nm} \otimes \sigma \otimes |\varphi_n\rangle\langle\varphi_m| - \mathrm{tr}_{\mathcal{H}_2}[\rho] \otimes \sigma \otimes \sigma \ + \tau_{122} \tag{7}$$

represents a source-operator for the state (5). Here, $\mathrm{tr}_{\mathcal{H}_2}[\rho] = \sum_n \rho_{nn}$ is the density operator on $\mathcal{H}_1$ reduced from ρ. The existence of a source-operator T_{112} is proved similarly. ∎

Consider now the main properties of source-operators:

1. $\mathrm{tr}[T] = 1$, for any source-operator T.
2. As any self-adjoint trace class operator, a source-operator admits the decomposition $T = T^{(+)} - T^{(-)}$ via non-negative operators $T^{(\pm)} = \frac{1}{2}(|T| \pm T)$ and $||T||_1 = \mathrm{tr}[T^{(+)}] + \mathrm{tr}[T^{(-)}]$. For a source-operator, the latter relation and property 1 imply $||T||_1 = 1 + 2\mathrm{tr}[T^{(-)}]$.
3. Any positive source-operator T is a density operator and we refer to it as a density source-operator (*DSO*). A source-operator is a DSO iff $\|T\|_1 = 1$.

DEFINITION 2. If a bipartite state has a density source-operator then we call this state a *density source-operator state* or a *DSO state*, for short.

Consider a separable state ρ_{sep}. Let $\sum_m \xi_m \rho_1^{(m)} \otimes \rho_2^{(m)}$, where $\xi_m > 0$, $\sum_m \xi_m = 1$, be a separable representation of ρ_{sep}. Then, for example, $T_{122} = \sum_m \xi_m \rho_1^{(m)} \otimes \rho_2^{(m)} \otimes \rho_2^{(m)}$ is a density source-operator for ρ_{sep}.

Hence, *any separable state is a DSO state*. However, the converse is not true and a DSO state may be nonseparable. In section 2.1.1, we consider examples of nonseparable DSO states, in particular, on an infinite dimensional Hilbert space.

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ then $\mathcal{K}_{122} = \mathcal{K}_{112} = \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ and in order to distinguish between source-operators T_{112} and T_{122} we further label ([9]) them as $T_{\blacktriangleleft}$ and $T_{\blacktriangleright}$, respectively. Moreover, if there exists a source-operator that satisfies both conditions in definition 1, then we denote it by $T_{\blacktriangleleft\blacktriangleright}$. The latter source-operator has the special dilation property:

$$\mathrm{tr}^{(1)}_{\mathcal{H}}[T_{\blacktriangleleft\blacktriangleright}] = \mathrm{tr}^{(2)}_{\mathcal{H}}[T_{\blacktriangleleft\blacktriangleright}] = \mathrm{tr}^{(3)}_{\mathcal{H}}[T_{\blacktriangleleft\blacktriangleright}] = \rho. \tag{8}$$

DEFINITION 3. If, for a density source-operator (DSO) state on $\mathcal{H} \otimes \mathcal{H}$, there exists a density source-operator with the special dilation property (8) then we refer to this DSO state as a *Bell class state.*

The Bell class includes both separable and nonseparable states. Separable states on $\mathcal{H} \otimes \mathcal{H}$ of the special form (49) introduced in [3], namely, of the form: $\sum_m \xi_m \rho^{(m)} \otimes \rho^{(m)}$, $\xi_m > 0$, $\sum_m \xi_m = 1$, constitute examples of separable Bell class states.

2.1.1. *Examples of DSO and Bell class states.* In this section, we present examples of nonseparable DSO and Bell class states on $\mathcal{H} \otimes \mathcal{H}$.

Consider the nonseparable Werner state [4]

$$\rho_W^{(d)} = \frac{d+1}{d^3} I_{\mathbb{C}^d \otimes \mathbb{C}^d} - \frac{1}{d^2} V_d \tag{9}$$

([9]) These labels indicate a "direction" of dilation.

on $\mathbb{C}^d \otimes \mathbb{C}^d$, $\forall d \geq 2$. Here, V_d is the permutation operator: $V_d(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1$, $\forall \psi_1, \psi_2 \in \mathbb{C}^d$. This operator is self-adjoint and has the properties: $(V_d)^2 = I_{\mathbb{C}^d \otimes \mathbb{C}^d}$, $\mathrm{tr}[V_d] = d$.

PROPOSITION 2. *The nonseparable Werner state* $\rho_W^{(d)}$, $\forall d \geq 2$, *represents a DSO state and is of the Bell class for any* $d \geq 3$.

Proof. Introduce on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, $\forall d \geq 3$, the orthogonal projection

$$Q_d^{(-)}(\psi_1 \otimes \psi_2 \otimes \psi_3) := \frac{1}{6}\{\psi_1 \otimes \psi_2 \otimes \psi_3 - \psi_2 \otimes \psi_1 \otimes \psi_3 - \psi_1 \otimes \psi_3 \otimes \psi_2 - \psi_3 \otimes \psi_2 \otimes \psi_1 + \psi_2 \otimes \psi_3 \otimes \psi_1 + \psi_3 \otimes \psi_1 \otimes \psi_2\}, \tag{10}$$

$\forall \psi_1, \psi_2, \psi_3 \in \mathbb{C}^d$. This projection has the form:

$$6Q_d^{(-)} = I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} - V_d \otimes I_{\mathbb{C}^d} - I_{\mathbb{C}^d} \otimes V_d - (I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d) + (I_{\mathbb{C}^d} \otimes V_d)(V_d \otimes I_{\mathbb{C}^d}) + (V_d \otimes I_{\mathbb{C}^d})(I_{\mathbb{C}^d} \otimes V_d) \tag{11}$$

and admits a representation:

$$6Q_d^{(-)} = I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} - \sum_{n,m} |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_n| \otimes I_{\mathbb{C}^d} - \sum_{n,m} I_{\mathbb{C}^d} \otimes |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_n| - \sum_{n,m} |e_n\rangle\langle e_m| \otimes I_{\mathbb{C}^d} \otimes |e_m\rangle\langle e_n| + \sum_{n,m,k} |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_k| \otimes |e_k\rangle\langle e_n| + \sum_{n,m,k} |e_m\rangle\langle e_n| \otimes |e_k\rangle\langle e_m| \otimes |e_n\rangle\langle e_k| \tag{12}$$

in an orthonormal basis $\{e_n\}$ in $\mathbb{C}^d$ (notice that $V_d = \sum_{n,m=1}^{d} |e_n\rangle\langle e_m| \otimes |e_m\rangle\langle e_n|$).

We have: $\mathrm{tr}_{\mathbb{C}^d}^{(j)}[Q_d^{(-)}] = \frac{d-2}{6}(I_{\mathbb{C}^d \otimes \mathbb{C}^d} - V_d)$, $\forall j = 1, 2, 3$. Hence, for the state $\rho_W^{(d)}$, $\forall d \geq 3$, the operator

$$R_{\blacktriangleleft\blacktriangleright}^{(d)} = \frac{1}{d^4} I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d} + \frac{6}{d^2(d-2)} Q_d^{(-)} \tag{13}$$

represents a density source-operator with the special dilation property (8), that is: $\mathrm{tr}_{\mathbb{C}^d}^{(j)}[R_{\blacktriangleleft\blacktriangleright}^{(d)}] = \rho_W^{(d)}$, $\forall j = 1, 2, 3$. If $d = 2$, then

$$R_{\blacktriangleright}^{(2)} = \frac{1}{4} I_{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2} - \frac{1}{8} V_2 \otimes I_{\mathbb{C}^2} - \frac{1}{8}(I_{\mathbb{C}^2} \otimes V_2)(V_2 \otimes I_{\mathbb{C}^2})(I_{\mathbb{C}^2} \otimes V_2) \tag{14}$$

is a density source-operator for $\rho_W^{(2)}$. The existence of the density source-operators (13) and (14) proves the statement. ■

Consider now examples of DSO and Bell class states on an infinite dimensional Hilbert space $\mathcal{H} \otimes \mathcal{H}$. Take the quantum states

$$\rho_1 = \frac{1}{4}|\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2\rangle\langle \psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2| + \frac{1}{4}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) \otimes |\psi_1\rangle\langle\psi_1| \tag{15}$$

and

$$\rho_2 = \frac{1}{6}|\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2| + \frac{1}{6}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) \otimes |\psi_1\rangle\langle\psi_1| + \frac{1}{6}|\psi_1\rangle\langle\psi_1| \otimes (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|), \tag{16}$$

where ψ_1, ψ_2 are any mutually orthogonal unit vectors in $\mathcal{H}$. The partial transpose $\rho_1^{T_1}$ has the negative eigenvalue $\lambda = \frac{1}{8}(1-\sqrt{5})$, corresponding to the eigenvector $\psi = c(\psi_1 \otimes \psi_2 + \frac{1-\sqrt{5}}{2}\psi_2 \otimes \psi_1)$. Therefore, due to the Peres separability criterion [6], the state ρ_1 is nonseparable. Nonseparability of ρ_2 is proved similarly. The operators

$$R_{\blacktriangleright} = \frac{1}{4}|\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2| \otimes |\psi_1\rangle\langle\psi_1| + \frac{1}{4}|\psi_1 \otimes \psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_1 \otimes \psi_2| \tag{17}$$

and

$$R_{\blacktriangleleft\blacktriangleright} = \frac{1}{6}|\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2| \otimes |\psi_1\rangle\langle\psi_1| + \frac{1}{6}|\psi_1 \otimes \psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_1 \otimes \psi_2| + \frac{1}{6}|\psi_1 \otimes \psi_1 \otimes \psi_1 + \psi_1 \otimes \psi_2 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_1 \otimes \psi_1 + \psi_1 \otimes \psi_2 \otimes \psi_2| \tag{18}$$

represent density source-operators for ρ_1 and ρ_2, respectively. Moreover, the DSO (18) has the special dilation property (8). Hence: (i) the state ρ_1 is a nonseparable DSO state; (ii) the state ρ_2 is a nonseparable Bell class state.

2.2. *Quantum Bell-form inequalities.* Based on the new notion of a source-operator introduced in section 2.1, consider now upper bounds of linear combinations (1) of quantum product averages in an arbitrary state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let T_{122} and T_{112} be any source-operators for a state ρ. According to proposition 1, for any bipartite state ρ, these operators exist. In view of definition 1, we have:

$$\begin{aligned} &\mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)} - W_1^{(a)} \otimes W_2^{(b_2)})] \\ &= \mathrm{tr}[T_{122}(W_1^{(a)} \otimes W_2^{(b_1)} \otimes I_{\mathcal{H}_2} - W_1^{(a)} \otimes I_{\mathcal{H}_2} \otimes W_2^{(b_2)})], \\ &\mathrm{tr}[\rho(W_1^{(a_1)} \otimes W_2^{(b)} - W_1^{(a_2)} \otimes W_2^{(b)})] \\ &= \mathrm{tr}[T_{112}(W_1^{(a_1)} \otimes I_{\mathcal{H}_1} \otimes W_2^{(b)} - I_{\mathcal{H}_1} \otimes W_1^{(a_2)} \otimes W_2^{(b)})], \end{aligned} \tag{19}$$

and these representations allow us to prove the following general statement.

PROPOSITION 3. *Let $W_1^{(a)}$ and $W_2^{(b)}$ be any bounded quantum observables on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, with operator norms $\|\cdot\| \leq 1$. An arbitrary state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfies the inequalities*

$$|\mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)})] - \mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_2)})]| \leq \|T_{122}\|_1\{1 - \mathrm{tr}[\sigma_{T_{122}}(W_2^{(b_1)} \otimes W_2^{(b_2)})]\} \tag{20}$$

and

$$|\mathrm{tr}[\rho(W_1^{(a_1)} \otimes W_2^{(b)})] - \mathrm{tr}[\rho(W_1^{(a_2)} \otimes W_2^{(b)})]| \leq \|T_{112}\|_1\{1 - \mathrm{tr}[\sigma_{T_{112}}(W_1^{(a_1)} \otimes W_1^{(a_2)})]\}, \tag{21}$$

where T_{122} and T_{112} are any source-operators for ρ and

$$\sigma_{T_{122}} := \frac{1}{\|T_{122}\|_1}\mathrm{tr}^{(1)}_{\mathcal{H}_1}[|T_{122}|], \qquad \sigma_{T_{112}} := \frac{1}{\|T_{112}\|_1}\mathrm{tr}^{(3)}_{\mathcal{H}_2}[|T_{112}|] \tag{22}$$

are density operators on $\mathcal{H}_2 \otimes \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_1$, respectively.

In the right-hand side of (20) (or (21)), the observables can be interchanged.

Proof. In order to prove (20), we notice that in (19):

$$\begin{aligned} W_1^{(a)} \otimes W_2^{(b_1)} \otimes I_{\mathcal{H}_2} &= (W_1^{(a)} \otimes I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_2})(I_{\mathcal{H}_1} \otimes W_2^{(b_1)} \otimes I_{\mathcal{H}_2}), \\ W_1^{(a)} \otimes I_{\mathcal{H}_2} \otimes W_2^{(b_2)} &= (W_1^{(a)} \otimes I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_2})(I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} \otimes W_2^{(b_2)}), \end{aligned} \tag{23}$$

and the bounded quantum observables

$$W_1^{(a)} \otimes I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_2}, \quad I_{\mathcal{H}_1} \otimes W_2^{(b_1)} \otimes I_{\mathcal{H}_2}, \quad I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} \otimes W_2^{(b_2)} \tag{24}$$

on $\mathcal{K}_{122}$ mutually commute. From the von Neumann theorem ([7], page 221) it follows that there exist:

(i) a bounded quantum observable $V_a^{(b_1,b_2)}$ on $\mathcal{K}_{122}$;

(ii) bounded Borel real-valued functions $\varphi_1^{(a)}, \varphi_2^{(b_1)}, \varphi_3^{(b_2)}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, with supremum norms $||\varphi_1^{(a)}||$, $||\varphi_2^{(b_1)}||$, $||\varphi_3^{(b_2)}|| \leq 1$;

such that

$$\begin{aligned} &W_1^{(a)} \otimes I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_2} = \varphi_1^{(a)}(V_a^{(b_1,b_2)}), \quad I_{\mathcal{H}_1} \otimes W_2^{(b_1)} \otimes I_{\mathcal{H}_2} = \varphi_2^{(b_1)}(V_a^{(b_1,b_2)}), \\ &I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2} \otimes W_2^{(b_2)} = \varphi_3^{(b_2)}(V_a^{(b_1,b_2)}). \end{aligned} \tag{25}$$

Let $P_{V_a^{(b_1,b_2)}}(\cdot)$, where $P_{V_a^{(b_1,b_2)}}(\mathbb{R}) = I_{\mathcal{K}_{122}}$, be the projection-valued measure corresponding uniquely to $V_a^{(b_1,b_2)}$ due to the spectral theorem. In view of (19) and (25),

$$\begin{aligned} \mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)})] &= \int_{\mathbb{R}} \varphi_1^{(a)}(\xi)\varphi_2^{(b_1)}(\xi)\nu_a^{(b_1,b_2)}(d\xi; T_{122}), \\ \mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_2)})] &= \int_{\mathbb{R}} \varphi_1^{(a)}(\xi)\varphi_3^{(b_2)}(\xi)\nu_a^{(b_1,b_2)}(d\xi; T_{122}), \end{aligned} \tag{26}$$

where we denote by $\nu_a^{(b_1,b_2)}(\cdot; Y)$ a σ-additive bounded real-valued measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, defined by the relation

$$\nu_a^{(b_1,b_2)}(\cdot; Y) := \mathrm{tr}[Y P_{V_a^{(b_1,b_2)}}(\cdot)], \quad \nu_a^{(b_1,b_2)}(\mathbb{R}; Y) = \mathrm{tr}[Y], \tag{27}$$

for any self-adjoint trace class operator Y on $\mathcal{K}_{122}$. For a source-operator T_{122}, the measure $\nu_a^{(b_1,b_2)}(\cdot; T_{122})$ is normalized but not, in general, positive. Due to property 2, section 2.1,

$$\nu_a^{(b_1,b_2)}(\cdot; T_{122}) = \nu_a^{(b_1,b_2)}(\cdot; T_{122}^{(+)}) - \nu_a^{(b_1,b_2)}(\cdot; T_{122}^{(-)}), \tag{28}$$

where $\nu_a^{(b_1,b_2)}(\cdot; T_{122}^{(\pm)})$ are unnormalized positive measures with

$$\nu_a^{(b_1,b_2)}(\mathbb{R}; T_{122}^{(+)}) + \nu_a^{(b_1,b_2)}(\mathbb{R}; T_{122}^{(-)}) = \nu_a^{(b_1,b_2)}(\mathbb{R}; |T_{122}|) = \|T_{122}\|_1. \tag{29}$$

Using (26), (28), the bound $||\varphi_1^{(a)}|| \leq 1$, and the inequality $|x-y| \leq 1-xy$, valid for any real numbers $|x| \leq 1$, $|y| \leq 1$, we derive:

$$\begin{aligned}(30)\qquad &|\mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)} - W_1^{(a)} \otimes W_2^{(b_2)})]| \\ &\qquad \leq \nu_a^{(b_1,b_2)}(\mathbb{R}; |T_{122}|) - \int_{\mathbb{R}} \varphi_2^{(b_1)}(\xi)\varphi_3^{(b_2)}(\xi)\nu_a^{(b_1,b_2)}(d\xi; |T_{122}|).\end{aligned}$$

Due to (25) and (27),

$$(31)\qquad \int_{\mathbb{R}} \varphi_2^{(b_1)}(\xi)\varphi_3^{(b_2)}(\xi)\nu_a^{(b_1,b_2)}(d\xi; |T_{122}|) = \|T_{122}\|_1 \mathrm{tr}[\sigma_{T_{122}}(W_2^{(b_1)} \otimes W_2^{(b_2)})],$$

where $\sigma_{T_{122}} := \frac{\mathrm{tr}^{(1)}_{\mathcal{H}_1}[|T_{122}|]}{||T_{122}||_1}$ is a density operator on $\mathcal{H}_2 \otimes \mathcal{H}_2$. Substituting (29) and (31) into (30), we finally have:

$$(32)\qquad |\mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b_1)} - W_1^{(a)} \otimes W_2^{(b_2)})]| \leq \|T_{122}\|_1 \left\{1 - \mathrm{tr}[\sigma_{T_{122}}(W_2^{(b_1)} \otimes W_2^{(b_2)})]\right\}.$$

The derivation of the inequality (21) is quite similar. ■

COROLLARY 1. *Let $W_1^{(a)}$ and $W_2^{(b)}$ be any bounded quantum observables with operator norms $|| \cdot || \leq 1$. For any state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the inequalities*

$$(33)\qquad \begin{aligned}|\mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b)})]| &\leq \frac{1}{2}\|T_{122}\|_1\{1 + \mathrm{tr}[\sigma_{T_{122}}(W_2^{(b)} \otimes W_2^{(b)})]\},\\ |\mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b)})]| &\leq \frac{1}{2}\|T_{112}\|_1\{1 + \mathrm{tr}[\sigma_{T_{112}}(W_1^{(a)} \otimes W_1^{(a)})]\}\end{aligned}$$

hold with arbitrary source-operators T_{122} and T_{112} for ρ in the right hand sides.

In particular, for a Bell class state ρ on $\mathcal{H} \otimes \mathcal{H}$, the relations (33) imply:

$$(34)\qquad \begin{aligned}|\mathrm{tr}[\rho(W_1 \otimes W_2)]| &\leq \frac{1}{2}\{1 + \mathrm{tr}[\rho(W_2 \otimes W_2)]\},\\ |\mathrm{tr}[\rho(W_1 \otimes W_2)]| &\leq \frac{1}{2}\{1 + \mathrm{tr}[\rho(W_1 \otimes W_1)]\},\end{aligned}$$

for any W_1 and W_2 on $\mathcal{H}$.

2.3. *Quantum CHSH-form inequalities.* Consider now upper bounds for a linear combination (2).

PROPOSITION 4. *Let $W_1^{(a_n)}$ and $W_2^{(b_m)}$, $n, m = 1, 2$, be any bounded quantum observables with operator norms $||\cdot|| \leq 1$ and γ_{nm}, $n, m = 1, 2$, be any real coefficients with $|\gamma_{nm}| \leq 1$.*

An arbitrary quantum state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfies the inequality

$$(35)\qquad \Big|\sum_{n,m=1,2} \gamma_{nm}\mathrm{tr}[\rho(W_1^{(a_n)} \otimes W_2^{(b_m)})]\Big| \leq 2\|T_{122}\|_1,$$

whenever $\gamma_{11}\gamma_{12} = -\gamma_{21}\gamma_{22}$, and the inequality

$$(36)\qquad \Big|\sum_{n,m=1,2} \gamma_{nm}\mathrm{tr}[\rho(W_1^{(a_n)} \otimes W_2^{(b_m)})]\Big| \leq 2\|T_{112}\|_1,$$

whenever $\gamma_{11}\gamma_{21} = -\gamma_{12}\gamma_{22}$. Here, T_{122} and T_{112} are any source-operators for a state ρ.

Proof. Due to the upper bounds (20) and (21), we have:

$$\Big|\sum_{n,m=1,2}\gamma_{nm}\mathrm{tr}[\rho(W_1^{(a_n)}\otimes W_2^{(b_m)})]\Big| \leq \|T_{122}\|_1\{2+(\gamma_{11}\gamma_{12}+\gamma_{21}\gamma_{22})\mathrm{tr}[\sigma_{T_{122}}(W_2^{(b_1)}\otimes W_2^{(b_2)})]\}, \tag{37}$$

$$\Big|\sum_{n,m=1,2}\gamma_{nm}\mathrm{tr}[\rho(W_1^{(a_n)}\otimes W_2^{(b_m)})]\Big| \leq \|T_{112}\|_1\{2+(\gamma_{11}\gamma_{21}+\gamma_{12}\gamma_{22})\mathrm{tr}[\sigma_{T_{112}}(W_1^{(a_1)}\otimes W_1^{(a_2)})]\}, \tag{38}$$

and these relations prove the statement. ■

3. Validity of classical Bell-type inequalities in the quantum case. Propositions 3 and 4 clearly indicate the cases where a bipartite quantum state satisfies a classical CHSH-form inequality and the original Bell inequality for *any* bounded quantum observables. Notice that, in our setting, bounded quantum observables may be of any spectral types.

THEOREM 1 (DSO states and the CHSH inequality). *A density source-operator* (10) *(DSO) state ρ on $\mathcal{H}_1\otimes\mathcal{H}_2$ satisfies the original CHSH inequality [2]:*

$$|\mathrm{tr}[\rho(W_1^{(a_1)}\otimes W_2^{(b_1)}+W_1^{(a_1)}\otimes W_2^{(b_2)}+W_1^{(a_2)}\otimes W_2^{(b_1)}-W_1^{(a_2)}\otimes W_2^{(b_2)})]|\leq 2, \tag{39}$$

for any bounded quantum observables $W_1^{(a_n)}, W_2^{(b_m)}$, $n,m=1,2$, with operator norms $||\cdot||\leq 1$.

If a DSO state on $\mathcal{H}\otimes\mathcal{H}$ is symmetric then, for this state, density source-operators $R_{\blacktriangleright}$ and $R_{\blacktriangleleft}$ exist simultaneously, and from proposition 4 there follows:

THEOREM 2. *Let γ_{nm}, $n,m=1,2$, be any real coefficients with $|\gamma_{nm}|\leq 1$ such that $\gamma_{11}\gamma_{12}=-\gamma_{21}\gamma_{22}$ or $\gamma_{11}\gamma_{21}=-\gamma_{12}\gamma_{22}$.*

A symmetric DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ satisfies the extended CHSH inequality [3]:

$$\Big|\sum_{n,m=1,2}\gamma_{nm}\mathrm{tr}[\rho(W_1^{(a_n)}\otimes W_2^{(b_m)})]\Big|\leq 2, \tag{40}$$

for any bounded quantum observables $W_1^{(a_n)}$, $W_2^{(b_m)}$, $n,m=1,2$, with operator norms $||\cdot||\leq 1$.

Due to proposition 3, we have the following general statement on Bell class states (11).

THEOREM 3 (Bell class states and the Bell inequality). *A Bell class state ρ on $\mathcal{H}\otimes\mathcal{H}$ satisfies the perfect correlation form of the original Bell inequality [1]:*

(10) See definition 2, section 2.1.
(11) See definition 3, section 2.1.

$$
\begin{aligned}
&|\mathrm{tr}[\rho(W_1 \otimes W_2)] - \mathrm{tr}[\rho(W_1 \otimes \widetilde{W}_2)]| \\
&\qquad \leq 1 - \mathrm{tr}[\rho(W_2 \otimes \widetilde{W}_2)], \\
&|\mathrm{tr}[\rho(W_1 \otimes W_2)] - \mathrm{tr}[\rho(\widetilde{W}_1 \otimes W_2)]| \\
&\qquad \leq 1 - \mathrm{tr}[\rho(W_1 \otimes \widetilde{W}_1)],
\end{aligned} \tag{41}
$$

for any bounded quantum observables $W_1, \widetilde{W}_1, W_2, \widetilde{W}_2$ *on* $\mathcal{H}$ *with operator norms* $\|\cdot\| \leq 1$.

Corollary 2. *Any Bell class state* ρ *on* $\mathcal{H} \otimes \mathcal{H}$ *satisfies the extended CHSH inequality* (40).

In the right-hand sides of the inequalities (41), the quantum observables can be interchanged.

It is necessary to underline that, in the physical literature, the validity of the perfect correlation form of the original Bell inequality for a bipartite state on $\mathcal{H} \otimes \mathcal{H}$ has been always linked with Bell's assumption of perfect correlations if the same quantum observable is measured on both sides (cf. in [1]).

In [3] (section 3.B.1, item 1) we proved that separable states of the special form ([12]) (49) in [3] satisfy (41) for any bounded quantum observables and do not necessarily exhibit perfect correlations. Theorem 3 generalizes this result of [3] and indicates that there exists a whole class of bipartite states, separable and nonseparable, where each state satisfies the perfect correlation form of the original Bell inequality for any three bounded quantum observables and does not necessarily exhibit perfect correlations.

In case of, for example, a dichotomic observable W_2, with eigenvalues ± 1, the latter means that a Bell class state ρ satisfies (41) even if the correlation function $\mathrm{tr}[\rho(W_2 \otimes W_2)] \neq 1$.

Due to theorem 3 and proposition 2, *the nonseparable Werner state* (9) *on* $\mathbb{C}^d \otimes \mathbb{C}^d$, $\forall d \geq 3$, *satisfies the perfect correlation form of the original Bell inequality for any bounded quantum observables and does not necessarily exhibit perfect correlations.*

The upper bounds in proposition 3 allow us to introduce also a condition sufficient for the validity of the original Bell inequality for a bipartite state and some three quantum observables.

Theorem 4 (General sufficient condition). *If, for a DSO state* ρ *on* $\mathcal{H} \otimes \mathcal{H}$*, there exists a density source-operator* $R_\blacktriangleright$ *such that*

$$
\mathrm{tr}[\sigma_{R_\blacktriangleright}(W_2 \otimes \widetilde{W}_2)] = \pm \mathrm{tr}[\rho(W_2 \otimes \widetilde{W}_2)], \quad \sigma_{R_\blacktriangleright} = \mathrm{tr}_{\mathcal{H}}^{(1)}[R_\blacktriangleright], \tag{42}
$$

for bounded quantum observables W_2, $\widetilde{W}_2$ *on* $\mathcal{H}$ *with operator norms* $\|\cdot\| \leq 1$ *then this DSO state* ρ *and these quantum observables* W_2, $\widetilde{W}_2$ *satisfy the original Bell inequality [1]:*

$$
\left| \mathrm{tr}[\rho(W_1 \otimes W_2)] - \mathrm{tr}[\rho(W_1 \otimes \widetilde{W}_2)] \right| \leq 1 \mp \mathrm{tr}[\rho(W_2 \otimes \widetilde{W}_2)], \tag{43}
$$

in its perfect correlation or anticorrelation form (minus or plus sign in (43)*, respectively). Here,* W_1 *is any bounded quantum observable on* $\mathcal{H}$ *with* $\|W_1\| \leq 1$.

([12]) As we discussed in section 2.1, these separable states belong to the Bell class.

Notice that, in theorem 3, the sufficient condition concerns only a bipartite state property and refers only to the perfect correlation form of the Bell inequality. A Bell class state satisfies the plus sign condition (42) for any quantum observables W_2, $\widetilde{W}_2$.

In theorem 4, the sufficient condition (42) establishes the restriction on the combination: *quantum observables and a DSO state*, and concerns both forms of the original Bell inequality. In general, a DSO state satisfying the condition (42) does not necessarily either belong to the Bell class or satisfy (42) for any W_2, $\widetilde{W}_2$.

For a symmetric DSO state, let us now prove that the sufficient condition (42) is more general than Bell's perfect correlation/anticorrelation restriction (44) and includes the latter only as a particular case.

PROPOSITION 5. *If a symmetric DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ satisfies the Bell perfect correlation or anticorrelation restriction*

$$\mathrm{tr}[\rho(W_2\otimes W_2)] = \pm 1 \tag{44}$$

then this DSO state satisfies the sufficient condition (42). The converse is not true.

Proof. If a DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ is symmetric then it has both density source-operators, $R_{\blacktriangleleft}$ and $R_{\blacktriangleright}$. We have:

$$\begin{aligned} \mathrm{tr}[\sigma_{R_{\blacktriangleright}}(W_2\otimes\widetilde{W}_2)] &= \mathrm{tr}[R_{\blacktriangleright}(I_{\mathcal{H}}\otimes W_2\otimes\widetilde{W}_2)],\\ \mathrm{tr}[\rho(W_2\otimes\widetilde{W}_2)] &= \mathrm{tr}[R_{\blacktriangleright}(W_2\otimes I_{\mathcal{H}}\otimes\widetilde{W}_2)],\\ \mathrm{tr}[\rho(W_2\otimes W_2)] &= \mathrm{tr}[R_{\blacktriangleright}(W_2\otimes W_2\otimes I_{\mathcal{H}})]. \end{aligned} \tag{45}$$

Using the arguments based on the von Neumann theorem [7] and quite similar to those in proposition 3, we derive:

$$\begin{aligned} \mathrm{tr}[\sigma_{R_{\blacktriangleright}}(W_2\otimes\widetilde{W}_2)] &= \int_{\mathbb{R}}\varphi_2(\xi)\varphi_3(\xi)\nu(d\xi;R_{\blacktriangleright}),\\ \mathrm{tr}[\rho(W_2\otimes\widetilde{W}_2)] &= \int_{\mathbb{R}}\varphi_1(\xi)\varphi_3(\xi)\nu(d\xi;R_{\blacktriangleright}),\\ \mathrm{tr}[\rho(W_2\otimes W_2)] &= \int_{\mathbb{R}}\varphi_1(\xi)\varphi_2(\xi)\nu(d\xi;R_{\blacktriangleright}), \end{aligned} \tag{46}$$

where:

(i) $\nu(\cdot;R_{\blacktriangleright}) := \mathrm{tr}[R_{\blacktriangleright}P_V(\cdot)]$ is a probability distribution on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$, induced ([13]) by the projection-valued measure P_V of a quantum observable V on $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$ (corresponding, due to the von Neumann theorem, to three mutually commuting observables $W_2\otimes I_{\mathcal{H}}\otimes I_{\mathcal{H}}$, $I_{\mathcal{H}}\otimes W_2\otimes I_{\mathcal{H}}$, $I_{\mathcal{H}}\otimes I_{\mathcal{H}}\otimes\widetilde{W}_2$);

(ii) φ_1, φ_2, φ_3 are bounded Borel real-valued functions on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$, with supremum norms $||\varphi_n|| \leq 1$, such that $\varphi_1(V) = W_2\otimes I_{\mathcal{H}}\otimes I_{\mathcal{H}}$, $\varphi_2(V) = I_{\mathcal{H}}\otimes W_2\otimes I_{\mathcal{H}}$ and $\varphi_3(V) = I_{\mathcal{H}}\otimes I_{\mathcal{H}}\otimes\widetilde{W}_2$.

If ρ satisfies the Bell restriction (44) then, due to (46),

$$\int_{\mathbb{R}}\varphi_1(\xi)\varphi_2(\xi)\nu(d\xi;R_{\blacktriangleright}) = \pm 1. \tag{47}$$

([13]) See proposition 3.

The latter implies $\varphi_1(\xi)\varphi_2(\xi) = \pm 1$, ν-a.e. Since $||\varphi_1||, ||\varphi_2|| \leq 1$, we have $\varphi_1(\xi) = \pm\varphi_2(\xi)$, ν-a.e., and, hence,

$$(48)\quad \mathrm{tr}[\sigma_{R_\blacktriangleright}(W_2 \otimes \widetilde{W}_2)] \mp \mathrm{tr}[\rho(W_2 \otimes \widetilde{W}_2)] = \int_{\mathbb{R}} \{\varphi_2(\xi) \mp \varphi_1(\xi)\}\varphi_3(\xi)\nu(d\xi; R_\blacktriangleright) = 0.$$

The converse statement is not true and a DSO state, satisfying (48), does not necessarily satisfy (47). ∎

Thus, a DSO state, satisfying the general sufficient condition (42) *and, therefore, the original Bell inequality* (43), *does not necessarily exhibit Bell's perfect correlations or anticorrelations.*

4. Generalized quantum measurements of Alice and Bob. In the physical literature, joint measurements on a bipartite system are usually referred to as measurements of Alice and Bob. Theorems 1–4 and proposition 5 specify the relations between the product expectation values under projective quantum measurements of Alice and Bob.

To analyze the situation under *generalized* joint quantum measurements on a bipartite quantum state, let us recall that an Alice/Bob joint generalized quantum measurement, with real-valued outcomes $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$ of any type, is described by the positive operator-valued (POV) measure

$$(49)\qquad M^{(a,b)}(B_1 \times B_2) = M_1^{(a)}(B_1) \otimes M_2^{(b)}(B_2), \quad \forall B_1 \subseteq \Lambda_1,\ \forall B_2 \subseteq \Lambda_2,$$

where "a" and Λ_1 refer to a setting and an outcome set on the side of Alice while "b" and Λ_2 on the side of Bob. For simplicity, we further suppose $|\lambda_1| \leq 1$, $|\lambda_2| \leq 1$.

For a quantum state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, the formula ([14])

$$(50)\qquad \begin{aligned} \langle\lambda_1\lambda_2\rangle_\rho^{(a,b)} &:= \int_{\Lambda_1\times\Lambda_2} \lambda_1\lambda_2 \mathrm{tr}[\rho(M_1^{(a)}(d\lambda_1) \otimes M_2^{(b)}(d\lambda_2))] \\ &= \mathrm{tr}[\rho(W_1^{(a)} \otimes W_2^{(b)})] \end{aligned}$$

represents the expectation value of the product $\lambda_1\lambda_2$ of outcomes observed by Alice and Bob. Here,

$$(51)\qquad W_1^{(a)} := \int_{\Lambda_1} \lambda_1 M_1^{(a)}(d\lambda_1), \quad W_2^{(b)} := \int_{\Lambda_2} \lambda_2 M_2^{(b)}(d\lambda_2)$$

are bounded quantum observables with operator norms $\|\cdot\| \leq 1$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Theorems 1–3 and the representation (50) imply:

THEOREM 5. *The product expectation values in a DSO state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfy the original CHSH inequality:*

$$(52)\qquad |\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} - \langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}| \leq 2,$$

under any generalized quantum measurements (49) of Alice and Bob with outcomes $|\lambda_1| \leq 1$, $|\lambda_2| \leq 1$ of any type.

([14]) See also [3], section 3.A.

THEOREM 6. *Let γ_{nm}, $n, m = 1, 2$, be any real coefficients with $|\gamma_{nm}| \leq 1$ and $\gamma_{11}\gamma_{12} = -\gamma_{21}\gamma_{22}$ or $\gamma_{11}\gamma_{21} = -\gamma_{12}\gamma_{22}$. If a DSO state ρ on $\mathcal{H} \otimes \mathcal{H}$ is either symmetric or of the Bell class then the product expectation values in this ρ satisfy the extended CHSH inequality:*

$$\Big| \sum_{n,m=1,2} \gamma_{nm} \langle \lambda_1 \lambda_2 \rangle_\rho^{(a_n, b_m)} \Big| \leq 2, \tag{53}$$

under any generalized quantum measurements (49) of Alice and Bob with outcomes $|\lambda_1| \leq 1$, $|\lambda_2| \leq 1$ of any type.

THEOREM 7. *If, under joint generalized quantum measurements (49) of Alice and Bob,*

$$\int_{\Lambda_1} \lambda_1 M_1^{(b_1)}(d\lambda_1) = \int_{\Lambda_2} \lambda_2 M_2^{(b_1)}(d\lambda_2) \tag{54}$$

then, for any Bell class state ρ on $\mathcal{H} \otimes \mathcal{H}$, the product expectation values satisfy the perfect correlation form of the original Bell inequality:

$$|\langle \lambda_1 \lambda_2 \rangle_\rho^{(a,b_1)} - \langle \lambda_1 \lambda_2 \rangle_\rho^{(a,b_2)}| \leq 1 - \langle \lambda_1 \lambda_2 \rangle_\rho^{(b_1,b_2)}. \tag{55}$$

The operator relation (54) does not necessarily imply the perfect correlations of outcomes on the sides of Alice and Bob and is always true in case of projective Alice and Bob measurements of the same quantum observable on both sides.

Theorem 4 can be also easily generalized to the case of joint generalized quantum measurements of Alice and Bob.

Acknowledgments. I am grateful to Marek Bożejko, Klaus Molmer and Asher Peres for valuable discussions.

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