

DECOMPOSABILITY OF EXTREMAL POSITIVE UNITAL MAPS ON $M_2(\mathbb{C})$

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Abstract. A map $\varphi : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is decomposable if it is of the form $\varphi = \varphi_1 + \varphi_2$ where φ_1 is a CP map while φ_2 is a co-CP map. It is known that if $m = n = 2$ then every positive map is decomposable. Given an extremal unital positive map $\varphi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ we construct concrete maps (not necessarily unital) φ_1 and φ_2 which give a decomposition of φ . We also show that in most cases this decomposition is unique.

1. Introduction. If \mathcal{A} is a C^* -algebra and $n \in \mathbb{N}$ then by $M_n(\mathcal{A})$ we denote the C^* -algebra of square $n \times n$ -matrices with coefficients in \mathcal{A} . In particular $M_n(\mathbb{C})$ is the algebra of matrices with complex entries. For each $m, n \in \mathbb{N}$ we have the following isomorphisms:

$$(1.1) \quad M_m(M_n(\mathbb{C})) \cong M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \cong M_{mn}(\mathbb{C}).$$

It follows that $M_m(M_n(\mathbb{C}))$ has the natural structure of a C^* -algebra. In particular one defines the conjugation of $\mathbf{A} = [A_{ij}]_{i,j=1}^m \in M_m(M_n(\mathbb{C}))$ (where $A_{ij} \in M_n(\mathbb{C})$ for $i, j = 1, \dots, m$) by the formula $\mathbf{A}^* = [A_{ji}^*]_{i,j=1}^m$. Recall (see for example [6, 2]) that \mathbf{A} is *positive* in $M_m(M_n(\mathbb{C}))$ if and only if $\sum_{i,j=1}^m \overline{\mu_i} \mu_j \langle v_i, A_{ij} v_j \rangle \geq 0$ (i.e. the matrix $[\langle v_i, A_{ij} v_j \rangle]_{i,j=1}^m$ is a positive element of $M_m(\mathbb{C})$) for every $v_1, \dots, v_m \in \mathbb{C}^n$ and $\mu_1, \dots, \mu_m \in \mathbb{C}$. We say that \mathbf{A} is *block-positive* if $\sum_{i,j=1}^m \overline{\mu_i} \mu_j \langle v, A_{ij} v \rangle \geq 0$ (i.e. the matrix $[\langle v, A_{ij} v \rangle]_{i,j=1}^m$ is positive in $M_m(\mathbb{C})$) for every $v \in \mathbb{C}^n$ and $\mu_1, \dots, \mu_m \in \mathbb{C}$. For every $\mathbf{A} = [A_{ij}]_{i,j=1}^m \in M_m(M_n(\mathbb{C}))$ we define the *partial transposition* of \mathbf{A} by $\mathbf{A}^\tau = [A_{ji}]_{i,j=1}^m$. Note the difference between this operation and the usual transposition $\mathbf{A} \mapsto \mathbf{A}^T$ on the

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algebra $M_{mn}(\mathbb{C})$ (cf. (1.1)): the usual transposition preserves positivity of \mathbf{A} while, for $n, m \geq 2$, partial transposition does not!

A linear map $\varphi : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is called a *positive map* if $\varphi(A)$ is a positive matrix for every positive matrix $A \in M_m(\mathbb{C})$. If $k \in \mathbb{N}$ then φ is called *k-positive map* (respectively *k-copositive map*) whenever $[\varphi(A_{ij})]_{i,j=1}^k$ (respectively $[\varphi(A_{ji})]_{i,j=1}^k$) is a positive element in the algebra $M_k(M_n(\mathbb{C}))$ for every positive element $[A_{ij}]_{i,j=1}^k$ from $M_k(M_m(\mathbb{C}))$. If φ is *k-positive* (respectively *k-copositive*) for every $k \in \mathbb{N}$ then φ is called *completely positive* or CP (respectively *completely copositive* or co-CP). A positive map which is a sum of completely positive and completely copositive maps is called *decomposable*¹.

Let $\{E_{ij}\}_{i,j=1}^m$ be a system of matrix units in $M_m(\mathbb{C})$ and $\mathbf{H}_\varphi = [\varphi(E_{ij})]_{i,j=1}^m \in M_m(M_n(\mathbb{C}))$ be *Choi matrix* of φ with respect to the system $\{E_{ij}\}$ ([1], see also [4]). Recall the following

THEOREM 1.1 ([1], see also [4]). *Let $\varphi : M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map. Then*

1. *the map φ is positive if and only if the matrix \mathbf{H}_φ is block-positive;*
2. *the map φ is completely positive (respectively completely copositive) if and only if \mathbf{H}_φ (respectively \mathbf{H}_φ^t) is a positive element of $M_m(M_n(\mathbb{C}))$.*

We say that a positive map φ is *unital* if $\varphi(\mathbb{I}) = \mathbb{I}$ where \mathbb{I} denotes the identity matrix of the respective algebra. The set of all positive (respectively completely positive, completely copositive, decomposable) maps from $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ will be denoted by $\mathcal{P}(m, n)$ (respectively $\mathcal{CP}(m, n)$, $\mathcal{CcP}(m, n)$, $\mathcal{D}(m, n)$). We will write simply \mathcal{P} , \mathcal{CP} , \mathcal{CcP} and \mathcal{D} instead of $\mathcal{P}(m, n)$, $\mathcal{CP}(m, n)$, $\mathcal{CcP}(m, n)$ and $\mathcal{D}(m, n)$ when no confusion can arise. Observe that all of these sets have the structure of a convex cone. By \mathcal{P}_1 , \mathcal{CP}_1 , \mathcal{CcP}_1 and \mathcal{D}_1 we will denote the subsets of unital maps from respective cones. All of them are convex subsets.

Let C be a convex cone and $c \in C$. We say that c is an *extreme point* of C if for every $c_1, c_2 \in C$ the equality $c = c_1 + c_2$ implies $c_1 = \lambda c$ and $c_2 = (1 - \lambda)c$ for some $0 \leq \lambda \leq 1$. The generalization of the notion of extremality leads to the face structure of the cone C . Namely, we say that a convex subcone $F \subset C$ is a *face* of C if for every $c_1, c_2 \in C$ the condition $c_1 + c_2 \in F$ implies $c_1, c_2 \in F$. Kye in [3] gave an interesting characterization of maximal faces of the cone $\mathcal{P}(m, n)$

THEOREM 1.2 ([3]). *A convex subset $F \subset \mathcal{P}(m, n)$ is a maximal face of $\mathcal{P}(m, n)$ if and only if there are vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$ such that $F = F_{\xi, \eta}$ where*

$$(1.2) \quad F_{\xi, \eta} = \{\varphi \in \mathcal{P}(m, n) : \varphi(P_\xi)\eta = 0\}$$

and P_ξ denotes the one-dimensional orthogonal projection in $M_m(\mathbb{C})$ onto the subspace generated by the vector ξ .

¹In this paper we follow the definition of decomposability given by Størmer in [5]. Note that there is no connection of this notion with decomposable maps considered by U. Haagerup in the theory of operator spaces (see [2]).

2. The case $m = n = 2$. In this section we analyze in detail the case $m = n = 2$. In [5] the following characterization of extremal points of \mathcal{P}_1 is given:

THEOREM 2.1. *A positive unital map $\varphi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is an extremal point of \mathcal{P}_1 if and only if there are unitary operators $V, W \in U(2)$ such that the Choi matrix of the map $\varphi_{V,W} : A \mapsto V^*\varphi(WAW^*)V$ has the form*

$$(2.1) \quad \left[\begin{array}{cc|cc} 1 & 0 & 0 & y \\ 0 & b & \bar{z} & t \\ \hline 0 & z & 0 & 0 \\ \bar{y} & \bar{t} & 0 & u \end{array} \right]$$

where the coefficients fulfil the following relations:

1. $b \geq 0, u \geq 0$ and $b + u = 1$,
2. $|t|^2 = 2b(u - |y|^2 - |z|^2)$ in the case when $b \neq 0$, and $|y| = 1$ or $|z| = 1$ when $b = 0$.

By $\{e_1, e_2\}$ we denote the canonical basis in \mathbb{C}^2 . Let φ be an extremal positive unital map as in Theorem 2.1. One can observe that $\varphi \in F_{\xi,\eta}$ for $\xi = We_2$ and $\eta = Ve_1$ where V, W are the unitary operators from Theorem 2.1. Suppose that $\varphi = \varphi_1 + \varphi_2$ where φ_1 is a completely positive map while φ_2 is a completely copositive one. Then both φ_1 and φ_2 should be elements of $F_{\xi,\eta}$ because $F_{\xi,\eta}$ is a face (cf. Theorem 1.2).

REMARK 2.2. There are extremal maps of the form (2.1) which are neither completely positive nor completely copositive (see Example 2.6 below). On the other hand theorem of Woronowicz (see [7]) asserts that every map from $\mathcal{P}(2, 2)$ is decomposable. Hence, the maps φ_1 and φ_2 giving the decomposition of an extremal element of $\mathcal{P}_1(2, 2)$ do not need be scalar multiples of unital maps.

In the sequel we will use the following lemmas.

LEMMA 2.3. *Let $\psi \in F_{\xi,\eta}$ for some $\xi, \eta \in \mathbb{C}^2$, and V and W be unitary operators from $M_2(\mathbb{C})$ such that $\xi = We_2$ and $\eta = Ve_1$. Then the Choi matrix of the map $\psi_{V,W}$ has the form*

$$(2.2) \quad \left[\begin{array}{cc|cc} a & c & 0 & y \\ \bar{c} & b & \bar{z} & t \\ \hline 0 & z & 0 & 0 \\ \bar{y} & \bar{t} & 0 & u \end{array} \right]$$

for some $a, b, u \geq 0$ and $c, y, z, t \in \mathbb{C}$. Moreover, the following conditions hold:

1. $|c|^2 \leq ab$,
2. $|t|^2 \leq bu$,
3. $(|y| + |z|)^2 \leq au$.

Proof. Let us write briefly ψ' instead of $\psi_{V,W}$. The Choi matrix of the map ψ' has the form

$$\mathbf{H} = \begin{bmatrix} \psi'(E_{11}) & \psi'(E_{12}) \\ \psi'(E_{21}) & \psi'(E_{22}) \end{bmatrix}.$$

From the condition (1.2) we get

$$\psi'(E_{22})e_1 = V^*\psi(WE_{22}W^*)Ve_1 = V^*\psi(P_\xi)\eta = 0.$$

Because $\psi'(E_{22})$ is a positive element of $M_2(\mathbb{C})$ we have $\psi'(E_{22}) = uE_{22}$ for some $u \geq 0$.

Hence $\psi'(E_{22}) = \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix}$. Positivity of ψ' implies also that $\psi'(E_{11})$ is a hermitian matrix

of the general form $\psi'(E_{11}) = \begin{bmatrix} a & c \\ \bar{c} & b \end{bmatrix}$ with $a, b \geq 0$ and $\det \psi'(E_{11}) = ab - |c|^2 \geq 0$, so

we have (1). Let $\psi'(E_{12}) = \begin{bmatrix} x & y \\ \bar{z} & t \end{bmatrix}$ for some $x, y, z, t \in \mathbb{C}$. Because ψ' is positive we have

$\psi'(E_{21}) = \psi'(E_{12}^*) = \psi'(E_{12})^* = \begin{bmatrix} \bar{x} & z \\ \bar{y} & \bar{t} \end{bmatrix}$. By Theorem 1.1 \mathbf{H} is block-positive, hence the matrix

$$\begin{bmatrix} \langle e_1, \psi'(E_{11})e_1 \rangle & \langle e_1, \psi'(E_{12})e_1 \rangle \\ \langle e_1, \psi'(E_{21})e_1 \rangle & \langle e_1, \psi'(E_{22})e_1 \rangle \end{bmatrix} = \begin{bmatrix} a & x \\ \bar{x} & 0 \end{bmatrix}$$

is positive and consequently $x = 0$. Thus we arrived at the form (2.2). Another application of block-positivity of H leads to the conclusion that the matrix

$$\begin{bmatrix} \langle e_2, \psi'(E_{11})e_2 \rangle & \langle e_2, \psi'(E_{12})e_2 \rangle \\ \langle e_2, \psi'(E_{21})e_2 \rangle & \langle e_2, \psi'(E_{22})e_2 \rangle \end{bmatrix} = \begin{bmatrix} b & t \\ \bar{t} & u \end{bmatrix}$$

is positive, hence we get inequality (2).

Let ω be a linear functional on $M_2(\mathbb{C})$. By Corollary 8.4 in [5] ω is a positive functional if and only if $\omega(E_{11}) \geq 0$, $\omega(E_{22}) \geq 0$, $\omega(E_{21}) = \overline{\omega(E_{12})}$ and $|\omega(E_{12})|^2 \leq \omega(E_{11})\omega(E_{22})$. Let us denote $\alpha = \omega(E_{11})$, $\beta = \omega(E_{22})$ and $\gamma = \omega(E_{12})$. Obviously $\omega \circ \psi'$ is a positive functional for every positive ω . From another application of this result of Størmer we get that the inequality

$$(2.3) \quad |\gamma y + \overline{\gamma z} + \beta t|^2 \leq \beta u(\alpha a + \beta b + 2\Re(\gamma c))$$

holds whenever $\alpha \geq 0$, $\beta \geq 0$ and $|\gamma|^2 \leq \alpha\beta$. Inequality (2.3) can be written in the form

$$(2.4) \quad |\gamma y + \overline{\gamma z}|^2 + \beta^2|t|^2 + 2\Re[(\gamma y + \overline{\gamma z})\beta \bar{t}] \leq \beta u(\alpha a + \beta b + 2\Re(\gamma c)).$$

Putting here $-\gamma$ instead of γ we get

$$(2.5) \quad |\gamma y + \overline{\gamma z}|^2 + \beta^2|t|^2 - 2\Re[(\gamma y + \overline{\gamma z})\beta \bar{t}] \leq \beta u(\alpha a + \beta b - 2\Re(\gamma c)).$$

If we add (2.4) and (2.5) and divide the result by 2, then we get

$$(2.6) \quad |\gamma y + \overline{\gamma z}|^2 + \beta^2|t|^2 \leq \beta u(\alpha a + \beta b).$$

This can be rewritten equivalently as

$$(2.7) \quad |\gamma|^2(|y|^2 + |z|^2) + 2\Re(yz\gamma^2) + \beta^2|t|^2 \leq \beta u(\alpha a + \beta b)$$

Let $\varepsilon > 0$ and take $\alpha = \varepsilon^{-1}$, $\beta = \varepsilon$ and γ such that $|\gamma| = 1$ and $yz\gamma^2 = |y||z|$. Then (2.7) has the form

$$(2.8) \quad (|y| + |z|)^2 \leq au + \varepsilon^2(bu - |t|^2)$$

Because ε can be chosen arbitrarily small, we get the inequality (3) and the proof is finished. ■

LEMMA 2.4. A map $\psi \in F_{\xi,\eta}$ is completely positive if and only if the coefficients of the matrix \mathbf{H} from (2.2) fulfil the following conditions:

- (A1) $z = 0$,
- (A2) $|y|^2 \leq au$,
- (A3) $|t|^2 \leq bu$,
- (A4) $|c|^2 \leq ab$,
- (A5) $a|t|^2 + u|c|^2 \leq b(au - |y|^2) + 2\Re(ct\bar{y})$.

Analogously, ψ is completely copositive if and only if the following conditions hold:

- (B1) $y = 0$,
- (B2) $|z|^2 \leq au$,
- (B3) $|t|^2 \leq bu$,
- (B4) $|c|^2 \leq ab$,
- (B5) $a|t|^2 + u|c|^2 \leq b(au - |z|^2) + 2\Re(c\bar{t}\bar{z})$.

Proof. By Theorem 1.1 and properties of unitary equivalence ψ is completely positive if and only if the matrix \mathbf{H} is positive. This is equivalent to the fact that all principal minors of \mathbf{H} are nonnegative. Conditions (A2), (A3) and (A4) follow from the fact that

$$\begin{vmatrix} a & y \\ \bar{y} & u \end{vmatrix} \geq 0, \begin{vmatrix} b & t \\ \bar{t} & u \end{vmatrix} \geq 0 \text{ and } \begin{vmatrix} a & c \\ \bar{c} & b \end{vmatrix} \geq 0. \text{ (A1) is a consequence of the equality } \det \mathbf{H} = -|z|^2(au - |y|^2) \text{ and (A2). Inequality in (A5) is equivalent to } \begin{vmatrix} a & c & y \\ \bar{c} & b & t \\ \bar{y} & \bar{t} & u \end{vmatrix} \geq 0.$$

The second part of the lemma follows in the similar way from positivity of the matrix \mathbf{H}^T in the case when ψ' is completely copositive. ■

REMARK 2.5. If φ is an extremal positive unital map described in Theorem 2.1 with the Choi matrix (2.1) then by condition (3) from Lemma 2.3 $|y| + |z| \leq u^{1/2}$. Lemma 8.11 in [5] claims that in the case $b > 0$ a stronger condition holds. Namely,

$$(2.9) \quad |y| + |z| = u^{1/2}.$$

Moreover, it follows from Lemma 8.8 in [5] that in this case

$$(2.10) \quad t^2 = -4(1 - u)y\bar{z}.$$

EXAMPLE 2.6. Consider the map $\psi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ with the Choi matrix

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}s \\ 0 & 1 - s^2 & \frac{1}{2}s & is(1 - s^2)^{1/2} \\ \hline 0 & \frac{1}{2}s & 0 & 0 \\ \frac{1}{2}s & -is(1 - s^2)^{1/2} & 0 & s^2 \end{bmatrix}$$

where $0 < s < 1$. It follows from Theorem 2.1 (compare also with (2.9) and (2.10)) and Lemma 2.4 that ψ is an extremal positive unital map which is neither completely positive nor completely copositive.

Now, we are ready to formulate our main theorem.

THEOREM 2.7. *Assume that φ is an extremal positive unital map with the Choi matrix given in (2.1) and $u > 0, y \neq 0, z \neq 0$. Then there are $\varphi_1, \varphi_2 \in F_{\xi, \eta}$ such that φ_1 is completely positive, φ_2 is completely copositive and $\varphi = \varphi_1 + \varphi_2$. Moreover, the pair φ_1, φ_2 is uniquely determined, and Choi matrices $\mathbf{H}_1, \mathbf{H}_2$ of maps φ_1, φ_2 have the following form:*

$$(2.11) \quad \mathbf{H}_1 = \left[\begin{array}{cc|cc} |y|u^{-1/2} & c & 0 & y \\ \bar{c} & |z|(1-u)u^{-1/2} & 0 & \frac{1}{2}t \\ \hline 0 & 0 & 0 & 0 \\ \bar{y} & \frac{1}{2}\bar{t} & 0 & |y|u^{1/2} \end{array} \right],$$

$$(2.12) \quad \mathbf{H}_2 = \left[\begin{array}{cc|cc} |z|u^{-1/2} & -c & 0 & 0 \\ -\bar{c} & |y|(1-u)u^{-1/2} & \bar{z} & \frac{1}{2}t \\ \hline 0 & z & 0 & 0 \\ 0 & \frac{1}{2}\bar{t} & 0 & |z|u^{1/2} \end{array} \right]$$

where u, y, z, t are the coefficients of the matrix (2.1) and c is a complex number such that $c^2 = -(1-u)u^{-1}yz$.

REMARK 2.8. The uniqueness of the decomposition of φ onto φ_1 and φ_2 does not hold if the assumptions of the above theorem are not fulfilled. To see this let us consider the following cases:

1. $u = 0$. Then the Choi matrix of the map φ has the following form

$$\mathbf{H} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and φ is completely positive and completely copositive.

2. $y = 0$. The Choi matrix of φ has the form

$$\mathbf{H} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 - |z|^2 & \bar{z} & 0 \\ \hline 0 & z & 0 & 0 \\ 0 & 0 & 0 & |z|^2 \end{array} \right]$$

and the map φ is completely copositive.

3. $z = 0$. The Choi matrix of φ has the form

$$\mathbf{H} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & y \\ 0 & 1 - |y|^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \bar{y} & 0 & 0 & |y|^2 \end{array} \right]$$

and the map φ is completely positive.

Turning to non-uniqueness of the decomposition of φ , let $\varepsilon > 0$ and put

$$\mathbf{A}_\varepsilon = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Then, it follows from Lemma 2.4 that \mathbf{A}_ε determines the map ψ_ε which is completely positive and completely copositive. Moreover, in each of the above three cases the equality $\mathbf{H} = (\mathbf{H} - \mathbf{A}_\varepsilon) + \mathbf{A}_\varepsilon$ describes the decomposition onto CP and co-CP parts for every sufficiently small ε .

Proof of Theorem 2.7. The existence of the decomposition follows from [4, 5] and the discussion after Theorem 2.1. To show that the decomposition is unique (and has the required form) assume that it is given by $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, where

$$(2.13) \quad \mathbf{H}_1 = \left[\begin{array}{cc|cc} a_1 & c & 0 & y \\ \bar{c} & b_1 & 0 & t_1 \\ \hline 0 & 0 & 0 & 0 \\ \bar{y} & \bar{t}_1 & 0 & u_1 \end{array} \right], \quad \mathbf{H}_2 = \left[\begin{array}{cc|cc} a_2 & -c & 0 & 0 \\ -\bar{c} & b_2 & \bar{z} & t_2 \\ \hline 0 & z & 0 & 0 \\ 0 & \bar{t}_2 & 0 & u_2 \end{array} \right].$$

Then the above coefficients fulfil the following set of relations:

$$(2.14) \quad a_1 + a_2 = 1,$$

$$(2.15) \quad b_1 + b_2 = 1 - u,$$

$$(2.16) \quad t_1 + t_2 = t,$$

$$(2.17) \quad u_1 + u_2 = u,$$

$$(2.18) \quad |y|^2 \leq a_1 u_1,$$

$$(2.19) \quad |t_1|^2 \leq b_1 u_1,$$

$$(2.20) \quad |c|^2 \leq a_1 b_1,$$

$$(2.21) \quad a_1 |t_1|^2 + u_1 |c|^2 \leq b_1 (a_1 u_1 - |y|^2) + 2\Re(ct_1 \bar{y}),$$

$$(2.22) \quad |z|^2 \leq a_2 u_2,$$

$$(2.23) \quad |t_2|^2 \leq b_2 u_2,$$

$$(2.24) \quad |c|^2 \leq a_2 b_2,$$

$$(2.25) \quad a_2 |t_2|^2 + u_2 |c|^2 \leq b_2 (a_2 u_2 - |z|^2) - 2\Re(\bar{c} t_2 \bar{z}).$$

We divide the rest of the proof into some lemmas.

LEMMA 2.9. *The coefficients a_1, a_2, u_1, u_2 have the following form*

$$(2.26) \quad a_1 = |y|u^{-1/2},$$

$$(2.27) \quad a_2 = |z|u^{-1/2},$$

$$(2.28) \quad u_1 = |y|u^{1/2},$$

$$(2.29) \quad u_2 = |z|u^{1/2}.$$

Proof. From (2.9) we have $|y|u^{-1/2} + |z|u^{-1/2} = 1$. Let $p = |y|u^{-1/2}$, $q = a_1$ and $r = u_1u^{-1}$. Then (2.18) gives

$$(2.30) \quad p^2 \leq qr$$

while (2.9), (2.22), (2.14) and (2.17) lead to

$$(2.31) \quad (1 - p)^2 \leq (1 - q)(1 - r).$$

The system of inequalities (2.30) and (2.31) is equivalent to

$$(2.32) \quad \frac{p^2}{q} \leq r \leq 1 - \frac{(1 - p)^2}{1 - q}.$$

By simple calculations one can show that the inequality between the first and the last terms in (2.32) is equivalent to $(q - p)^2 \leq 0$, so $q = p$. So, putting p instead of q in (2.32) we obtain also $r = p$. Hence, we have

$$\begin{aligned} a_1 &= q = p = |y|u^{-1/2}, \\ a_2 &= 1 - a_1 = 1 - |y|u^{-1/2} = |z|u^{-1/2}, \\ u_1 &= ru = pu = |y|u^{1/2}, \\ u_2 &= u - u_1 = (u^{1/2} - |y|)u^{1/2} = |z|u^{1/2}. \blacksquare \end{aligned}$$

LEMMA 2.10. *The following relations hold:*

$$(2.33) \quad |y|t_1 = y\bar{c}u^{1/2},$$

$$(2.34) \quad |z|t_2 = -\bar{z}cu^{1/2}.$$

Proof. Observe that application of (2.26)-(2.29) reduces inequalities (2.21) and (2.25) to

$$(2.35) \quad |y|u^{-1/2}|t_1|^2 + |y|u^{1/2}|c|^2 - 2\Re(ct_1\bar{y}) \leq 0$$

and

$$(2.36) \quad |z|u^{-1/2}|t_2|^2 + |z|u^{1/2}|c|^2 + 2\Re(\bar{c}t_2\bar{z}) \leq 0$$

respectively. Let $y_1, z_1 \in \mathbb{C}$ be such that $y_1^2 = y$ and $z_1^2 = z$. Then (2.35) and (2.36) can be rewritten in the form

$$|\bar{y}_1u^{-1/4}t_1 - y_1u^{1/4}\bar{c}|^2 \leq 0$$

and

$$|\bar{z}_1u^{-1/4}\bar{t}_2 + z_1u^{1/4}\bar{c}|^2 \leq 0.$$

These inequalities are equivalent to $\bar{y}_1u^{-1/4}t_1 = y_1u^{1/4}\bar{c}$ and $\bar{z}_1u^{-1/4}\bar{t}_2 = -z_1u^{1/4}\bar{c}$. Multiplication of both sides of the first equality by $y_1u^{1/4}$ leads to (2.33) while multiplication of the second one by $z_1u^{1/4}$ gives (2.34). \blacksquare

COROLLARY 2.11. $|t_1| = |t_2| \geq \frac{1}{2}|t|$.

Proof. The equality follows from (2.33) and (2.34) while the inequality is a consequence of (2.16) and the triangle inequality. ■

LEMMA 2.12. *The following relations hold:*

$$(2.37) \quad b_1 = |z|(1-u)u^{-1/2},$$

$$(2.38) \quad b_2 = |y|(1-u)u^{-1/2}.$$

Proof. From (2.10), (2.19), (2.23), (2.28), (2.29) and Corollary 2.11 we have the following inequalities

$$(1-u)|y||z| = \frac{1}{4}|t|^2 \leq |t_1|^2 \leq b_1u_1 = |y|u^{1/2}b_1$$

and

$$(1-u)|y||z| = \frac{1}{4}|t|^2 \leq |t_2|^2 \leq b_2u_2 = |z|u^{1/2}b_2.$$

From the first inequality we obtain

$$b_1 \geq |z|(1-u)u^{-1/2}$$

while from the second one and (2.15) we have

$$\begin{aligned} b_1 &= 1-u-b_2 \leq 1-u-|y|(1-u)u^{-1/2} = \\ &= (u^{1/2}-|y|)(1-u)u^{-1/2} = |z|(1-u)u^{-1/2}. \end{aligned}$$

Thus we obtain (2.37). In a similar way we get (2.38). ■

LEMMA 2.13. $|t_1| = |t_2| = \frac{1}{2}|t|$.

Proof. It follows from (2.19), Lemma 2.12, (2.28) and (2.10) that

$$|t_1|^2 \leq b_1u_1 = |z|(1-u)u^{-1/2} \cdot |y|u^{1/2} = (1-u)|y||z| = \frac{1}{4}|t|^2.$$

The converse inequality is included in Corollary 2.11. ■

COROLLARY 2.14. $t_1 = t_2 = \frac{1}{2}t$.

Proof. It easily follows from (2.16) and Lemma 2.13. ■

LEMMA 2.15. $c^2 = -(1-u)u^{-1}yz$.

Proof. From (2.34) and Corollary 2.14 we obtain $c = -\frac{1}{2}z|z|^{-1}tu^{-1/2}$. Thus, (2.10) implies $c^2 = -z^2|z|^{-2} \cdot (1-u)y\bar{z}u^{-1} = -(1-u)u^{-1}yz$. ■

The coefficient c in (2.11) and (2.12) is uniquely determined. It can be described in the following way. Let $y_1, z_1 \in \mathbb{C}$ be such that $y_1^2 = y, z_1^2 = z$ and $t = 2i(1-u)^{1/2}y_1\bar{z}_1$ (cf. (2.10)). The numbers y_1, z_1 are not uniquely determined by these conditions but the expression y_1z_1 is. Then, by (2.34)

$$(2.39) \quad c = -i(1-u)^{1/2}u^{-1/2}y_1z_1.$$

Combining the results of Lemmas 2.9, 2.12, 2.15 and Corollary 2.10 we end the proof of Theorem 2.7. ■

COROLLARY 2.16. *If φ is extremal positive unital map with the Choi matrix of the form (2.1) and assumptions of Theorem 2.7 are fulfilled then $\varphi(A) = U_1AU_1^* + U_2A^TU_2^*$ for every $A \in M_2(\mathbb{C})$, where $U_1, U_2 \in M_2(\mathbb{C})$ are of the form*

$$U_1 = \begin{bmatrix} y_1u^{-1/4} & 0 \\ i\bar{z}_1(1-u)^{1/2}u^{-1/4} & \bar{y}_1u^{1/4} \end{bmatrix}, \quad U_2 = \begin{bmatrix} z_1u^{-1/4} & 0 \\ -i\bar{y}_1(1-u)^{1/2}u^{-1/4} & \bar{z}_1u^{1/4} \end{bmatrix},$$

where y_1 and z_1 are as in the proof of Theorem 2.7.

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