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SOME CLASS OF POLYNOMIAL HYPERGROUPS

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Abstract. We provide explicit formulas for linearizing coefficients for some class of orthogonal polynomials.

Introduction. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials, $\deg P_n = n$, which are orthogonal with respect to a probability measure μ on $\mathbf{R}$ with infinite support. Then, under the convention that $P_{-1} = 0$, they satisfy the recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x), \qquad n \geq 1, \tag{1}$$

with $P_0(x) = 1$, $\gamma_n > 0$ and β_m real (see [Ch]). We denote by $\mathcal{L}$ the linear functional on $\mathbf{R}[x]$ given by $\mathcal{L}(P) := \int P(x)d\mu(x)$.

Now we define the *linearization coefficients* by the relation

$$P_n(x)P_m(x) = \sum_j c(j,m,n)P_j(x). \tag{2}$$

We say that $\{P_n\}_{n=0}^{\infty}$ admits *nonnegative product linearization* if all these coefficients $c(k,m,n)$ are nonnegative. In this case one can define a hypergroup in the following way: Choose x_0 such that $P_m(x_0) > 0$ for every m (this holds if and only if $x_0 \geq \sup(\mathrm{supp}\mu)$) and put

$$\delta_m * \delta_n := \sum_k \frac{c(k,m,n)P_k(x_0)}{P_m(x_0)P_n(x_0)}\,\delta_k.$$

Extending this to convex combinations one obtains an associative and commutative operation on the class of probability measures on the set $\{0,1,2,\dots\}$ (see [BH], [Ko]). Many of the classical orthogonal polynomials do admit nonnegative product linearization (see

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[BH]). On the other hand, there are some general criteria stated in terms of the coefficients β_n, γ_n, which imply this property (see [As, Sz1, Sz2, MS]).

The main result. Let us denote $L(k,m,n) := \mathcal{L}(P_k P_m P_n)$. Then multiplying both sides of (2) by P_k and applying $\mathcal{L}$ we get $L(k,m,n) = c(k,m,n)\mathcal{L}(P_k^2)$, so from now on we will be studying the numbers $L(k,m,n)$ instead of $c(k,m,n)$. We will be using the following properties:

(3) $L(k,m,k+m) = \gamma_0\gamma_1\ldots\gamma_{k+m-1}$,

(4) $L(k_1,k_2,k_3) = L(k_{\sigma_1},k_{\sigma_2},k_{\sigma_3})$ for every permutation σ of the set $\{1,2,3\}$,

(5) $L(k,m,n) = 0$ whenever $n > k+m$,

and

$$L(k,m,n) = L(k-1,m,n+1) + (\beta_n - \beta_{k-1})L(k-1,m,n) + \gamma_{n-1}L(k-1,m,n-1) - \gamma_{k-2}L(k-2,m,n) \tag{6}$$

for every k,m,n.

To prove them we first note that if $\deg P < n$ then $\mathcal{L}(P\cdot P_n) = 0$ because P can be expressed as a linear combination of $P_0, P_1, \ldots, P_{n-1}$. In particular, putting $P = P_k P_m$ we prove (5).

We use this remark again to note that

$$\mathcal{L}(xP_{k-1}P_m P_{k+m}) = L(k,m,k+m)$$

and

$$\mathcal{L}(xP_{k-1}P_m P_{k+m}) = L(k-1,m+1,k+m),$$

so we obtain $L(k,m,m+n) = L(k-1,m+1,m+n)$. Hence it is sufficient to prove (3) in the case when $k=0$. Now, by orthogonality we have

$$\mathcal{L}(P_m P_m) = \mathcal{L}(xP_{m-1}P_m) = \gamma_{m-1}\mathcal{L}(P_{m-1}P_{m-1}),$$

which completes the proof of (3).

For the last formula one only needs to compare the right hand sides of the equalities:

$$\mathcal{L}(xP_{k-1}P_m P_n) = L(k,m,n) + \beta_{k-1}L(k-1,m,n) + \gamma_{k-2}L(k-2,m,n),$$
$$\mathcal{L}(xP_{k-1}P_m P_n) = L(k-1,m,n+1) + \beta_n L(k-1,m,n) + \gamma_{n-1}L(k-1,m,n-1).$$

Note in passing that in view of (6) the coefficients $L(k,m,n)$ can be expressed as a sum of weights of so-called Motzkin paths on the plane connecting points $(0,m)$ and (k,n). Namely, the step from the point $(i-1,j+1)$, $(i-1,j)$, $(i-1,j-1)$ or $(i-2,j)$, respectively, to (i,j) is equal to 1, $\beta_j - \beta_{i-1}$, γ_{j-1} or $-\gamma_{i-2}$, respectively, and the weight of a path is the product of weights of its steps (see [MS]).

From now on we will assume that

$$\gamma_n = \begin{cases} a & \text{if } n = 0, \\ b & \text{if } n \text{ is odd}, \\ c & \text{if } n > 0 \text{ is even}, \end{cases} \qquad \beta_n = \begin{cases} u & \text{if } n = 0, \\ v & \text{if } n \text{ is odd}, \\ w & \text{if } n > 0 \text{ is even}. \end{cases}$$

This class contains orthogonal polynomials related to some interesting distributions which appear in noncommutative probability (see [BLS]). In the case when $b = c$ and $v = w$ the corresponding measure was thoroughly studied in [SY].

Our aim is to provide explicit formulas for the coefficients $L(k,m,n)$ in this case. Note, that by (4) and (5) we can assume that $k \le m \le n \le k+n$. Set $\Gamma(s) := \prod_{i=0}^{s-1} \gamma_i$. If $s > 1$ then $\Gamma(s) = \Gamma(s-1)b$ if s is even and $\Gamma(s) = \Gamma(s-1)c$ if s is odd.

THEOREM. *Assume that* $0 \le k \le m \le n \le k+m$ *and put* $s := [(k+m+n)/2]$. *Then* $L(k,m,n)$ *is equal to:*

- e1) $\Gamma(s)$ *if* $n = k+m$,
- e2) $\Gamma(s-1)(b+c-a+(w-u)(w-v))$ *if* k,m,n *are even,* s *is odd,*
- e3) $\Gamma(s-1)(b+c-a)$ *if only one of the numbers* k,m,n *is even,* s *is even and* $n < k+m$,
- e4) $\Gamma(s-1)(2c-a)$ *if* $n < k+m$ *and either* k,m,n,s *are all even or only one of the numbers* k,m,n *is even,* s *is odd,*
- o1) $\Gamma(s-1)(a(w-v)+c(v-u))$ *if* k,m,n *are odd,* s *is even*
- o2) $\Gamma(s)(v-u)$ *if* k,m,n *are odd,* s *is odd,*
- o3) $\Gamma(s)(w-u)$ *if only one of the numbers* k,m,n *is odd.*

Note that in cases (e1)–(e4) the sum $k+m+n$ is even, while in cases (o1)–(o3) it is odd.

Proof. We will proceed by induction on k. First we examine a few particular cases. For $n = k+m$ the formula is a consequence of (3) so we will assume that $k \ge 1$ and $n < k+m$.

If $k = 1 \le m$ then (6) yields

$$L(1,m,m) = (\beta_m - u)L(0,m,m) = \begin{cases} \Gamma(m)(v-u) & \text{if } m \text{ is odd,} \\ \Gamma(m)(w-u) & \text{if } m \text{ is even,} \end{cases}$$

which completes the proof for $k = 1$.

Now let us consider the special case when $n = k+m-1$, $k \ge 2$. Then $k+m+n = 2(k+m-1)+1$ is odd, $s = k+m-1 = n$ and $L(k-1,m,n+1) = L(k-2,m,n) = 0$. Moreover, if m is even then $\beta_n - \beta_{k-1} = 0$ so we get

$$L(k,m,k+m-1) = \gamma_{k+m-2}\Gamma(k+m-2)(w-u) = \Gamma(k+m-1)(w-u).$$

If m is odd, k is even then n is even so

$$\begin{aligned} L(k,m,k+m-1) &= (w-v)\Gamma(k+m-1) + b\Gamma(k+m-2)(v-u) \\ &= \Gamma(k+m-1)(w-u), \end{aligned}$$

and if k,m are odd then n is odd, hence

$$\begin{aligned} L(k,m,k+m-1) &= (v-w)\Gamma(k+m-1) + c\Gamma(k+m-2)(w-u) \\ &= \Gamma(k+m-1)(v-u). \end{aligned}$$

Now let us check the case $k = 2$ and $n = m$. If m is even, we have

$$\begin{aligned} L(2,m,m) &= L(1,m,m+1) + (w-v)L(1,m,m) + bL(1,m,m-1) - aL(0,m,m) \\ &= \Gamma(m+1) + \Gamma(m)(w-u)(w-v) + \Gamma(m)b - \Gamma(m)a \\ &= \Gamma(m)(c+(w-u)(w-v)+b-a), \end{aligned}$$

while if m odd then $\beta_n - \beta_{k-1} = v - v = 0$ and we get

$$L(2,m,m) = L(1,m,m+1) + cL(1,m,m-1) - a\Gamma(0,m,m)$$
$$= \Gamma(m+1) + c\Gamma(m) - a\Gamma(m) = \Gamma(m)(b+c-a).$$

Therefore we have completed the case $k = 2$.

From now on we assume that $2 < k \le m < k+m-1$. We will consider four cases.

I. Assume that k, n are even. Then

$$L(k,m,n) = L(k-1,m,n+1) + (w-v)L(k-1,m,n) + bL(k-1,m,n-1) - cL(k-1,m,n).$$

a) If m, s are even then the sum of the first, second and forth summand is zero and we get

$$L(k,m,n) = \Gamma(s-1)(b+c-a) + (w-v)\Gamma(s-1)(w-u) + b\Gamma(s-3)b(2c-a)$$
$$-c\Gamma(s-2)\big(b+c-a+(w-u)(w-v)\big) = \Gamma(s-2)b(2c-a)$$

b) If m is even and s is odd then the first summand cancels with the fourth one (including the special case $n = k+m-2$) so

$$L(k,m,n) = (w-v)\Gamma(s-1)(w-u) + b\Gamma(s-2)(b+c-a) = \Gamma(s-1)\big(b+c-a+(w-u)(w-v)\big).$$

c) Now assume that m is odd and s is even. Then

$$L(k,m,n) = \Gamma(s-1)\big(a(w-v)+c(v-u)\big) + (w-v)\Gamma(s-1)(b+c-a)$$
$$+b\Gamma(s-1)(v-u) - c\Gamma(s-1)(w-u) + \Gamma(s-1)b(w-u) = \Gamma(s)(w-u)$$

d) Finally, assume that m, s are odd. Then

$$L(k,m,n) = \Gamma(s)(v-u) + (w-v)\Gamma(s-2)b(2c-a)$$
$$+b\Gamma(s-2)\big(a(w-v)+c(v-u)\big) - c\Gamma(s-1)(w-u) + \Gamma(s-1)c(w-u) = \Gamma(s)(w-u).$$

II. If k is even and n is odd then $\beta_n - \beta_{k-1} = v - v = 0$, $\gamma_{n-1} = \gamma_{k-2} = c$ and $L(k-1,m,n-1) = L(k-2,m,n)$ (unless $n = k+m-2$), which leads

$$L(k,m,n) = L(k-1,m,n+1).$$

In the case $n = m+k-2$ we get

$$L(k,m,n+k-2) = \Gamma(k+m-1) + c\Gamma(k+m-4)b(2c-a) - c\Gamma(k+m-2)$$
$$= \Gamma(k+m-2)(b+c-a).$$

III. Similarly, if k is odd and n is even then $\beta_n - \beta_{k-1} = w - w = 0$, $\gamma_{n-1} = \gamma_{k-2} = b$ and $L(k-1,m,n-1) = L(k-2,m,n)$ (again, unless $n = k+m-2$), which, similarly as before, leads

$$L(k,m,n) = L(k-1,m,n+1),$$

and for $n = k+m-2$ we get

$$L(k,m,k+m-2) = \Gamma(k+m-1) + b\Gamma(k+m-3)(b+c-a) - b\Gamma(k+m-2)$$
$$= \Gamma(k+m-2)(2c-a).$$

IV. Finally, assume that k, n are odd. Then

$$L(k,m,n) = L(k-1,m,n+1) + (v-w)L(k-1,m,n) + cL(k-1,m,n-1) - bL(k-2,m,n).$$

a) If m, s are even then, similarly as in the case (Ib), the first summand cancels with the fourth one:

$$L(k,m,n) = (v-w)\Gamma(s-1)(w-u)+c\Gamma(s-2)\big(b+c-a+(w-u)(w-v)\big) = \Gamma(s-1)(b+c-a).$$

b) The case when m is even and s is odd is analogous to case (Ia), i.e. the sum of the first, second and forth summand is zero:

$$L(k,m,n) = \Gamma(s-1)\big(b+c-a+(w-u)(w-v)\big)$$

$$+(v-w)\Gamma(s-1)(w-u)+c\Gamma(s-3)b(2c-a)-b\Gamma(s-2)(b+c-a) = \Gamma(s-2)b(2c-a).$$

c) Now assume that m is odd and s is even. Then

$$L(k,m,n) = \Gamma(s)(w-u)+(v-w)\Gamma(s-1)(b+c-a)$$

$$+c\Gamma(s-1)(w-u)-b\Gamma(s-1)(v-u) = \Gamma(s-1)\big(a(w-v)+c(v-u)\big).$$

d) Finally, if m, s are odd then

$$L(k,m,n) = \Gamma(s)(w-u)+(v-w)\Gamma(s-2)b(2c-a)$$

$$+c\Gamma(s-1)(w-u)-c\Gamma(s-2)\big(a(w-v)+c(v-u)\big) = \Gamma(s)(v-u),$$

which completes the whole proof. ∎

COROLLARY. *The sequence* $\{P_n\}_{n=0}^{\infty}$ *admits nonnegative product linearization if and only if:*

$$a \le b+c, \quad a \le 2c, \quad a \le b+c+(w-u)(w-v),$$

$$u \le v, \quad u \le w, \quad \textit{and} \quad 0 \le a(w-v)+c(v-u). \ \blacksquare$$

For example, if either $w = v$ or $w = u$ then $\{P_n\}_{n=0}^{\infty}$ admits nonnegative product linearization if and only if $a \le b+c$, $a \le 2c$ and $u \le v$.

Finally let us specify our results to orthogonal polynomials related to limit measures with respect to conditionally free indendence introduced by Bożejko, Leinert and Speicher [BLS].

1. The central limit theorem (Theorem 4.3 in [BLS]): $a = \alpha^2$, $b = c = \beta^2$, $u = v = w = 0$. Here $\{P_m\}_{m=0}^{\infty}$ admit nonnegative product linearization if and only if $\alpha^2 \le 2\beta^2$ and this holds if and only if the corresponding measure has no atom.

2. The Poisson limit theorem (Theorem 4.4 in [BLS]): $a = \alpha^2$, $b = c = \beta^2$, $u = \alpha^2$, $v = w = \beta^2+1$ (these recurrence coefficients are not calculated explicitly in [BLS] but they can be derived from the final formula in [M]). Here the necessary and sufficient condition for nonnegative product linearization for these polynomials $\{R_n\}_{n=0}^{\infty}$ is $\alpha^2 \le \beta^2+1$ and $\alpha^2 \le 2\beta^2$.

3. Let us now consider the symmetrization of the Poisson measure. It means that we are dealing with polynomials $\{Q_n\}_{n=0}^{\infty}$ which are related to $\{R_n\}_{n=0}^{\infty}$ by $R_n(x^2) = Q_{2n}(x)$, so we get $a = \alpha^2$, $b = 1$, $c = \beta^2$ and $u = v = w = 0$. Here the corresponding condition is the same as in the previous case: $\alpha^2 \le \beta^2+1$ and $\alpha^2 \le 2\beta^2$.

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