QUANTUM PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 73
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2006

QUANTUM RANDOM WALK REVISITED

KALYAN B. SINHA
Indian Statistical Institute
7, SJS Sansanwal Marg
New Delhi 110016, India
E-mail: kbs@isid.ac.in

Abstract. In the framework of the symmetric Fock space over $L^2(\mathbb{R}_+)$, the details of the approximation of the four fundamental quantum stochastic increments by the four appropriate spin-matrices are studied. Then this result is used to prove the strong convergence of a quantum random walk as a map from an initial algebra $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}\,(\mathrm{Fock}\,(L^2(\mathbb{R}_+)))$ to a *-homomorphic quantum stochastic flow.

1. Introduction. As a sequel to the early works of Parthasarathy [Par 1987] and of Lindsay and Parthasarathy [L-P 1988], Attal and Pautrat [A-P 2006] revived the idea of quantum random walk and showed the weak convergence of a sequence of unitaries to the unitary process solution of a Hudson-Parthasarathy equation with constant bounded operator coefficients in an initial Hilbert space. However, unlike in [Par 1987], they work in one Hilbert space, viz. the symmetric (Bosonic) Fock space, with the 'walks' taking place in an infinite dimensional 'toy Fock space' which is a subspace of the Fock space. However, in both cases the authors only prove weak convergence. They therefore *cannot* conclude unitarity of the limiting operator process and thus do not have the homomorphic property for the resulting mapping process. For the motivation from classical Markov chains set in algebraic language, we refer the reader to the work of Lindsay and Parthasarathy [L-P 1988]; we also use mostly their notations and work only in the Heisenberg picture, i.e. with the processes as maps on the algebra of the observables of the system [A-F-L 1982].

Here we consider only the case with bounded coefficients of the stochastic process coming in a natural way from the one-step quantum random walk. However, we have

2000 *Mathematics Subject Classification*: 81S25, 46N50, 60H10.

The author acknowledges gratefully the discussions with Professor S. Attal during a visit of the author to the Institute Joseph Fourier, University of Grenoble, France as a part of the research project 2301-1 under the auspices of Indo-French Centre for the promotion of Advanced Research.

The paper is in final form and no version of it will be published elsewhere.

found it useful to separate the two issues in the study of quantum stochastic diffusion processes viz. the one on the existence of a solution of a quantum stochastic differential equation (q.s.d.e.) and other on *-homomorphic property of the solution as a map on the algebra of observables. In fact, we assume the existence of a strongly continuous solution of the q.s.d.e. for the diffusion, and also assume the existence of a one-step quantum walk given by *-homomorphisms depending on the step-size which in a suitable sense approximates the coefficients driving the diffusion as the step-size converges to zero. Here we work with bounded coefficients so that the assumption on existence of solutions holds, by standard quantum stochastic theory.

The motivation behind such a separation of assumptions is that in the case of unbounded coefficients one may be able to find a solution of the q.s.d.e. by some other methods (see e.g. [A-K 2001] and [L-W 2004]). Then the hope is that under suitable conditions the method indicated here may be modified to show strong convergence of the associated quantum random walk, thereby proving the *-homomorphic property of the solution.

2. Notations and preliminaries. *All tensor products between von Neumann algebras are meant in the ultraweak sense.* We begin by fixing a basis $\{N_k\}_{k=1}^4$ of $\mathcal{B}(\mathbb{C}^2)$ so that $\sum_{j=1}^4 b_j N_j = I$ for some constants $b_1, \ldots, b_4$, in particular, we choose

$$N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, N_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, N_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In such a case, $b_2 = b_3 = 0$ and $b_1 = b_4 = 1$ and we also note that the N_j's satisfy the following algebraic relations:

$$\begin{aligned} & N_3 = N_2^*, \ N_2^2 = N_3^2 = 0, \\ & N_1 = N_2 N_3, \ N_4 = N_3 N_2, \\ & N_2 N_3 + N_3 N_2 = N_1 + N_4 = I. \end{aligned} \tag{1}$$

We also observe that these are similar to the canonical anti-commutation relations (CAR) or equivalently the spin algebra relations satisfied by the Pauli matrices, and this is the reason why the authors of [L-P 1988] named the associated random walk as quantum spin random walk.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{h})$ be a von Neumann algebra of system observables where $\mathfrak{h}$ is a separable Hilbert space and let $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathcal{C}^2)$ be the basic unital *-homomorphism or the *one step random walk*. This means that we can write for $x \in \mathcal{A}$,

$$\alpha(x) = \sum_{j=1}^{4} \alpha_j(x) \otimes N_j, \tag{2}$$

and this also means that the equalities $\alpha(xy) = \alpha(x)\alpha(y)$ and $\alpha(x)^* = \alpha(x^*)$ will imply

$$\begin{aligned} & \alpha_j(xy) = \sum C_{kl}^j \alpha_k(x)\alpha_l(y), \qquad \alpha_j(I) = b_j, \\ & \alpha_1(x)^* = \alpha_1(x^*), \alpha_2(x) = \alpha_3(x^*), \alpha_3(x) = \alpha_2(x^*), \alpha_4(x) = \alpha_4(x^*), \end{aligned} \tag{3}$$

where C_{kl}^j is determined by $N_k N_l = \sum_j C_{kl}^j N_j$ as can be seen in (1).

We can now construct the n-step quantum spin random walk by the following recursive process:

$$J_0(x) = x, \quad J_1(x) = \alpha(x), \quad J_n(x) = \sum_{j=1}^{4} J_{n-1}(\alpha_j(x) \otimes N_j). \tag{4}$$

It is clear that (as in [L-P 1988]) for each n, $J_n : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathbb{C}^2)^{\otimes n}$ is a *-homomorphism.

Next, in order to pass to the diffusion limit of a quantum stochastic process, we need to embed the structure in a Fock space. Thus consider the symmetric (Bosonic) Fock space $\mathcal{H} \equiv \Gamma(L^2(\mathbb{R}_+)) = \mathcal{H} \equiv \mathbb{C} \oplus \sum_{n=1}^{\oplus} L^2(\mathbb{R}_+)^{\textcircled{s}^n}$, where $\textcircled{s}^n$ denotes n-fold symmetric tensor product. For any $f \in L^2(\mathbb{R}_+)$, we denote by $e(f)$ the exponential vector given by

$$e(f) = 1 \oplus f \oplus \frac{f^{\otimes 2}}{\sqrt{2}} \oplus \cdots \oplus \frac{f^{\otimes n}}{\sqrt{n}} \oplus \cdots$$

so that

$$\langle e(f), e(g) \rangle = \exp(< f, g >).$$

We also note the standard Fock space properties:

(i) continuous tensor product property: $L^2(\mathbb{R}_+) \cong L^2(0,a) \oplus L^2(a,b) \oplus L^2(b,\infty)$ implies $\Gamma(L^2(\mathbb{R}_+)) \simeq \Gamma(L^2(0,a)) \otimes \Gamma(L^2(a,b)) \otimes \Gamma(L^2(b,\infty))$ for $0 \leq a \leq b \leq \infty$;

(ii) if we define, for $0 \leq a \leq b < \infty$, $\theta(a,b)$ to be the unitary isomorphism from $L^2[0,1]$ onto $L^2[a,b]$ given by $(\theta(a,b)f)(s) = (b-a)^{-1/2} f((b-a)^{-1}(s-a))$, where $f \in L^2[0,1]$, then this lifts to a unitary isomorphism of $\mathcal{H}(0,1)$ onto $\mathcal{H}(a,b)$ by second quantization where we have written $\mathcal{H}(a,b) = \Gamma(L^2(a,b))$.

We have already observed that (1) is a representation of the CAR. By the results of Hudson and Parthasarathy [H-P 1986] the CAR also admit a representation in the Fock space $\mathcal{H}$ as follows:

$$N_2 = \int_0^1 \Gamma(R_s) A(ds), \quad N_3 = \int_0^1 \Gamma(R_s) A^\dagger(ds), \tag{5}$$

where $A, A^\dagger$ and Λ are the three fundamental (martingale) processes in $\mathcal{H}$ (see Par 1992), $\Gamma(R_s)$ is the second quantization of the reflection operator R_s in $L^2(\mathbb{R}_+)$ defined as

$$(R_s f)(t) = \begin{cases} -f(t) & \text{if } t \leq s, \\ f(t) & \text{if } t > s, \end{cases} \tag{6}$$

and N_1 and N_4 are obtained by using the relations in (1). In the standard notation of the CAR, $N_2 = a$ and $N_3 = a^+$, moreover $N_4 = a^+ a$, the number operator, and the N_k's so represented are bounded operators in $\mathcal{H}(0,1)$.

We also set, for $0 \leq a \leq b \leq \infty$ and $l = 1,2,3,4$,

$$N_l(a,b) = \Gamma(\theta(a,b)) N_l \Gamma(\theta(a,b))^{-1} \tag{7}$$

so that the $N_l(a,b)$'s are bounded operators in $\mathcal{H}(a,b)$.

Next suppose that for each non-zero positive number h, we are given a basic*-homomorphism $\alpha(h,\cdot) : \mathcal{A} \to \mathcal{A} \otimes B(0,h)$ as in (2). Then we can proceed to construct the n-step random walk as follows: for $t \geq 0$,

$$J_0^{(h)}(x) = x, \qquad J_h^{(h)}(x) = \sum_{k=1}^{4} \alpha_k(h,x) \otimes N_k(0,h),$$

$$J_t^{(h)}(x) = \sum_{k=1}^{4} J_{\overline{n-1}h}^{(h)}(\alpha_k(h,x)) \otimes N_k(\overline{n-1}h, nh), \tag{8}$$

where $\overline{n-1}h < t \leq nh$.

From the construction above, two facts follow easily:

(i) for each $x \in \mathcal{A}$ and $t \geq 0$

$$J_t^{(h)}(x) \in \mathcal{A} \otimes \mathcal{B}(0,h) \otimes \mathcal{B}(h,2h) \otimes \cdots \otimes \mathcal{B}(\overline{n-1}h, nh) \subseteq \mathcal{A} \otimes \mathcal{B}(0,nh) \subseteq \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$$

where we have used the notation $\mathcal{B}(a,b) = \mathcal{B}(\mathcal{H}(a,b))$;

(ii) since each $J_t^{(h)}$ is *-homomorphic and unital, in particular it satisfies

$$\|J_t^{(h)}(x)\| \leq \|x\| \tag{9}$$

Here we also collect a result from pages 185–186 of [Par 1992] which will be used more than once in the sequel.

Proposition 2.1 *Let $\{X_j(t)\}_{j=1,2}$ be stochastic processes given by*

$$X_j(t) = \sum_{k=1}^{n} L_j(t_{k-1}) \left[M_j(t_k) - M_j(t_{k-1})\right]$$

where $t_0 < t_1 < \cdots < t_n = t$ and M_j is any martingale process (such as a linear combination of A, A^+ and Λ). Then for $u, v \in \mathfrak{h}$ and $f, g \in L^2(\mathbb{R}_+)$,

$$\begin{aligned}
&\langle X_1(t)ue(f), X_2(t)ve(g)\rangle \\
&= \sum_{j=1}^{n} \langle L_1(t_{j-1})ue(f_{t_{j-1}]}), L_2(t_{j-1})ve(g_{t_{j-1}]})\rangle \\
&\quad \cdot\langle (M_1(t_j) - M_1(t_{j-1}))\, e(f_{[t_{j-1}}), (M_2(t_j) - M_2(t_{j-1}))\, e(g_{[t_{j-1}})\rangle \\
&+ \sum_{j=1}^{n} \langle X_1(t_{j-1})ue(f_{t_{j-1}]}), L_2(t_{j-1})ve(g_{t_{j-1}]})\rangle \left\langle e\left(f_{[t_{j-1}}\right), (M_2(t_j) - M_2(t_{j-1}))\, e\left(g_{[t_{j-1}}\right)\right\rangle \\
&+ \sum_{j=1}^{n} \langle L_1(t_{j-1})ue(f_{t_{j-1}]}), X_2(t_{j-1})ve(g_{t_{j-1}]})\rangle \\
&\quad \cdot\langle (M_1(t_j) - M_1(t_{j-1}))\, e(f_{[t_{j-1}}), e(g_{[t_{j-1}})\rangle.
\end{aligned}$$

3. Approximation of the fundamental processes. Here we derive some useful estimates regarding the behaviour of the N_k's in relation to the fundamental processes in the Fock space and for this it is convenient to use the notation:

$$\Lambda_1(s) = s, \qquad \Lambda_2(s) = A(s), \qquad \Lambda_3(s) = A^\dagger(s), \qquad \Lambda_4(s) = \Lambda(s), \tag{10}$$

and

$$\varepsilon_1 = 1, \quad \varepsilon_2 = \varepsilon_3 = 1/2, \quad \varepsilon_4 = 0.$$

LEMMA 3.1. *Let $f, g \in \mathcal{M} \equiv BC^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and l be a natural number. Then for $0 < h < 1$, $k = 1, 2, 3, 4$,*

$$\text{(i)} \quad |\langle e(g_{[\overline{l-1}h,lh]}), [h^{\varepsilon_k} N_k(\overline{l-1}h, lh) - \Lambda_k(\overline{l-1}h, lh)]e(f_{[\overline{l-1}h,lh]})\rangle| \leq C_1 h^2,$$

$$\text{(ii)} \quad \|[h^{\varepsilon_k} N_k(\overline{l-1}h, lh) - \Lambda_k(\overline{l-1}h, lh)]e(g_{[\overline{l-1}h,lh]}\| \leq \begin{cases} C_2 h^{\frac{3}{2}} & \text{for } k = 1, 2, \\ C_2 h & \text{for } k = 3, 4. \end{cases}$$

where C_1 and C_2 are two positive constants depending on g, f and on g respectively. Here also for an interval $\Delta \subseteq \mathbb{R}_+$, g_Δ means $g\chi_\Delta$ and $BC^1(\mathbb{R}_+)$ means the space of once continuously differentiable functions on $\mathcal{R}_+$ with the function as well as its derivative bounded. In the following for the brevity of writing, we shall write $g_{l-1]}$, $g_{[l]}$, $g_{[l}$ for $g_{[0,\overline{l-1}h]}$, $g_{[\overline{l-1}h,lh]}$ and $g_{[lh}$ respectively and $N_k[l]$ and $\Lambda_k[l]$ for $N_k(\overline{l-1}h, lh)$ and $\Lambda_k(\overline{l-1}h, lh)$ respectively.

Proof. (i)

$$\begin{aligned} &\langle e(g_{[l]}), (N_4[l] - \Lambda[l])e(f_{[l]})\rangle \\ &\qquad = \frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} f(\tau)d\tau \; e^{\langle R_s g_{[l]}, R_\tau f_{[l]}\rangle} - \int_{\overline{l-1}h}^{lh} \overline{g}(s)f(s)ds \; e^{\langle g_{[l]} f_{[l]}\rangle}. \end{aligned}$$

Noting that

$$R_s R_\tau f(t) = \begin{cases} -f(t) & \text{if } t \in [s \wedge \tau, s \vee \tau], \\ f(t) & \text{otherwise,} \end{cases}$$

we get that the L.H.S. in (i) equals

$$\begin{aligned} &\frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} f(\tau)d\tau \left[\exp\left\{\int_{\overline{l-1}h}^{s\Lambda\tau} \overline{g}f - \int_{s\Lambda\tau}^{s\vee\tau} \overline{g}f + \int_{s\vee\tau}^{lh} \overline{g}f\right\} - \exp\left\{\int_{\overline{l-1}h}^{lh} \overline{g}f\right\}\right] \\ &\qquad + \exp\left\{\int_{\overline{l-1}h}^{lh} \overline{g}f\right\} \cdot \left[\frac{1}{h}\int_{\overline{l-1}h}^{lh} ds\overline{g}(s) \int_{\overline{l-1}h}^{lh} f(\tau)d\tau - \int_{\overline{l-1}h}^{lh} \overline{g}(s)f(s)ds\right] \\ &\qquad \equiv \mathbf{I}_1 + \mathbf{I}_2. \end{aligned}$$

We have

$$\begin{aligned} |\mathbf{I}_1| &\leq h\|g\|_\infty \|f\|_\infty \exp(\|g\|\,\|f\|) \sup_{s,\tau,\in[\overline{l-1}h,lh]} \left|1 - \exp\left\{-2\int_{s\wedge\tau}^{s\vee\tau} \overline{g}f\right\}\right| \\ &\leq 2h^2 \|g\|_\infty^2 \|f\|_\infty^2 \exp(3\|g\|\,\|f\|), \end{aligned}$$

and

$$\begin{aligned} |\mathbf{I}_2| &\leq \left|\exp\left(\int_{\overline{l-1}h}^{lh} \overline{g}f\right)\right| \left|\frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} [f(\tau) - f(s)]d\tau\right| \\ &\leq 2\exp(\|g\|\,\|f\|)\,\|g\|_\infty \|f'\|_\infty h^2, \end{aligned}$$

proving that

$$|\langle e(g_{[l]}), (N_4[l] - \Lambda[l])e(f_{[l]}\rangle| \leq C_1 h^2.$$

Next,

$$\langle e(g_{[l]}), (hN_1[l] - h)e(f_{[l]})\rangle = -h\{\langle e(g_{[l]})\{N_4[l] - \Lambda[l]\}e(f_{[l]})\rangle + \langle e(g_{[l]}), \Lambda[l]e(f_{[l]})\rangle\}$$

and thus by the previous estimates, $|L.H.S.| \leq C_1'' h^3 + C_1' h^2$ for small h.

Next,

$$\langle e(g_{[l]}), (h^{\frac{1}{2}} N_3[l] - A^\dagger[l]) e(f_{[l]})\rangle$$
$$= \int_{\overline{l-1}h}^{lh} \overline{g}(s) ds \langle e(g_{[l]}), (\Gamma(R_s) - I) e(f_{[l]})\rangle$$
$$= \int_{\overline{l-1}h}^{lh} \overline{g}(s) ds \Big\{ \exp\Big(- \int_{\overline{l-1}h}^{s} \overline{g} f + \int_{s}^{lh} \overline{g} f \Big) - \exp\Big(\int_{\overline{l-1}h}^{lh} \overline{g} f \Big) \Big\}$$

and an estimate similar to the earlier one leads to

$$|\langle e(g_{[l]}), (h^{\frac{1}{2}} N_3[l] - A^\dagger[l]) e(f_{[l]})\rangle| \leq 2 \|g\|_\infty^2 \exp(3 \|g\| \|f\|) \|f\|_\infty h^2.$$

Finally,

$$|\langle e(g_{[l]}), (h^{\frac{1}{2}} N_2[l] - A[l]) e(f_{[l]})\rangle| = |\langle e(f_{[l]}), (h^{\frac{1}{2}} N_3[l] - A^\dagger[l]) e(g_{[l]})\rangle|$$

and a similar estimate follows from the previous one.

(ii) We have

$$\begin{aligned}\|(hN_1[l] - h) e(g_{[l]})\|^2 &= h^2 \langle e(g_{[l]}), N_4[l] e(g_{[l]})\rangle \\ &= h^2 \{\langle e(g_{[l]}), (N_4[l] - \Lambda[l]) e(g_{[l]})\rangle + \langle e(g_{[l]}), \Lambda[l] e(g_{[l]})\rangle\} \\ &\leq h^2 (C' h^2 + C'' h) \leq C_2^2 h^3\end{aligned}$$

giving the required estimate. Similarly,

$$\|\{h^{\frac{1}{2}} N_2[l] - A[l]\} e(g_{[l]})\| = \left\| \int_{\overline{l-1}h}^{lh} g(s) ds \left[e(R_s g_{[l]}) - e(g_{[l]}) \right] \right\| \leq \text{Const } h^{\frac{3}{2}},$$

since

$$\begin{aligned}\|e(R_s g_{[l]}) - e(g_{[l]})\|^2 &= 2 \left[\exp\left(\int_{\overline{l-1}h}^{lh} |g|^2 \right) - \exp\left(- \int_{\overline{l-1}h}^{s} |g|^2 + \int_{s}^{lh} |g|^2 \right) \right] \\ &\leq 4 \exp(\|g\|^2) \|g\|_\infty^2 h.\end{aligned}$$

Next,

$$\begin{aligned}\|(\sqrt{h} N_3[l] - A^\dagger[l]) e(g_{[l]})\| &= \left\| \int_{\overline{l-1}h}^{lh} dA^\dagger(s) \left\{ e(R_s g_{[l]}) - e(g_{[l]}) \right\} \right\| \\ &\leq C''' \sqrt{\int_{\overline{l-1}h}^{lh} (1 + |g(s)|^2) \left\| e(R_s g_{[l]}) - e(g_{[l]}) \right\|^2} \leq C_2 h,\end{aligned}$$

where we have used the estimate in the previous paragraph. Finally,

$$\begin{aligned}(11) \quad & \left\| (N_4[l] - \Lambda[l]) e(g_{[l]}) \right\|^2 \\ &= \langle e(g_{[l]}), N_4[l] e(g_{[l]})\rangle + \langle e(g_{[l]}), \Lambda[l]^2 e(g_{[l]})\rangle - 2 Re \langle N_4[l] e(g_{[l]}), \Lambda[l] e(g_{[l]})\rangle.\end{aligned}$$

By the quantum Ito formula, we have

$$d_t \Lambda[a,t]^2 = 2\Lambda[a,t] d\Lambda(t) + d\Lambda(t)$$

or,

$$\Lambda[l]^2 = \int_{\overline{l-1}h}^{lh} (2\Lambda(\overline{l-1}h, t) + 1) d\Lambda(t).$$

Therefore,

$$\begin{aligned}\langle e(g_{[l]}(,\Lambda[l]^2 e(g_{[l]})\rangle &= \int_{\overline{l-1}h}^{lh} |g(s)|^2 ds \langle e(g_{[l]}), \left\{2\Lambda(\overline{l-1}h,s)+1\right\} e(g_{[l]})\rangle \\ &= \int_{\overline{l-1}h}^{lh} |g(s)|^2 ds \left\{2\int_{\overline{l-1}h}^{s} |g(\tau)|^2 d\tau + 1\right\} \exp(\left\|g_{[l]}\right\|^2),\end{aligned}$$

and also note that the first term $\leq e^{\|g\|^2} \|g\|_\infty^2 h^2$. Next,

$$\langle N_4[l]e[g_{[l]}), \Lambda[l]e[g_{[l]})\rangle = h^{-1}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \left\langle \int_{\overline{l-1}h}^{lh} \Gamma(R_\tau)dA^\dagger(\tau)e(R_s g_{[l]}), \Lambda[l]e(g_{[l]})\right\rangle. \tag{12}$$

Using the relation that for $\Delta, \Delta' \subseteq \mathbb{R}_+$ and $f \in L^2(\mathbb{R}_+)$,

$$A(\Delta)\Lambda(\mathcal{X}_{\Delta'})e(f) = \left[\int_{\Delta\cap\Delta'} f(\tau)d\tau + \int_\Delta f(\tau)d\tau \Lambda(\mathcal{X}_{\Delta'})\right] e(f),$$

we have that

$$\begin{aligned}\left\langle \int_{\overline{l-1}h}^{lh} \Gamma(R_\tau)dA^+(\tau)e(R_s g_{[l]}), \Lambda[l]e(g_{[l]})\right\rangle &= \int_{\overline{l-1}h}^{lh} \langle e(R_\tau R_s g_{[l]}), e(g_{[l]})\rangle g(\tau)d\tau \\ &+ \int_{\overline{l-1}h}^{lh} g(\tau)d\tau \left(\int_{\overline{l-1}h}^{lh} \overline{R_\tau R_s g(t)} g(t)dt\right) \langle e(R_\tau R_s g_{[l]}), e(g_{[l]})\rangle,\end{aligned}$$

and therefore (12) is equal to

$$h^{-1}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} g(\tau)d\tau \left\{1 + \int_{\overline{l-1}h}^{lh} \overline{R_\tau R_s g(t)} g(t)dt\right\} \cdot e^{\langle R_\tau R_s g_{[l]}, g_{[l]}\rangle}.$$

Finally we note that

$$\begin{aligned}&\overline{\langle N_4[l]e(g_{[l]}), \Lambda(l)e(g_{[l]})\rangle} \\ &\quad = h^{-1}\int_{\overline{l-1}h}^{lh} g(s)ds \int_{\overline{l-1}h}^{lh} \overline{g(\tau)}d\tau \left\{1 + \int_{\overline{l-1}h}^{lh} R_\tau R_s g(t)\overline{g(t)}dt\right\} \cdot e^{\overline{\langle R_\tau R_s g_{[l]}, g_{[l]}\rangle}} \\ &\quad = \langle N_4[l]e(g_{[l]}), \Lambda(l)e(g_{[l]})\rangle,\end{aligned}$$

since $R_\tau^* = R_\tau$. Putting all this together in (11) we observe that

$$\begin{aligned}\left\|[N_4[l] - \Lambda(l)]\, e(g_{[l]})\right\|^2 &= \frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} g(\tau)d\tau e^{\langle R_\tau R_s g_{[l]}, g_{[l]}\rangle} \\ &+ \int_{\overline{l-1}h}^{lh} |g(s)|^2 ds\, e^{\left\|g_{[l]}\right\|^2} - \frac{2}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} g(\tau)d\tau\, e^{\langle R_\tau R_s g_{[l]}, g_{[l]}\rangle} + O(h^2) \\ &\leq \int_{\overline{l-1}h}^{lh} |g(s)|^2 ds\, \exp(\left\|g_{[l]}\right\|^2) - \frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} g(\tau)d\tau\, e^{\langle R_\tau R_s g_{[l]}, g_{[l]}\rangle} + O(h^2) \\ &= \left(\frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} [g(s) - g(\tau)]d\tau\right) e^{\left\|g_{[l]}\right\|^2} \\ &+ \frac{1}{h}\int_{\overline{l-1}h}^{lh} \overline{g}(s)ds \int_{\overline{l-1}h}^{lh} g(\tau)d\tau \left(e^{\left\|g_{[l]}\right\|^2} - e^{\langle R_\tau R_s g_{[l]}, g_{[l]}\rangle}\right) + O(h^2).\end{aligned} \tag{13}$$

The first term in (13) in absolute value does not exceed

$$\|g\|_\infty \|g'\|_\infty \frac{e^{\|g\|^2}}{h} \int_{\overline{l-1}h}^{lh} ds \int_{\overline{l-1}h}^{lh} |s-\tau|\, d\tau = \frac{1}{6} \exp(\|g\|^2) \|g\|_\infty \|g'\|_\infty h^2,$$

while the second term in (13) in absolute value does not exceed

$$\frac{1}{h} \int_{\overline{l-1}h}^{lh} |g(s)| ds \int_{\overline{l-1}h}^{lh} |g(\tau)| d\tau \Big| \exp\Big(\int_{\overline{l-1}h}^{lh} |g|^2 \Big) - \exp\Big\{ \int_{\overline{l-1}h}^{s\wedge\tau} |g|^2 - \int_{s\wedge\tau}^{s\vee\tau} |g|^2 + \int_{s\vee\tau}^{lh} |g|^2 \Big\} \Big|$$

$$\leq \frac{1}{h} \exp(\|g\|^2) \int_{\overline{l-1}h}^{lh} |g(s)| ds \int_{\overline{l-1}h}^{lh} |g(\tau)| d\tau |1 - e^{-2\int_{s\wedge\tau}^{s\vee\tau} |g|^2}| \leq 2 \|g\|_\infty^4 \exp(\|g\|^2) h^2.$$

Combining all these together in (13), we get finally,

$$\|(N_4[l] - \Lambda[l]) e(g_{[l]})\| \leq C_2 h.$$

REMARK 3.2. From Lemma 3.1 (ii), it is clear that for $k = 1, 2$;

$$\Big[\sum_{l=1}^{n} h^{\epsilon_k} N_k(\overline{l-1}h, lh) - \Lambda_k(0,t)\Big] e(g) \to 0$$

as $h \to 0$ for $g \in BC^1(\mathbb{R}_+)$. In order to prove a similar result for $k = 3, 4$, one has to be a bit more careful.

COROLLARY 3.3. *For $g \in \mathcal{M}$ and $k = 1, 2, 3, 4$ with the choices of ϵ_k as given in Lemma 3.1,*

$$\Big[\sum_{l=1}^{n} h^{\epsilon_k} N_k(\overline{l-1}h, lh) - \Lambda_k(0,t)\Big] e(g) \to 0$$

as $h \to 0+$.

Proof. As we have observed above the proof for $k = 1, 2$ is straightforward. For the rest, we note from Proposition 2.1 that if we set

$$\begin{aligned} X_{t,h}^k e(g) &\equiv \Big[\sum_{l=1}^{n} h^{\epsilon_k} N_k(\overline{l-1}h, lh) - \Lambda_k(0,t)\Big] e(g) \\ &= \sum_{l=1}^{n} \big[h^{\epsilon_k} N_k(\overline{l-1}h, lh) - \Lambda_k(\overline{l-1}h, lh)\big]\, e(g), \\ &\equiv \sum_{l=1}^{n} M_k(\overline{l-1}h, lh) e(g), \end{aligned}$$

then

$$\begin{aligned} \|X_{t,h}^k e(g)\|^2 &= \sum_{l=1}^{n} \|M_k(\overline{l-1}h, lh) e(g_{[l]})\|^2 \|e(g - g_{[l]})\|^2 \\ &\quad + 2Re \sum_{l=1}^{n} \langle X_{\overline{l-1}h,h}^{(k)} e(g_{\overline{l-1}h]}), e(g_{\overline{l-1}h]})\rangle \langle e(g_{[\overline{l-1}h}), M_k(\overline{l-1}h, lh) e(g_{[\overline{l-1}h})\rangle \\ &= I_3 + I_4. \end{aligned} \tag{14}$$

In the above we have used the fact that each M_k is a martingale w.r.t. the Fock filtration, i.e. for $s < t$

$$\langle e(g_{s]}), M_k(s,t)e(g_{s]})\rangle = 0.$$

This is because the L.H.S. above equals

$$\langle e(\theta(s,t)^{-1}g_{s]}), M_k(0,1)e(\theta(s,t)^{-1}g_{s]})\rangle = \langle e(0), M_k(0,1), e(0)\rangle = 0 \quad \text{for } k = 2,3$$

in an obvious fashion and for $k = 4$,

$$\langle e(0), M_4(0,1), e(0)\rangle = \langle e(0), N_4(0,1), e(0)\rangle = \left\| \int_0^1 \Gamma(R_s)dA(s)e(0) \right\|^2 = 0,$$

while for $k = 1$

$$\langle e(0), M_1(0,1), e(0)\rangle = \left\| \int_0^1 \Gamma(R_s)dA^\dagger(s)e(0) \right\|^2 - 1 = \|A^\dagger(0,1)e(0)\|^2 - 1 = 0.$$

Next we note that, by the estimates in Lemma 3.1(ii), for $k = 3,4$, $\|X^{(k)}_{t,h}e(g)\| \le$ constant, independent of h, and therefore by using Lemma 3.1(i) in I_4 in (14), we get that $|I_4| \to 0$ as $h \to 0$, On the other hand, since by Lemma 3.1(ii)

$$\|M_k(\overline{l-1}h, lh)e(g)\|^2 \le \text{Constant} \cdot h^2,$$

we have $|I_3| \to 0$ as $h \to 0$.

Corollary 3.3 essentially produces the result obtained by Attal and Pautrat [A-P2006] on exponential vectors though they used an entirely different construction. This also makes clear in what sense the "random walks" approximate various quantum noises.

4. Quantum diffusion as a limit of random walk. In this section we shall address the central result of this paper, viz. that the quantum n-step random walk $J^{(h)}_t(x)$ with step-size h in (8) converges as $h \to 0, n \to \infty$ with $nh \to t$ strongly to the solution $j_t(x)$ of a particular quantum stochastic differential equation, provided the basic 1-step homomorphism $\alpha(h,.)$ has suitable limiting properties. Unlike in [L-P 1988] or [A-P 2006], the strong convergence allows one to conclude that $(j_t)_{t\ge 0}$ is a flow of *-homomorphisms on the algebra $\mathcal{A}$ of observables and therefore is a "quantum diffusion".

To present the results in this section, we make two basic assumptions separately so that one knows which assumption leads to exactly which result.

ASSUMPTIONS.

A1: The quantum stochastic initial value problem: for $x \in \mathcal{A}, t \ge 0, u \in h, f \in L^2(\mathbb{R}_+)$,

$$j_t(x)\, ue(f) = x\, ue(f) + \int_0^t \sum_{k=1}^4 j_s(\beta_k(x))d\Lambda_k(s).ue(f) \tag{15}$$

has a unique strongly continuous solution $j_t : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$. Furthermore, it is assumed that for fixed $u \in \mathfrak{h}$ and $f \in \mathcal{M} \subseteq L^2(\mathbb{R}^+)$ the solution satisfies the bound:

$$\|j_t(x)ue(f)\| \le C(t,f)\|x\|\,\|u\|. \tag{16}$$

A2: Assume furthermore that the basic *-homomorphism $\alpha(h,.)$ satisfies for $k = 1,2,3,4$,

$$h^{-\eta_k}\{\alpha_k(h,x) - b_k x - h^{\epsilon_k}\beta_k(x)\} \to 0 \quad \text{as } h \to 0 \tag{17}$$

uniformly with respect to $x \in \mathcal{A}$, where $\eta_k = 1$ for $k = 1,3$ and $= 1/2$ for $k = 2,4$.

Since we are assuming in this paper that the structure maps (β_k) are *bounded*, assumption A1 holds ([Par 1992]).

THEOREM 4.1. *Assume A1 and A2. Then $J_t^{(h)}(x)$, as defined in (8), converges strongly to $j_t(x)$ for each $x \in \mathcal{A}$ and $t \geq 0$. Thus $j_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$ is a unital *-homomorphic flow.*

Proof. (a) From A1 it follows that for $u \in \mathfrak{h}$, $f \in \mathcal{M}$ and $0 \leq s < t < T, x \in \mathcal{A}$

$$\begin{aligned}\| [j_t(x) - j_s(x)] \, ue(f)\|^2 &= \left\| \sum_{k=1}^{4} \int_s^t j_t \left(\beta_k(x)\right) \Lambda_k(dt) ue(f) \right\|^2 \\ &\leq C'(T,f) \sum_{k=1}^{4} \int_s^t \|j_t \left(\beta_k(x)\right) ue(f)\|^2 ds \\ &\leq C''(T,f) \|u\|^2 \sum_{k=1}^{4} \|\beta_k(x)\|^2 (t-s) \\ &\leq \widetilde{C}(t,f)^2 (t-s) \|u\|^2 \, \|x\|^2.\end{aligned}$$

Thus in the definition of the strong integral in the right-hand side of (15) with respect to the partition of $[0,t)$ as given in (8), one has that

$$\begin{aligned}&\left(\sum_{k=1}^{4} \int_0^t j_s \left(\beta_k(x)\right) \Lambda_k(ds) - \sum_{k=1}^{4} \sum_{l=1}^{n} j_{\overline{l-1}h} \left(\beta_k(x)\right) \Lambda_k[l] \right) ue(f) \\ &\qquad = \sum_{k=1}^{4} \int_0^t \left[j_s \left(\beta_k(x)\right) - j_{s,n} \left(\beta_k(x)\right) \right] \Lambda_k(ds) ue(f),\end{aligned}$$

with $j_{s,n}(y) = j_{\overline{l-1}h}(y)$ for $\overline{l-1}h \leq s < lh$, $l = 1, 2, \ldots, n$.

By using the earlier estimate, for $u \in \mathfrak{h}$, $f \in \mathcal{M}$, $x \in \mathcal{A}$ we have that

$$\begin{aligned}&\left\| \left(\sum_{k=1}^{4} \int_0^t j_s \left(\beta_k(x)\right) \Lambda_k(ds) - \sum_{k=1}^{4} \sum_{l=1}^{n} j_{\overline{l-1}h}(\beta_k(x)) \Lambda_k[l] \right) ue(f) \right\|^2 \\ &\quad \leq D'(T,f) \sum_{k=1}^{4} \int_0^t \| \left[j_s \left(\beta_k(x)\right) - j_{s,n} \left(\beta_k(x)\right) \right] ue(f)\|^2 \left(1 + |f(s)|^2\right) ds \\ &\quad \leq \widetilde{D}(T,f) \sum_{k=1}^{4} \sum_{l=1}^{n} \int_{\overline{l-1}h}^{lh} ds \| [j_s \left(\beta_k(x)\right) - j_{\overline{l-1}h}(\beta_k(x))] ue(f)\|^2 \\ &\quad \leq \widetilde{D}\widetilde{C}^2 \|u\|^2 \sum_{k=1}^{4} \|\beta_k(x)\|^2 \sum_{l=1}^{n} h^2 \leq D(T,f) \|u\|^2 \, \|x\|^2 h.\end{aligned}$$

This establishes the fact that for u, f and x as above,

$$\begin{aligned}&\left(j_t(x) - x - \sum_{k=1}^{4} \sum_{l=1}^{n} j_{\overline{l-1}h}(\beta_k(x)) \Lambda_k[l] \right) ue(f) \\ &\qquad = \left(\sum_{k=1}^{4} \int_0^t j_s(\beta_k(x)) \Lambda_k(ds) - \sum_{k=1}^{4} \sum_{l=1}^{4} j_{\overline{l-1}h}(\beta_k(x)) \Lambda_k[l] \right) ue(f)\end{aligned}$$

converges to 0, uniformly w.r.t. $x \in \mathcal{A}$, as $h \to 0$, $n \to \infty$ with $nh \to t$. From (8), it follows that

$$\begin{aligned} J_t^{(h)}(x) - x &= \sum_{l=1}^{n} \{J_{lh}^{(h)}(x) - J_{\overline{l-1}h}^{(h)}(x)\} \\ &= \sum_k \sum_{l=1}^{n} \{J_{\overline{l-1}h}^{(h)}(\alpha_k(h,x)) \otimes N_k(\overline{l-1}h, lh) - J_{\overline{l-1}h}^{(h)}(x) \otimes b_k N_k(\overline{l-1}h, lh)\} \\ &= \sum_k \sum_{l=1}^{n} J_{\overline{l-1}h}^{(h)}(\alpha_k(h,x) - b_k x) \otimes N_k(\overline{l-1}h, lh) \\ &= \sum_k \sum_{l=1}^{n} J_{\overline{l-1}h}^{(h)}(\alpha_k(h,x) - b_k x - h^{\epsilon_k}\beta_k(x)) \otimes N_k[l] \\ &\quad + \sum_k \sum_{l=1}^{n} J_{\overline{l-1}h}^{(h)}(\beta_k(x)) \otimes \{h^{\epsilon_k} N_k[l] - \Lambda_k[l]\} + \sum_k \sum_{l=1}^{n} J_{\overline{l-1}h}^{(h)}(\beta_k(x)) \otimes \Lambda_k[l]. \end{aligned}$$

Thus for $u \in \mathfrak{h}$, $f \in \mathcal{M}$, $x \in \mathcal{A}$, and using (a), we get

(18)
$$\begin{aligned} &[J_t^{(h)}(x) - j_t(x)]ue(f) \\ &= \sum_k \sum_{l=1}^{n} (J_{\overline{l-1}h}^{(h)}(\alpha_k(h,x) - b_k x - h^{\epsilon_k}\beta_k(x)) \otimes N_k[l])ue(f) \\ &\quad + \sum_k \sum_{l=1}^{n} (J_{\overline{l-1}h}^{(h)}(\beta_k(x)) \otimes \{h^{\epsilon_k} N_k[l] - \Lambda_k[l]\})ue(f) \\ &\quad - \Big[\sum_k \int_0^t j_s(\beta_k(x))\Lambda_k(ds) - \sum_k \sum_{l=1}^{n} j_{\overline{l-1}h}(\beta_k(x))\Lambda_k[l]\Big]ue(f) \\ &\quad + \Big(\sum_k \sum_{l=1}^{n} \{J_{\overline{l-1}h}^{(h)}(\beta_k(x)) - j_{\overline{l-1}h}(\beta_k(x))\} \otimes \Lambda_k[l]\Big)ue(f) \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where we have used the notations used in the proof of Lemma 3.1.

Next we fix $u \in \mathfrak{h}$, $f \in \mathcal{M}$, $x \in \mathcal{A}$ and then using (9) and assumption A2, we can estimate I_1 in (18):

$$\|I_1\| \le \|u\| \sum_{k=1}^{4} \sum_{l=1}^{n} \|\alpha_k(h,x) - b_k x - h^{\epsilon_k}\beta_k(x)\| \, \|N_k[l]e(f)\|$$

which goes to zero uniformly w.r.t. $x \in \mathcal{A}$, as $h \to 0$ once we observe from Lemma 3.1 that

$$\|N_{1,3}[l]e(f_{[l]})\| \le C_1', \;\; \|N_{2,4}[l]e(f_{[l]})\| \le C_2' h^{\frac{1}{2}}$$

with constants C_1' and C_1' depending on f only.

For I_2, we shall use again Proposition 2.1 as we did in the proof of Corollary 3.3. For this we set

$$Y^k_{t,h} \equiv \sum_{l=1}^n J^{(h)}_{\overline{l-1}h}(\beta_k(x)) \otimes \{h^{\epsilon_k} N_k[l] - \Lambda_k[l]\}$$

and observe first that by Lemma 3.1(ii)

$$\|Y^k_{t,h} ue(f_{[l]})\| \le \text{Constant} \cdot \|x\| \text{ (independent of } h) \tag{19}$$

Thus

$$\begin{aligned}\|Y^k_{t,h} ue(f)\|^2 = & \sum_{l=1}^n \|J^{(h)}_{\overline{l-1}h}(\beta_k(x)) ue(f_{\overline{l-1}h]})\|^2 \\ & \cdot \|(h^{\epsilon_k} N_k[l] - \Lambda_k[l]) e(f_{[l]})\|^2 \|e(f_{[lh})\|^2 \\ & + 2Re \sum_{l=1}^n \langle Y^k_{\overline{l-1}h,h} ue(f_{\overline{l-1}h]}), ue(f_{\overline{l-1}h]})\rangle \\ & \cdot \langle e(f_{[l]}), \{h^{\epsilon_k} N_k[l] - \Lambda_k[l]\}\, e(f_{[l]})\rangle \cdot \|e(f_{[lh})\|^2\end{aligned}$$

which, again by virtue of Lemma 3.1 (i) and (ii) and (19) goes to zero uniformly w.r.t. $x \in \mathcal{A}$ as $h \to 0$.

Thus using the observation in the part (a) of this proof and (18), we note that given an arbitrary $\epsilon > 0$, we can choose $h > 0$ sufficiently small so that

$$\|I_1 + I_2 + I_3\| < \epsilon c \|x\|, \tag{20}$$

where c is a constant depending only on u, f, T but independent of h and x.

(c) This last part of the proof is similar to that of theorem 3.3 in [L-P 1988]. For fixed $u \in \mathfrak{h}$ and $f \in \mathcal{M}$, define

$$T^{(h)}_t(x) = J^{(h)}_t(x) ue(f), \qquad \mathcal{T}_t(x) = j_t(x) ue(f). \tag{21}$$

First of all, note that both the maps $T^{(h))}_t$ and $\mathcal{T}_t$ are well-defined bounded linear maps from $\mathcal{A}$ into $h \otimes \mathcal{H}$. Next, we get from (18) and (20) that

$$\|T^{(h)}_t(x) - \mathcal{T}_t(x)\|^2 \le 2c^2\epsilon^2 \|x\|^2 + 2C^2 \sum_{l=1}^n h \sum_{k=1}^4 \|T^{(h)}_{\overline{l-1}h}(\beta_k(x)) - \mathcal{T}_{\overline{l-1}h}(\beta_k(x))\|^2,$$

where we have used the standard estimate (page 223 [Par 1992]) for I_3 and C is a constant which depends only on f. Noting that β'_k's are all bounded, the above leads to

$$\begin{aligned}\|T^{(h)}_t - \mathcal{T}_t\|^2 & \le 2c^2\epsilon^2 + 2C^2 \sum_{k=1}^4 \|\beta_k\|^2 \sum_{l=1}^n h \|T^{(h)}_{\overline{l-1}h} - \mathcal{T}_{\overline{l-1}h}\|^2 \\ & = 2c^2\epsilon^2 + D \sum_{l=1}^n \|T^{(h)}_{\overline{l-1}h} - \mathcal{T}_{\overline{l-1}h}\|^2 .h\end{aligned} \tag{22}$$

where we have set $D = 2C^2 \sum_{k=1}^4 \|\beta_k\|^2$. As in [L-P 1988], we note that since the initial values $T^{(h)}_0(x) = xue(f) = \mathcal{T}_0(x)$, it follows that

$$\|T^{(h)}_t - \mathcal{T}_t\|^2 \le (1 + hD)^n 2c^2\epsilon^2 \le 2e^{tD} c^2\epsilon^2$$

and since ϵ was arbitrary, it follows that

$$\lim_{h\to 0} \|T_t^{(h)} - \mathcal{T}_t\|^2 \equiv 0$$

or equivalently

$$\lim_{h\to 0} \|J_t^{(h)}(x)ue(f) - j_t(x)ue(f)\| = 0, \text{ for } u \in \mathfrak{h},\ f \in \mathcal{M},\ x \in \mathcal{A}.$$

Next, since for $t>0$ fixed and for each $h>0$, $J_t^{(h)}$ is contractive, and $\{J_t^{(h)}(x)ue(f)\}_{h>0}$ is strongly Cauchy in h, for $u \in \mathfrak{h}$ and $f \in \mathcal{M}$, it easily follows by the density of the algebraic tensor product of $\mathfrak{h}$ and vectors of the type $e(f)$ (with $f \in \mathcal{M}$) in $\mathfrak{h}\otimes\Gamma\left(L^2(\mathbb{R}_+)\right)$ that:

(i) for every $t > 0$ and $x \in \mathcal{A}$, $\{J_t^{(h)}(x)\}_{h>0}$ is strongly Cauchy on $\mathfrak{h} \otimes \mathcal{H}$,
(ii) for every $t > 0$ and $x \in \mathcal{A}$, $J_t^{(h)}(x)$ converges strongly on $\mathfrak{h} \otimes \mathcal{H}$ to $j_t(x)$,
(iii) since each $J_t^{(h)}$ is a *-homomorphic unital map from $\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{B}(\mathcal{H})$, $j_t(.)_{t\geq 0}$ is also a *-homomorphic unital map from $\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{B}(\mathcal{H})$,
(iv) since furthermore by A1, $j_t(x)$ satisfies the quantum stochastic differential equation (15), $j_t : \mathcal{A} \to \mathcal{A}\otimes\mathcal{B}(\mathcal{H})$ is a *-homomorphic flow and is the limit of the quantum random walk (8).

REMARK 4.2. (a) In the case β_k's are bounded maps, as has been assumed here, the hypothesis A1 can actually be verfied directly by an iterative construction of the solution (see for example, [E 1989] or [Par 1992] on the exponential domain i.e. on vectors of the type $ue(f), u \in \mathfrak{h}, f \in L^2(\mathbb{R}_+)$.

(b) In the original papers ([E 1989], [M-S 1990] and [G-S 1999]), the homomorphism property was also deduced by using a kind of iterative procedure, depending heavily on the boundedness of all the "structure maps" β_k's. Here we have separated the two issues, viz, the existence and the property of homomorphism. This is more natural because for various unbounded β_k's, the existence problem may be handled by using various theories of semigroups and of evolutions, whereas the scheme of approximation of the solution by a sequence of homomorphisms (called "random walks"), if it can be adapted for unbounded β_k's, will prove the homomorphism property of the solution. However, it should be emphasised that the step (c) in the proof of the theorem 4.1 has to be modified appropriately for this to succeed. This will be addressed elsewhere.

(c) From (2), (8) and (16) we find that the map $\alpha(h, .) : \mathcal{A} \to \mathcal{A}\otimes\mathcal{B}(\mathbb{C}^2)$ is given as:

$$\alpha(h,x) = \begin{pmatrix} \alpha_1(h,x) & \alpha_2(h,x) \\ \alpha_3(h,x) & \alpha_4(h,x) \end{pmatrix} = \begin{pmatrix} x + h\beta_1(x) + O(h) & \sqrt{h}\beta_2(x) + O(h^{\frac{1}{2}}) \\ \sqrt{h}\beta_3(x) + O(h) & x + \beta_4(x) + O(h^{\frac{1}{2}}) \end{pmatrix} \tag{23}$$

and the property of homomorphism for $\alpha(h, \cdot)$ gives:

$$\begin{aligned} \alpha_1(h,xy) &= \alpha_1(h,x)\alpha_1(h,y) + \alpha_2(h,x)\alpha_3(h,y), \\ \alpha_2(h,xy) &= \alpha_1(h,x)\alpha_2(h,y) + \alpha_2(h,x)\alpha_4(h,y), \\ \alpha_3(h,xy) &= \alpha_3(h,x)\alpha_1(h,y) + \alpha_4(h,x)\alpha_3(h,y), \\ \alpha_4(h,xy) &= \alpha_3(h,x)\alpha_2(h,y) + \alpha_4(h,x)\alpha_4(h,y). \end{aligned} \tag{24}$$

Combining (23) and (24) we get, for fixed $x, y \in \mathcal{A}$,

$$xy + h\beta_1(xy) + O(h) = (x + h\beta_1(x) + O(h))(y + h\beta_1(y) + O(h)),$$

or,

$$h\left[\beta_1(xy) - \beta_1(x)y - x\beta_1(y) - \beta_2(x)\beta_3(y)\right] + O(h) = 0,$$

and similarly,

$$\begin{aligned}
&h^{\frac{1}{2}}\left[\beta_2(xy) - x\beta_2(y) - \beta_2(x)\left\{y + \beta_4(y)\right\}\right] + O(h^{\frac{1}{2}}) = 0,\\
&h^{\frac{1}{2}}\left[\beta_3(xy) - \beta_3(x)y - \left\{x + \beta_4(x)\right\}\beta_3(y)\right] + O(h^{\frac{1}{2}}) = 0,\\
&\left[\beta_4(xy) - \beta_4(x)y - x\beta_4(y)\right] + h\beta_3(x)\beta_2(y) + O(h) = 0 - \beta_4(x)\beta_4(y),
\end{aligned}$$

leading to the familiar structure relations in the limit of $h \to 0$,

$$\begin{aligned}
\beta_1(xy) &= \beta_1(x)y + x\beta_1(y) + \beta_2(x)\beta_3(y),\\
\beta_2(xy) &= x\beta_2(y) + \beta_2(x)(y + \beta_4(y)),\\
\beta_3(xy) &= \beta_3(x)y + (x + \beta_4(x))\beta_3(y),\\
\beta_4(xy) &= \beta_4(x)y + x\beta_4(y) + \beta_4(x)\beta_4(y).
\end{aligned}$$

Similarly, the *-preservation property of $\alpha(h, .)$ leads to

$$\beta_1(x)^* = \beta_1(x^*), \quad \beta_2(x)^* = \beta_3(x^*), \quad \beta_4(x)^* = \beta_4(x^*).$$

References

[A-F-L 1982] L. Accardi, A. Frigerio and J. Lewis, *Quantum Stochastic Processes*, Publ. Res. Inst. Math. Sci. 18, 97–133.

[A-K 2001] L. Accardi and S. V. Kozyrev, *On the structure of Markov flows*, Chaos, Solution, Fractals, 12, 2639–2655.

[A-P 2006] S. Attal and Y. Pautrat, *From repeated to continuous quantum interactions*, Ann. H. Poincaré 7, 59–104.

[E 1989] M. P. Evans, *Existence of quantum diffusions*, Probab. Theory Related Fields 81, 473–483.

[G-S 1999] D. Goswami and K. B. Sinha, *Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra*, Comm. Math. Phys. 205, 377–403.

[H-P 1986] R. L. Hudson and K. R. Parthasarathy, *Unification of fermion and boson stochastic calculus*, Comm. Math. Phys. 104, 457–470.

[L-P 1988] J. M. Lindsay and K. R. Parthasarathy, *The passage from random walk to diffusion in quantum probability. II*, Sankhyā Ser. A 50, 151–170.

[L-W 2004] J. M. Lindsay and S. Wills, *Quantum stochastic operator cocycles via associated semigroups*, math.FA/0512398, Math. Proc. Cambridge Phil. Soc. (to appear).

[M-S 1990] A. Mohari and K. B. Sinha, *Quantum stochastic flows with infinite degrees of freedom and countable state Markov processes*, Sankhyā Ser. A 52, 43–57.

[Par 1988] K. R. Parthasarathy, *The passage from random walk to diffusion in quantum probability. A celebration of applied probability*, J. Appl. Probab. 1988, Special Vol. 25A, 151–166.

[Par 1992] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Monographs in Mathematics 85, Birkhäuser Verlag, Basel.