QUANTUM PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 73
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2006

ON THE TRANSIENT AND RECURRENT PARTS OF A QUANTUM MARKOV SEMIGROUP

VERONICA UMANITÀ
Dipartimento di Matematica "F. Brioschi", Politecnico di Milano
Piazza Leonardo da Vinci 32, I-20133, Milano, Italy
E-mail: umanita@mate.polimi.it

Abstract. We define the transient and recurrent parts of a quantum Markov semigroup $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$ and we show that, when $\mathcal{A}$ is σ-finite, we can write $\mathcal{T}$ as the sum of such semigroups. Moreover, if $\mathcal{T}$ is the countable direct sum of irreducible semigroups each with a unique faithful normal invariant state ρ_n, we find conditions under which any normal invariant state is a convex combination of ρ_n's.

1. Introduction. If P is a Markov chain with finite state space, T is the set of its transient states and $R_1, \dots, R_k$ denote the different classes of recurrent states, then we can think of P as the sum of its transient part (i.e. the one relative to T) and its recurrent part, given by a block-diagonal matrix where any block is irreducible and it corresponds to a recurrent class R_i.

In their work [6], Evans and Høegh-Krohn have generalized this decomposition for a positive stochastic map Φ on a finite-dimensional C*-algebra $\mathcal{A}$ by introducing a recurrent and a transient projection in terms of invariant states; they show that the recurrent projection is subharmonic and that, if Φ is recurrent (i.e. its recurrent projection is equal to $\mathbf{1}$), there exists a resolution of the identity $\{p_1, \dots, p_s\}$ such that the restriction of Φ to each of the subalgebras $p_i \mathcal{A} p_i$ is irreducible.

Our intention here is to extend such results to the case of a quantum Markov semigroup (QMS) $\mathcal{T}$ on a σ-finite von Neumann algebra $\mathcal{A}$. As in [6], we define the *fast recurrent projection* p_R as the supremum of the supports of the normal invariant states, but, to distinguish between fast and slow recurrence, we decompose $p_R^{\perp}$ further as the sum of a *transient projection* p_T (determined by range projections of potentials, see [10]) and a *slow recurrent projection* p_{R_0}. As in the case of Markov chains with finite state space, in the finite-dimensional setting we shall have $p_{R_0} = 0$. Therefore, we shall call a QMS

2000 *Mathematics Subject Classification*: 46L55, 82C10.

The paper is in final form and no version of it will be published elsewhere.

transient or recurrent according to $p_T = \mathbf{1}$ or $p_T = 0$, respectively. Further, we show that, when $\mathcal{A}$ is σ-finite, the subalgebra $p_T\mathcal{A}p_T$ is invariant under the action of $\mathcal{T}$ (see Cor. 11) and its restriction to this subalgebra is a transient semigroup; on the other hand, the reduced semigroup associated with $p_T^\perp$ is a recurrent QMS (see Thm. 15). Moreover, under appropriate conditions, we can decompose the semigroup $\mathcal{T}^{p_R}$ associated with p_R into the direct sum of irreducible "sub"-QMS's each one supporting a unique faithful normal invariant state (Prop. 19).

Finally, in the last part we analyze a typical situation occurring in many examples known in the literature: we assume that there exists an orthogonal sequence (p_n) of $\mathcal{T}^{p_R}$-invariant projections such that the restriction of $\mathcal{T}^{p_R}$ to the subalgebra $p_n\mathcal{A}p_n$ is irreducible and possesses a (faithful) normal invariant state ρ_n for all n. Then, under this hypothesis, we investigate if we can write any normal invariant state as a convex combination of ρ_n, and we show that this is equivalent to a condition on the set of fixed points of $\mathcal{T}$ (Thm. 24).

2. Preliminaries. In this paper $\mathcal{A}$ is a von Neumann algebra with unit $\mathbf{1}$ acting on a complex Hilbert space $\mathcal{H}$. A *quantum dynamical semigroup* (QDS) is a w*-continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of normal completely positive maps on $\mathcal{A}$; if $\mathcal{T}_t(\mathbf{1}) = \mathbf{1}$ for all $t \geq 0$, then it is *Markov* (i.e. it is a QMS). The *infinitesimal generator* of $\mathcal{T}$ is the operator $\mathcal{L}$ whit domain $D(\mathcal{L})$ which is the vector space of elements a in $\mathcal{A}$ such that the $\lim_{t\to 0} t^{-1}(\mathcal{T}_t(a) - a)$ exists in the weak* topology. For $a \in D(\mathcal{L})$, $\mathcal{L}(a)$ is defined as the limit above. In many cases (for instance $\mathcal{T}$ uniformly continuous, i.e. such that there exists $\lim_{t\to 0} \|\mathcal{T}_t - \mathcal{T}_0\| = 0$), the generator $\mathcal{L}$ of a QMS $\mathcal{T}$ can be represented in the Lindblad form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}\sum_k (L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k),$$

where H, L_k, G are bounded operators on $\mathcal{H}$, H self-adjoint.

A state ω on $\mathcal{A}$ is *normal* if it is σ-weakly continuous or, equivalently, if $\omega(\sup_\alpha a_\alpha) = \sup_\alpha \omega(a_\alpha)$ for any increasing net $(a_\alpha)_\alpha$ of positive elements in $\mathcal{A}$ with an upper bound; we denote by $\mathcal{A}_*$ the *predual* of $\mathcal{A}$, that is the space of all σ-weakly continuous linear functionals on $\mathcal{A}$. We recall also that ω is a normal state if and only if there exists a density matrix ρ, that is, a positive trace-class operator of $\mathcal{H}$ with a unit trace, such that $\omega(a) = tr(\rho a)$ for all $a \in \mathcal{A}$.

ω is *faithful* if $\omega(a) > 0$ for all non-zero positive elements $a \in \mathcal{A}$.

For any normal state ω on $\mathcal{A}$, the *support projection* $s(\omega)$ is the smallest projection in $\mathcal{A}$ such that $\omega(s(\omega)a) = \omega(as(\omega)) = \omega(a)$ for any $a \in \mathcal{A}$ (cf. [5], Prop. 3); since it is easy to check that any normal state ω is faithful on $s(\omega)\mathcal{A}s(\omega)$, it follows that ω is faithful if and only if $s(\omega) = \mathbf{1}$.

If $\mathcal{T}$ is a QDS on $\mathcal{A}$, its *predual semigroup* is the semigroup $\mathcal{T}_*$ of operators in $\mathcal{A}_*$ defined by $(\mathcal{T}_{*t}(\omega))(a) = \omega(\mathcal{T}_t(a))$ for every $a \in \mathcal{A}$ and $\omega \in \mathcal{A}_*$. Since any map $\mathcal{T}_{*t}$ is clearly weakly continuous on $\mathcal{A}_*$, $\mathcal{T}_*$ is a strongly continuous semigroup in the Banach space $\mathcal{A}_*$ (see, for instance [3] Cor. 3.1.8); moreover, if $\mathcal{T}$ is Markov, $\mathcal{T}$ and $\mathcal{T}_*$ are semigroups of contractions (see [7], Prop. 2.10.3).

We say that a normal state ω on $\mathcal{A}$ is *invariant* if $\mathcal{T}_{*t}(\omega) = \omega$ for all $t \geq 0$ and we denote by $\mathcal{F}(\mathcal{T}_*)_1$ the set of normal invariant states on $\mathcal{A}$.

A family $\mathcal{G}$ of normal states on $\mathcal{A}$ is called *faithful* if $a \in \mathcal{A}$, a positive and $\omega(a) = 0$ for all $\omega \in \mathcal{G}$ implies $a = 0$; given a family $\mathcal{G}$ of normal invariant state and put $p = \sup\{\, s(\omega) : \omega \in \mathcal{G}\,\}$, then $\mathcal{G}$ is faithful on the subalgebra $p\mathcal{A}p$.

We recall that a von Neumann algebra $\mathcal{A}$ on $\mathcal{H}$ is *σ-finite* if there exists a countable subset S of $\mathcal{H}$ which is separating for $\mathcal{A}$ (i.e. for any $a \in \mathcal{A}$, $au = 0$ for all $u \in S$ implies $a = 0$).

We shall often make use of the following elementary remark. Given a positive element $x \in \mathcal{A}$ and a projection p, then $pxp = 0$ implies $p^\perp xp = pxp^\perp = 0$ (see Lemma $II.1$ of [9]).

3. The fast recurrent projection and the transient projection. Following the theory of classical Markov processes and [6], we first introduce the fast recurrent projection p_R in such a way that the set of fast recurrent states is invariant for the system and the reduced semigroup is mean ergodic; therefore, p_R will be determined by the supports of the normal invariant states.

We call a positive operator a *subharmonic* (resp. *superharmonic*, resp. *harmonic*) if $\mathcal{T}_t(a) \geq a$ (resp. $\mathcal{T}_t(a) \leq a$, resp. $\mathcal{T}_t(a) = a$) for all $t \geq 0$; we denote by $\mathcal{F}(\mathcal{T})$ the set of harmonic elements of $\mathcal{T}$. Subharmonic projections play an important role in the study of QMSs. For example, we have the following

Proposition 1 ([10]). *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. If $\omega \in \mathcal{A}_*$ is an invariant state, then its support projection is subharmonic.*

Proof. Let ω be a normal invariant state, $p := s(\omega)$, and fix $t \geq 0$. From the invariance of ω it follows $\omega(p - p\mathcal{T}_t(p)p) = \omega(p - \mathcal{T}_t(p)) = 0$, and then $p\mathcal{T}_t(p)p = p$, because $p\mathcal{T}_t(p)p \leq p$ and ω is faithful on $p\mathcal{A}p$. Therefore, the projection $p^\perp$ satisfies $p\mathcal{T}_t(p^\perp)p = 0$, so $\mathcal{T}_t(p^\perp) = p^\perp\mathcal{T}_t(p^\perp)p^\perp$. This implies $\mathcal{T}_t(p^\perp) \leq p^\perp$ and then $\mathcal{T}_t(p) \geq p$. ∎

Notation. For any $\omega \in \mathcal{A}_*$ and p projection of $\mathcal{A}$, we denote by $p\omega p$ the element of $\mathcal{A}_*$ defined as $(p\omega p)(a) = \omega(pap)$ for all $a \in \mathcal{A}$, and by $p\mathcal{A}_*p$ the set of $p\omega p$ as ω varies in $\mathcal{A}_*$. Then the subalgebra $p\mathcal{A}p$ is canonically isomorphic to the dual space of $p\mathcal{A}_*p$ and we can identify the normal states on $p\mathcal{A}p$ with the normal states on $\mathcal{A}$ whose support is smaller than p.

Given a subharmonic projection p, we can construct a QMS on the subalgebra $p\mathcal{A}p$ in the following way: since p subharmonic implies that $p\mathcal{A}_*p$ is $\mathcal{T}_*$-invariant (see Prop. $II.1$ of [9]), we can restrict $\mathcal{T}_*$ to such a Banach space and obtain a weakly continuous semigroup. If we denote by $\mathcal{T}^p = \{\mathcal{T}^p_t\}_t$ its dual semigroup, taking $a \in p\mathcal{A}p = (p\mathcal{A}_*p)^*$ and $\omega \in p\mathcal{A}_*p$, we have

$$\omega((\mathcal{T}_{*t|p\mathcal{A}_*p})^*(a)) = (\mathcal{T}_{*t}(\omega))(a) = \omega(\mathcal{T}_t(a)) = \omega(p\mathcal{T}_t(a)p), \quad \forall\; t \geq 0,$$

that is,

$$\mathcal{T}^p_t(a) = p\mathcal{T}_t(a)p, \quad \forall\; a \in p\mathcal{A}p,\; t \geq 0. \tag{1}$$

$\mathcal{T}_t^p$ is a QMS on $p\mathcal{A}p$ because any $\mathcal{T}_t^p$ is clearly normal, completely positive and

$$p = p\mathcal{T}_t(\mathbf{1})p \geq p\mathcal{T}_t(p)p \geq p.$$

Definition 1 ([13]). $\mathcal{T}^p$ is called the *reduced semigroup* associated with p.

If $\{p_i\}_i$ is an arbitrary family of projections, then we denote by $\sup_i p_i$ the projection (in $\mathcal{A}$) onto the closure of the linear space of $\mathcal{H}$ generated by the ranges of p_i's.

Definition 2 ([13]). The *fast recurrent projection* associated with a QMS $\mathcal{T}$ is the projection $p_R = \sup_i p_i$ where the p_i's are the support projections of all invariant states of $\mathcal{T}$.

Theorem 2. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. Then its fast recurrent projection is subharmonic.*

Proof. It follows immediately from the definition, p_R being the least upper bound of subharmonic projections. ∎

We can then consider the reduced semigroup associated with p_R.

We have $p_R = 0$ when the semigroup has no normal invariant states, and $p_R = \mathbf{1}$ when $\mathcal{T}$ has a faithful family of normal invariant states; in particular, if $\mathcal{A}$ is σ-finite, then $p_R = \mathbf{1}$ if and only if there exists a faithful normal invariant state (Corollary 1 of [15]). However, since $\mathcal{F}(\mathcal{T}_*)_1$ is a faithful family on $p_R\mathcal{A}p_R$ and any $\mathcal{T}$-invariant state is clearly also $\mathcal{T}^{p_R}$-invariant, the family $\mathcal{F}(\mathcal{T}_*)_1$ is faithful for $\mathcal{T}^{p_R}$; so, applying the mean ergodic Thm. of [12] to $\mathcal{T}^{p_R}$ we get the following

Theorem 3 ([13]). *For all $a \in \mathcal{A}$ the limit*

$$\mathcal{E}(a) := \text{w}^*\text{-}\lim_t \frac{1}{t}\int_0^t p_R\mathcal{T}_s(a)p_R\, ds$$

exists and it defines a $p_R\mathcal{T}p_R$-invariant normal conditional expectation onto the von Neumann subalgebra $\mathcal{F}(\mathcal{T}^{p_R})$ of $p_R\mathcal{A}p_R$ such that $\mathcal{E}\circ\mathcal{T}_t = \mathcal{E}$ for all $t \geq 0$. A normal state ω on $\mathcal{A}$ is $\mathcal{T}$-invariant if and only if $\omega\circ\mathcal{E} = \omega$.

We now introduce the projection in which the system spends a small amount of time; for this purpose, we need to define a *potential* associated to $\mathcal{T}$, which really represents the time of sojourn of a pure state in a projection.

Our reference on quadratic forms is the book of Kato [14].

Definition 3 ([10]). Given a positive operator $x \in \mathcal{A}$ we define the *form-potential of x* as a quadratic form $\mathfrak{U}(x)$ by

$$\mathfrak{U}(x)[u] = \int_0^\infty \langle u, \mathcal{T}_s\,(x)\,u\rangle ds, \quad \forall\; u \in D(\mathfrak{U}(x)),$$

where the domain $D(\mathfrak{U}(x))$ is the set of all $u \in \mathcal{H}$ s.t. $\int_0^\infty \langle u, \mathcal{T}_s\,(x)\,u\rangle ds < \infty$.

This is clearly a symmetric and positive form; moreover, by Thm. 3.13a p. 461 and Lemma 3.14a p. 461 of [14] it is also closed. Therefore, when it is densely defined, it is represented by a self-adjoint operator (see Thm.2.1, p. 322, Thm. 2.6, p. 323 and Thm. 2.23 p. 331 of [14]). This motivates the following definition.

DEFINITION 4 ([10]). For all positive $x \in \mathcal{A}$ such that $D(\mathfrak{U}(x))$ is dense, the *potential of* x is the self-adjoint operator $\mathcal{U}(x)$ which represents $\mathfrak{U}(x)$.

We put also $\mathcal{A}_{int} := \{x \in \mathcal{A}_+ : \mathcal{U}(x) \text{ is bounded}\}$ and we call its elements $\mathcal{T}$-*integrable* (or *integrable*).

Since $D(\mathcal{U}(x)^{1/2}) = D(\mathfrak{U}(x))$ by [14] Th. 2.23 p. 331, given $x \in \mathcal{A}_{int}$, we have $D(\mathfrak{U}(x)) = \mathcal{H}$ and then $\langle u, \mathcal{U}(x)u\rangle = \int_0^\infty \langle u, \mathcal{T}_s(x)u\rangle ds$ for all $u \in \mathcal{H}$.

PROPOSITION 4. *If $\mathcal{T}$ is a QMS and $x \in \mathcal{A}$ is positive, then the orthogonal projections onto the closure of $D(\mathfrak{U}(x))$ and onto $\mathcal{K}(x) = \{u \in D(\mathfrak{U}(x)) : \mathfrak{U}(x)[u] = 0\}$ are subharmonic.*

Proof. See Prop. 2 and 4 of [10]. ∎

For each operator x on $\mathcal{H}$, we call the orthogonal projection onto the closure of $x(\mathcal{H})$ the *range projection* of x and denote it by $[x]$; it is well-known that $x \in \mathcal{A}$ implies $[x] \in \mathcal{A}$.

Inspired by the notion of transient QMS given in [10] we give the following

DEFINITION 5. The *transient projection* associated with the QMS $\mathcal{T}$ is the projection p_T in $\mathcal{A}$ defined by $p_T := \sup_{p\in\mathcal{P}} p$, where $\mathcal{P} = \{\,[\mathcal{U}(x)] : x \in \mathcal{A}_{int}\,\}$.

This definition is original, as are all the next results.

The transient projection is orthogonal to p_R, indeed

PROPOSITION 5. *If $\mathcal{T}$ is a QMS on $\mathcal{A}$, then $p_T \le p_R^\perp$.*

Proof. Let ω be a normal invariant state and put $p = [\mathcal{U}(x)]$ with $x \in \mathcal{A}_{int}$; then

$$\int_0^\infty \omega(x)ds = \int_0^\infty \omega(\mathcal{T}_s(x))ds = \omega(\mathcal{U}(x)) \le \|\omega\| \cdot \|\mathcal{U}(x)\|$$

implies $\omega(\mathcal{U}(x)) = 0$. But ω is faithful on the subalgebra $s(\omega)\mathcal{A}s(\omega)$, so that this means $s(\omega)\mathcal{U}(x) = 0$, i.e. $\overline{\mathcal{U}(x)(\mathcal{H})} \subseteq \ker s(\omega)$; from the arbitrariness of ω it follows $p(\mathcal{H}) \subseteq \ker p_R$, so $p \le p_R^\perp$ for all $p \in \mathcal{P}$. Hence $p_T \le p_R^\perp$. ∎

By Prop. 4 any projection $[\mathcal{U}(x)]$ with x integrable is superharmonic, but it is not clear whether the supremum of a family of superharmonic projections is still superharmonic. However, when $\mathcal{A}$ is σ-finite, we will prove that p_T is superharmonic because we can write it as the supremum of an increasing sequence of superharmonic projections. We shall make use of the following

LEMMA 6. *If $e \in p_T(\mathcal{H})$, then there exists $x \in \mathcal{A}_{int}$ such that $e \in \overline{\mathrm{Ran}(\mathcal{U}(x))}$.*

Proof. By definition of p_T, for any $n \ge 1$ there exists $u_n \in p_n(\mathcal{H})$, $p_n = [\mathcal{U}(x_n)]$ ($x_n \in \mathcal{A}_{int}$), such that $\|e - u_n\| < n^{-1}$; therefore, if we put

$$x := \sum_{n\ge1} 2^{-n}(\|x_n\| + \|\mathcal{U}(x_n)\| + 1)^{-1}x_n,$$

we obtain an integrable element with $\ker \mathcal{U}(x) = \bigcap_{n\ge1} \ker \mathcal{U}(x_n)$ and $p := \sup_n p_n = [\mathcal{U}(x)]$. Moreover, since $u_n \in p(\mathcal{H})$, we get

$$\|e - pe\| \le \|e - u_n\| + \|pu_n - pe\| < 2n^{-1} \quad \forall\, n \ge 1,$$

which implies $e \in p(\mathcal{H})$. ∎

THEOREM 7. *Suppose $\mathcal{A}$ is σ-finite and let $\mathcal{T}$ be a QMS on $\mathcal{A}$. Then there exists an increasing sequence $(p_n)_{n\geq 0}$ in $\mathcal{P}$ such that $p_T = \sup_{n\geq 0} p_n$. Moreover $p_T \in \mathcal{P}$.*

Proof. Let $\{e_n\}_{n\geq 0}$ be a countable subset of $\mathcal{H}$ which is separating for $\mathcal{A}$; then, for all $n \geq 0$ there exists $x_n \in \mathcal{A}_{int}$ such that $p_T e_n \in \overline{\mathrm{Ran}(\mathcal{U}(x_n))}$ (see Lemma 6). If $y_n := \sum_{k=0}^{n} x_k$ and $p_n := [\mathcal{U}(y_n)]$ $(n \geq 0)$, we obtain an increasing sequence $(p_n)_{n\geq 0}$ in $\mathcal{P}$ with $p_T e_n \in \overline{\mathcal{U}(x_n)(\mathcal{H})} \subseteq \overline{\mathcal{U}(y_n)(\mathcal{H})} = p_n(\mathcal{H})$; therefore we have $(p_T - \sup_{m\geq 0} p_m)p_T e_n = 0$ for all $n \geq 0$, so $p_T = \sup_{n\geq 0} p_n$ because $\{p_T e_n\}_{n\geq 0}$ is separating for $p_T \mathcal{A} p_T$ and $p_T - \sup_{n\geq 0} p_n \in p_T \mathcal{A} p_T$.

Finally, put

$$y := \sum_{n\geq 0} 2^{-n}(\|y_n\| + \|\mathcal{U}(y_n)\| + 1)^{-1} y_n,$$

it is clear that $y \in \mathcal{A}_{int}$ and $\ker \mathcal{U}(y) = \bigcap_{n\geq 0} \ker \mathcal{U}(y_n) = \ker p_T$, so that $[\mathcal{U}(y)] = p_T$, i.e. $p_T \in \mathcal{P}$. ∎

COROLLARY 8. *If $\mathcal{A}$ is σ-finite and $\mathcal{T}$ is a QMS on $\mathcal{A}$, then its transient projection p_T is superharmonic. In particular, the subalgebra $p_T \mathcal{A} p_T$ is $\mathcal{T}$-invariant.*

We put $\mathcal{T}^T := \mathcal{T}_{|p_T \mathcal{A} p_T}$; then it is a submarkovian QDS on $p_T \mathcal{A} p_T$. If $(p_n)_{n\geq 0}$ is a sequence of projections as in Thm. 7, then the map $t \mapsto \langle u, \mathcal{T}_t(p_n) u\rangle$ is integrable on $[0, \infty)$ for all $u \in \mathcal{H}$; this implies that $\mathcal{T}_t(p_n)$ is strongly convergent to 0 as $t \to \infty$. Using this fact and the uniform boundeness in t of $\mathcal{T}_t$ we can easily show that $\mathcal{T}^T$ has no normal invariant states.

DEFINITION 6. The projection $p_{R_0} = p_R^{\perp} - p_T$ is called *slow recurrent projection* associated with the QMS $\mathcal{T}$.

4. Decomposition of QMSs

DEFINITION 7. We call a QMS $\mathcal{T}$

1. *irreducible* if it has no non-trivial subharmonic projections;
2. *transient* if $p_T = \mathbf{1}$;
3. *recurrent* if $p_T = 0$;
4. *fast recurrent* if $p_R = \mathbf{1}$;
5. *slow recurrent* if $p_{R_0} = \mathbf{1}$.

Notice that we can also define p_T, p_R and p_{R_0} for a QDS $\mathcal{T}$ on $\mathcal{A}$ such that $\mathcal{T}_t(\mathbf{1}) \leq \mathbf{1}$ for all $t \geq 0$; since it is easy to check that these projections satisfy the same properties, we can introduce the concepts of transience and recurrence for such semigroups too.

PROPOSITION 9. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. If $\mathcal{T}$ is irreducible, then it is either transient, or fast recurrent, or slow recurrent.*

Proof. If $\mathcal{T}$ is irreducible, since p_T is superharmonic we have either $p_T = \mathbf{1}$ or $p_T = 0$, that is, $\mathcal{T}$ is either transient or recurrent. On the other hand, if $p_T = 0$, since p_R is subharmonic we get either $p_R = \mathbf{1}$ or $p_R = 0$, that is, $\mathcal{T}$ is either fast or slow recurrent. ∎

Instead, in general a QMS $\mathcal{T}$ is not type $2, 3, 4, 5$ but, if $\mathcal{A}$ is σ-finite, we can write it as a sum of a transient and a recurrent semigroup. Indeed, we have the following

THEOREM 10. *If $\mathcal{A}$ is σ-finite and $\mathcal{T}$ is a QMS on $\mathcal{A}$, then $\mathcal{T}^T$ is a transient QDS on $p_T\mathcal{A}p_T$, while $\mathcal{T}^{p_T^\perp}$ is a recurrent QMS on $p_T^\perp\mathcal{A}p_T^\perp$. Moreover $\mathcal{T}^{p_R}$ is a fast recurrent semigroup on $p_R\mathcal{A}p_R$.*

We refer to Thm. 9 of [15] for the proof.

It is not yet clear if we can associate a semigroup with the slow recurrent projection p_{R_0} (we don't know if p_{R_0} is superharmonic or subharmonic) and, in this case, if such a semigroup is slow recurrent.

Since, for all projections $p \in \mathcal{A}$, $\mathfrak{U}(p)[u] = \int_0^\infty \langle u, \mathcal{T}_s(p)u\rangle ds$ represents the time of sojourn of the state $tr(|u\rangle\langle u|\cdot)$ ($\|u\| = 1$) in p (see [10]) and any normal state ω is defined by a density matrix $\sum_k \lambda_k |e_k\rangle\langle e_k|$ with $e_k \in s(\omega)(\mathcal{H})$, we can read the above theorem as follows:

- starting from a transient state (support in $p_T\mathcal{A}p_T$), the semigroup $\mathcal{T}_*$ spends a finite or an infinite amount of time in p_T but, if it leaves p_T to come into $p_T^\perp$ (i.e. its support is in $p_T^\perp\mathcal{A}p_T^\perp$), it stays there forever;
- starting from a recurrent state, the semigroup $\mathcal{T}_*$ cannot leave $p_T^\perp$.

In particular, starting from a fast recurrent state, the semigroup $\mathcal{T}_*$ cannot leave p_R.

We want now to decompose p_R as a sum of an arbitrary family of orthogonal $\mathcal{T}^{p_R}$-invariant projections $\{p_i\}$ such that any restriction of $\mathcal{T}^{p_R}$ to the subalgebra $p_i\mathcal{A}p_i$ is irreducible; such a decomposition is given in [6] for finite-dimensional algebras. We prove that this is possible if and only if there exists a faithful family of extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ with orthogonal supports. In this case, since $p_i\mathcal{A}p_i$ is $\mathcal{T}^{p_R}$-invariant, the equation

$$\mathcal{T}_t^{p_R}(x) = p_i\mathcal{T}_t^{p_R}(x)p_i = p_i\mathcal{T}_t(x)p_i = \mathcal{T}_t^{p_i}(x)$$

holds for all $x \in p_i\mathcal{A}p_i$, so that the restriction of $\mathcal{T}^{p_R}$ to $p_i\mathcal{A}p_i$ is the reduced semigroup $\mathcal{T}^{p_i}$ for all i. Moreover, given $\omega \in \mathcal{F}(\mathcal{T}_*)_1$ with $\omega(p_i) \neq 0$, we have that

$$(p_i\omega p_i)(\mathcal{T}_t^{p_i}(x)) = \omega(\mathcal{T}_t(x)) = \omega(x)$$

for all $x \in p_i\mathcal{A}p_i$. Hence, $\omega_i := \omega(p_i)^{-1}p_i\omega p_i$ is a normal $\mathcal{T}^{p_i}$-invariant state; also, from the irreducibility of $\mathcal{T}^{p_i}$, it follows that ω_i is faithful on $p_i\mathcal{A}p_i$, so that it is the unique normal invariant state on $p_i\mathcal{A}p_i$ by Thm. 1 of [11]. As a consequence, $\mathcal{T}^{p_R}$ is the direct sum of the irreducible "sub-QMS" $\mathcal{T}^{p_i}$ each one supporting a unique faithful normal invariant state.

LEMMA 11. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$; if ω is a normal state on $\mathcal{A}$ and p is a subharmonic projection such that $p \geq s(\omega)$, then:*

1. *ω is $\mathcal{T}$-invariant if and only if ω is $\mathcal{T}^p$-invariant;*
2. *ω is extremal in $\mathcal{F}(\mathcal{T}_*)_1$ if and only if ω is extremal in $\mathcal{F}(\mathcal{T}_*^p)_1$.*

THEOREM 12. *Let $\mathcal{T}$ be a QMS on $\mathcal{A}$. The following facts are equivalent:*

1. *there exists a set $\{p_i\}_{i\in I}$ of pairwise orthogonal projections such that:*
   a) $p_R = \sum_{i\in I} p_i$;
   b) $\mathcal{T}_t^{p_R}(p_i) = p_i$ for all $i \in I$;
   c) the restriction of $\mathcal{T}^{p_R}$ to the subalgebra $p_i\mathcal{A}p_i$ is irreducible for all $i \in I$.

2. there exists a faithful family of normal invariant states $\{\omega_i\}_{i\in I}$ *such that:*

a′) each ω_i *is an extremal point of* $\mathcal{F}(\mathcal{T}_*)_1$*;*

b′) $s(\omega_j)s(\omega_i)=0$ *for* $i\neq j,\ i,j\in I$.

Proof. $1\Rightarrow 2$. Fix $i\in I$; by the above remarks, there exists a unique faithful normal $\mathcal{T}^{p_i}$-invariant state ω_i on $p_i\mathcal{A}p_i$. Since ω_i must be extremal in $\mathcal{F}(\mathcal{T}_*^{p_i})_1$ by Thm. 1 of [11], we can conclude the proof by virtue of Lemma 11.

$2\Rightarrow 1$. Define $p_i:=s(\omega_i)$ $(i\in I)$; we obtain a set of pairwise orthogonal $\mathcal{T}^{p_R}$-harmonic projections, ω_i being a faithful invariant state on $p_i\mathcal{A}p_i$ and p_i subharmonic. Moreover, since $\omega_j(p_R-\sum_{i\in I}p_i)=0$ for all $j\in I$ and $\{\omega_i\}_{i\in I}$ is faithful, we get $p_R=\sum_{i\in I}p_i$.

Finally, for $i\in I$, ω_i is extremal in $\mathcal{F}(\mathcal{T}_*^{p_i})_1$ by virtue of Lemma 11 and $\mathcal{T}^{p_i}=\mathcal{T}^{p_R}_{|_{p_i\mathcal{A}p_i}}$ is irreducible by Thm. 1 of [11]. ■

Remark 1. If $\mathcal{A}$ is σ-finite, then we have card(I) $\leq\aleph_0$ by virtue of Prop. 2.5.6 of [3]. Therefore, in this case, $\mathcal{T}^{p_R}$ is a countable direct sum of irreducible semigroups.

We find now some conditions under which such a decomposition holds.

Proposition 13. *Let* $\mathcal{T}$ *be a QMS on* $\mathcal{A}$*. The equivalent conditions of Thm. 12 are satisfied if at least one of the following assumptions holds:*

- $\mathcal{F}(\mathcal{T}_*)$ *is finite dimensional,*
- $\mathcal{A}$ *is commutative and the family of extremal states of* $\mathcal{F}(\mathcal{T}_*)_1$ *is faithful on* $p_R\mathcal{A}p_R$.

Proof. Let $\{\omega_i\}_{i\in I}$ be a maximal family of extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ with pairwise orthogonal supports. Set $q:=\sum_{i\in I}s(\omega_i)$, which is majorized by p_R, and define $q':=p_R-q$. If $q\neq p_R$ we show that there exists an extremal state σ of $\mathcal{F}(\mathcal{T}_*)_1$ such that $s(\sigma)s(\omega_i)=0$ for all $i\in I$. Since this contradicts the maximality of $\{\omega_i\}_{i\in I}$, we would obtain that $q=p_R$ (and then $(\omega_i)_{i\in I}$ is a faithful family of extremal points of $\mathcal{F}(\mathcal{T}_*)_1$).

Let $\rho\in\mathcal{F}(\mathcal{T}_*)_1$ be such that $\rho(q')\neq 0$. Since $\mathcal{T}^{p_R}(q'\mathcal{A}q')\subseteq q'\mathcal{A}q'$ and $s(\rho)\leq p_R$, we have

$$q'\rho q'(\mathcal{T}_t(a))=\rho(\mathcal{T}_t^{p_R}(q'aq'))=\rho(\mathcal{T}_t(q'aq'))=q'\rho q'(a)$$

for all $a\in\mathcal{A}$, $t\geq 0$, that is, $\omega:=\rho(q')^{-1}q'\rho q'$ is a normal invariant state. Therefore, if $\mathcal{F}(\mathcal{T}_*)$ is finite-dimensional, and since ω is a convex combination of extremal points of $\mathcal{F}(\mathcal{T}_*)_1$ by Thm. 2.3.15 of [3], we have $q'\geq s(\omega)\geq s(\sigma)$ for some σ extremal in $\mathcal{F}(\mathcal{T}_*)_1$, which means $s(\sigma)s(\omega_i)=0$ for all $i\in I$.

On the other hand, if $\mathcal{A}$ commutative and the family of extremal states of $\mathcal{F}(\mathcal{T}_*)_1$ is faithful on $p_R\mathcal{A}p_R$, we can choose an extremal state ρ such that $\rho(q')\neq 0$. Fix $i\in I$: the condition $[s(\omega_i),s(\rho)]=0$ implies $s(\omega_i)\wedge s(\rho)=s(\omega_i)s(\rho)$; since $s(\omega_i)\wedge s(\rho)\leq s(\omega_i)$ is $\mathcal{T}^{s(\omega_i)}$-invariant and ω_i,ρ are extremal, by Thm. 1 of [11] we have that $\mathcal{T}^{s(\omega_i)}$ is irreducible, so $s(\omega_i)\wedge s(\rho)=0$, for $s(\omega_i)\leq q$. This means $s(\rho)s(\omega_i)=0$. ■

5. The finite-dimensional case. If $\mathcal{A}$ acts on a finite-dimensional Hilbert space $\mathcal{H}$, as in the case of Markov chains with finite state space, we get $p_R\neq 0$ and $p_{R_0}=0$. Moreover, p_T is integrable.

Proposition 14. *Suppose* $\dim\mathcal{H}<+\infty$*. If* $\mathcal{T}$ *is a QMS on* $\mathcal{A}$*, then its fast recurrent projection* p_R *is not zero.*

Proof. It is a trivial consequence of the Markov-Kakutani Theorem. ∎

COROLLARY 15. *If $\mathcal{T}$ is an irreducible QMS on $\mathcal{A}$ and $\mathcal{H}$ is finite-dimensional, then $\mathcal{T}$ is fast recurrent.*

Proof. Since $\mathcal{T}$ is irreducible, it can be either transient, or fast or slow recurrent by Prop. 9; but Prop. 14 implies $p_R \neq 0$, so that $p_R = \mathbf{1}$, i.e. $\mathcal{T}$ is fast recurrent. ∎

LEMMA 16. *If $\dim \mathcal{H} < +\infty$ and $\mathcal{T}$ is a QMS on $\mathcal{A}$, then $p_R^\perp \in \mathcal{A}_{int}$. In particular, $\mathcal{T}_t(p_R^\perp) \overset{t\to\infty}{\to} 0$.*

Proof. Let $x_0 \in p_R^\perp \mathcal{A} p_R^\perp$ be the positive limit of the decreasing net $\{\mathcal{T}_t(p_R^\perp)\}_{t\geq 0}$ ($p_R^\perp$ is superharmonic); therefore x_0 is harmonic. If we put

$$\mathcal{S}(\omega) = \lim_n \frac{1}{n}\sum_{k=1}^{n} \mathcal{T}_{*k}(\omega)$$

for all $\omega \in \mathcal{A}_* = \mathcal{A}^*$, then $\mathcal{S}(\omega) \in \mathcal{F}(\mathcal{T}_*)_+$, so $s(\mathcal{S}(\omega)) \leq p_R$. Hence

$$\omega(x_0) = \lim_n \frac{1}{n}\sum_{k=1}^{n} \omega(\mathcal{T}_1^k(x_0)) = \mathcal{S}(\omega)(x_0) = \mathcal{S}(\omega)(s(\mathcal{S}(\omega))x_0) = 0,$$

so that $x_0 = 0$. But $\mathcal{H}$ finite-dimensional implies that $\mathcal{T}_t(p_R^\perp)$ is also norm-convergent to 0, and then there exists $t_0 > 0$ such that $\|\mathcal{T}_{t_0}(p_R^\perp)\| < 1$; therefore, by $\|\mathcal{T}_t(p_R^\perp)\| \leq \|\mathcal{T}_{t_0}(p_R^\perp)\| < 1$ for all $t \geq t_0$, it follows that $\|\mathcal{T}_t(p_R^\perp)\| \leq c\exp(-t\alpha)$ for some $\alpha > 0$, $c > 0$, and for all $t \geq t_0$, so that finally

$$\int_0^\infty \|\mathcal{T}_t(p_R^\perp)\| dt \leq t_0 + \int_{t_0}^\infty \|\mathcal{T}_t(p_R^\perp)\| dt < \infty,$$

i.e. $p_R^\perp$ is integrable. ∎

THEOREM 17. *If $\dim \mathcal{H} < +\infty$ and $\mathcal{T}$ is a QMS on $\mathcal{A}$, then $p_{R_0} = 0$.*

Proof. Since by virtue of Lemma 16 $p_R^\perp$ is integrable, we have $p_T + p_{R_0} = p_R^\perp \leq [\mathcal{U}(p_R^\perp)] \leq p_T$, i.e. $p_{R_0} = 0$. ∎

COROLLARY 18. *Suppose $\dim \mathcal{H} < +\infty$. If $\mathcal{T}$ is a QMS on $\mathcal{A}$, then its transient projection p_T is integrable.*

We conclude this section with an application to a physical model: this is the open BCS model, where the system is described by spin variables and the reservoir is given in terms of bosonic operators (see [2]). It is contained in a box with N sites.

We show first some preliminary results which will be very useful to analyze the open BCS model.

We recall that, given a QMS $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$, we can consider the subalgebra

$$\mathcal{N}(\mathcal{T}) = \bigcap_{t\geq 0} \{\, a \in \mathcal{A} : \mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a),\ \mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*)\,\}.$$

If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{T}$ is uniformly continuous and its generator is represented in the Lindblad form, we have $\mathcal{N}(\mathcal{T}) = \{L_k, L_k^* : k \geq 0\}'$ (see Prop. 2.33 of [8]); moreover, if there exists a faithful normal invariant state, Prop. 2.32 of [8] implies that $\mathcal{F}(\mathcal{T}) = \{L_k, L_k^*, H : k \geq 0\}' \subseteq \mathcal{N}(\mathcal{T})$. We shall use these facts in the following

PROPOSITION 19. *Let* $\mathcal{A} = M_2(\mathbb{C})^{\otimes^N}$*, with* $N \geq 1$*. If* $\omega_1, \ldots, \omega_N$ *are faithful states on* $M_2(\mathbb{C})$*, then* $\omega_1 \otimes \ldots \otimes \omega_N$ *is a faithful state on* $\mathcal{A}$*.*

Proof. We denote by E_1 and E_2 the partial traces over $(\mathbb{C}^2)^{\otimes^{(N-1)}}$ and $\mathbb{C}^2$ respectively. It is clear that $(\omega_1 \otimes \ldots \otimes \omega_N)(\mathbf{1}) = (\omega_1 \otimes \ldots \otimes \omega_N)(\mathbf{1} \otimes \ldots \otimes \mathbf{1}) = 1$. We prove by induction on N that $\omega_1 \otimes \ldots \otimes \omega_N$ is positive and faithful: for $N = 1$, it is trivial. Suppose now $\omega_2 \otimes \ldots \otimes \omega_N$ positive and faithful on $M_2(\mathbb{C})^{\otimes^{N-1}}$, and denote by ρ its density; hence, if we denote by tr, tr_1 and tr_2 the normalized traces on the Hilbert spaces $(\mathbb{C}^2)^{\otimes^N}$, $\mathbb{C}^2$ and $(\mathbb{C}^2)^{\otimes^{(N-1)}}$ respectively, we have

$$(\omega_2 \otimes \ldots \otimes \omega_N)(b) = \mathrm{tr}_2(\rho b) = \mathrm{tr}_2(\rho^{1/2} b \rho^{1/2}) \tag{2}$$

for all $b \in M_2(\mathbb{C})^{\otimes^{N-1}}$. Let F be the $(\omega_1 \otimes \ldots \otimes \omega_N)$-preserving conditional expectation onto $M_2(\mathbb{C})$ given by

$$\begin{array}{rcl} F : M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{\otimes^{N-1}} & \rightarrow & M_2(\mathbb{C}) \otimes \mathbb{C} \\ a \otimes b & \mapsto & (\omega_2 \otimes \ldots \otimes \omega_N)(b)\, a \otimes \mathbf{1}. \end{array}$$

Therefore, identifying $M_2(\mathbb{C}) \otimes \mathbb{C}$ with $M_2(\mathbb{C})$, we have

$$(\omega_1 \otimes \ldots \otimes \omega_N)(a) = \omega_1(F(a))$$

for all $a \in \mathcal{A}$. In particular, if a is positive, $F(a)$ is also positive in $M_2(\mathbb{C})$, so that $\omega_1 \otimes \ldots \otimes \omega_N$ is a positive functional on $\mathcal{A}$.

Assume now that $(\omega_1 \otimes \ldots \otimes \omega_N)(a) = 0$, $a \in \mathcal{A}_+$. With the identification $\mathbb{C} \otimes M_2(\mathbb{C})^{\otimes^{(N-1)}} \simeq M_2(\mathbb{C})^{\otimes^{(N-1)}}$, the faithfulness of ω_1 and (2) imply

$$0 = F(a) = E_2((\mathbf{1} \otimes \rho^{1/2}) a (\mathbf{1} \otimes \rho^{1/2})).$$

Since $(\mathbf{1} \otimes \rho^{1/2}) a (\mathbf{1} \otimes \rho^{1/2})$ is positive and E_2 is faithful, we obtain that $(\mathbf{1} \otimes \rho^{1/2}) a (\mathbf{1} \otimes \rho^{1/2}) = 0$, and so

$$0 = \mathrm{tr}((\mathbf{1} \otimes \rho) a) = \mathrm{tr}_2(\rho E_1(a)) = (\omega_2 \otimes \ldots \otimes \omega_N)(E_1(a)).$$

Due to the faithfulness of $\omega_2 \otimes \ldots \otimes \omega_N$ and E_1, we have $a = 0$. This proves that $\omega_1 \otimes \ldots \otimes \omega_N$ is a faithful state on $\mathcal{A}$. ∎

PROPOSITION 20. *Let* $\mathcal{A} = M_2(\mathbb{C})^{\otimes^N}$ *and* $\mathcal{L}$ *be the linear map on* $\mathcal{A}$ *given by*

$$\mathcal{L}(x_1 \otimes \ldots \otimes x_N) = \sum_{j=1}^{N} x_1 \otimes \ldots \otimes \underbrace{\mathcal{L}_j(x_j)}_{j} \otimes \ldots x_N \quad \forall\, x_i \in M_2(\mathbb{C}), \tag{3}$$

where each $\mathcal{L}_j$ *is the generator of a uniformly continuous QMS* $\mathcal{T}^{(j)}$ *on* $M_2(\mathbb{C})$*. Then* $\mathcal{L}$ *generates a uniformly continuous QMS* $\mathcal{T}$ *on* $\mathcal{A}$ *defined by*

$$\mathcal{T}_t(x_1 \otimes \ldots \otimes x_N) = \mathcal{T}_t^{(1)}(x_1) \otimes \ldots \otimes \mathcal{T}_t^{(N)}(x_N). \tag{4}$$

Moreover, if we assume that:

1. *each* $\mathcal{T}^{(j)}$ *is irreducible and it possesses a (unique) faithful invariant state* ω_j*,*
2. $\mathcal{N}(\mathcal{T}^{(j)}) = \mathcal{F}(\mathcal{T}^{(j)})$ *for all* $j = 1, \ldots, N$*,*

then $\mathcal{T}$ *is irreducible and* $\omega_1 \otimes \ldots \otimes \omega_N$ *is the unique faithful invariant state of* $\mathcal{T}$*.*

Proof. If

$$\mathcal{L}_j(x) = \frac{1}{2}\Big(\sum_{k\geq 0}(L_k^{(j)})^* L_k^{(j)} x - 2\sum_{k\geq 0}(L_k^{(j)})^* x L_k^{(j)} + \sum_{k\geq 0} x L_k^{(j)*} L_k^{(j)}\Big) + i[H^{(j)}, x]$$

is the Lindblad form of $\mathcal{L}_j$, then $\mathcal{L}$ can be represented in the Lindblad form too taking

$$L_{j,k} = \mathbf{1} \otimes \ldots \otimes \underbrace{L_k^{(j)}}_{j} \otimes \ldots \mathbf{1},$$

$$H = \sum_{i=1}^{N} \mathbf{1} \otimes \ldots \otimes H^{(i)} \otimes \ldots \otimes \mathbf{1},$$

for all $j = 1, \ldots, N$ and $k \geq 0$. Therefore $\mathcal{L}$ generates a uniformly continuous QMS $\mathcal{T}$ on $\mathcal{A}$; it is easy to prove that $\mathcal{T}$ is given by (4) and

$$\mathcal{L}_*(\sigma_1 \otimes \ldots \otimes \sigma_N) = \sum_{j=1}^{N} \sigma_1 \otimes \ldots \otimes \underbrace{\mathcal{L}_{j*}(\sigma_j)}_{j} \otimes \ldots \sigma_N \quad \forall\, \sigma_i \in M_2(\mathbb{C}). \tag{5}$$

Suppose now that conditions 1, 2 hold. Hence $\omega_1 \otimes \ldots \otimes \omega_N$ is a faithful $\mathcal{T}$-invariant state thanks to (5) and Proposition 19. To conclude, it is enough to prove that $\mathcal{F}(\mathcal{T}) = \mathbb{C}\mathbf{1}$: indeed, in this case, since $\omega_1 \otimes \ldots \otimes \omega_N$ is a faithful $\mathcal{T}$-invariant state, Thm. 1 and Lemma 2 of [11] imply that $\mathcal{T}$ is irreducible and $\omega_1 \otimes \ldots \otimes \omega_N$ is the unique invariant state.

If $x \in \mathcal{F}(\mathcal{T})$, then in particular x commutes with each $L_{j,k}$, so that

$$\sum_{i_j,k_j=1}^{2} x_{(i_1,k_1),\ldots,(i_j,k_j),\ldots,(i_N,k_N)} E_{i_j}^{k_j} \in \{L_k^{(j)}, L_k^{(j)*} : k \geq 0\}' = \mathcal{N}(\mathcal{T}^{(j)})$$

for all $j = 1, \ldots, N$; since $\mathcal{N}(\mathcal{T}^{(j)}) = \mathcal{F}(\mathcal{T}^{(j)})$ and this last space is equal to $\mathbb{C}\mathbf{1}$ by Thm. 1 and Lemma 2 of [11], this means that $x_{(i_1,k_1),\ldots,(i_N,k_N)} = 0$ for $i_j \neq k_j$ and

$$x_{(i_1,k_1),\ldots\underbrace{(1,1)}_{j},\ldots,(i_N,k_N)} = x_{(i_1,k_1),\ldots,\underbrace{(2,2)}_{j},\ldots,(i_N,k_N)}$$

for all $j = 1, \ldots, N$. Therefore, we get $x_{(i_1,i_1),\ldots,(i_N,i_N)} = x_{(k_1,k_1),\ldots,(k_N,k_N)}$ for all $i_j, k_j \in \{1, 2\}$ and $j = 1, \ldots, N$, i.e. $\mathcal{F}(\mathcal{T}) = \mathbb{C}(\mathbf{1} \otimes \ldots \otimes \mathbf{1}) = \mathbb{C}\mathbf{1}$. ■

EXAMPLE 1. Let $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, $N \geq 1$, and $\mathcal{A} = \mathcal{B}(\mathcal{H}) \simeq M_2(\mathbb{C})^{\otimes N}$; denote $\sigma_i^\epsilon = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \underbrace{\sigma^\epsilon}_{i} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$, where $\epsilon = 0, \pm$, $i = 1, \ldots, N$ and

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We recall that $[\sigma_i^+, \sigma_j^-] = \delta_{ij}\sigma_i^0$ and $[\sigma_i^\pm, \sigma_j^0] = \mp 2\delta_{ij}\sigma_i^\pm$. The index i represents the discrete values of the momentum that an electron in a fixed volume can have, σ_i^+ creates a Cooper pair with given momentum while σ_i^- annihilates the same pair.

We define

$$\mathcal{L}(x) = \sum_{j=1}^{N} \sum_{\alpha=0,\pm} \{\Gamma_\alpha[\rho_\alpha^j, x]\rho_\alpha^{j*} + \Lambda_\alpha[\rho_\alpha^{j*}, x]\rho_\alpha^j - \overline{\Gamma}_\alpha \rho_\alpha^j[\rho_\alpha^{j*}, x] - \overline{\Lambda}_\alpha \rho_\alpha^{j*}[\rho_\alpha^j, x]\}$$

for all $x \in \mathcal{A}$, where $\Gamma_\alpha, \Lambda_\alpha \in \mathbb{C}$, $\Re\Gamma_\alpha, \Re\Lambda_\alpha > 0$, and $\rho_\alpha^j = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \underbrace{\rho_\alpha}_{j} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$ with

$$\rho_0 = \frac{g^2 S^+}{\omega^2}(2S^-\sigma^+ + S^0\sigma^0 + 2S^+\sigma^-),$$
$$\rho_+ = \frac{gS^+}{\omega^2}\left(gS^-\frac{\omega - gS^0}{\omega + gS^0}\sigma^+ + \frac{\omega - gS^0}{2}\sigma^0 - gS^+\sigma^-\right),$$
$$\rho_- = \frac{gS^+}{\omega^2}\left(gS^-\frac{\omega + gS^0}{\omega - gS^0}\sigma^+ - \frac{\omega + gS^0}{2}\sigma^0 - gS^+\sigma^-\right),$$

$\omega, S^0 \in \mathbb{R}$, $S^+, S^- \in \mathbb{C}$, $\omega \pm gS^0, S^+, S^-, \omega \neq 0$; $-g < 0$ is the interaction close to the Fermi surface.

Notice that $\mathcal{L}$ assumes the form (3) with $\mathcal{L}_k = \tilde{\mathcal{L}}$ for all $k = 1, \ldots, N$ and

$$\tilde{\mathcal{L}}(x) := \sum_{\alpha=0,\pm}\left\{\Gamma_\alpha[\rho_\alpha, x]\rho_\alpha^* + \Lambda_\alpha[\rho_\alpha^*, x]\rho_\alpha - \overline{\Gamma}_\alpha\rho_\alpha[\rho_\alpha^*, x] - \overline{\Lambda}_\alpha\rho_\alpha^*[\rho_\alpha, x]\right\}$$

for any $x \in M_2(\mathbb{C})$. Since $\tilde{\mathcal{L}}$ can be represented in the Lindblad form taking

$$L_{1,\alpha} = \sqrt{2\Re\Gamma_\alpha}\rho_\alpha^*, \qquad L_{2,\alpha} = \sqrt{2\Re\Lambda_\alpha}\rho_\alpha$$

for all $\alpha = 0, \pm$ and $H = 0$, it is the generator of a uniformly continuous QMS $\tilde{\mathcal{T}}$. Therefore, it follows from Prop. 20 that $\mathcal{L}$ generates a uniformly continuous QMS $\mathcal{T}$ on $\mathcal{A}$. We want to prove that $\tilde{\mathcal{T}}$ satisfies conditions $1, 2$ of the same Proposition, so that $\mathcal{T}$ is irreducible and it possesses a unique faithful normal invariant state.

Denote by p_R the fast recurrent projection of $\tilde{\mathcal{T}}$ and analyze the subharmonic projections of this semigroup; notice that, if p is such a projection, then $pL_{j,\alpha}p = L_{j,\alpha}p$ for $j = 1, 2$ means $p\rho_\alpha p = \rho_\alpha p$ and $p\rho_\alpha^* p = \rho_\alpha^* p$, $\alpha = 0, \pm$, that is,

$$p \in \{\rho_\alpha, \rho_\alpha^*\}' = \{L_{j,\alpha}, L_{j,\alpha}^* : j = 1, 2,\ \alpha = 0, \pm\}' = \mathcal{N}(\tilde{\mathcal{T}}).$$

But $H = 0$ implies $\mathcal{N}(\tilde{\mathcal{T}}) = \mathcal{F}(\tilde{\mathcal{T}})$, so that any subharmonic (and therefore superharmonic) projection is harmonic; in particular, the non-zero superhamonic projections are not integrable. As a consequence, since p_T is superharmonic and integrable (because $\mathcal{H}$ is finite-dimensional), we get $p_T = 0$ and then $p_R = p_T^\perp = \mathbf{1}$, i.e. $\tilde{\mathcal{T}}$ is fast recurrent. In particular there exists a faithful $\tilde{\mathcal{T}}$-invariant state ω (Corollary 1 of [15], $M_2(\mathbb{C})$ being a σ-finite algebra). Since a straightforward calculation shows that $\mathcal{F}(\tilde{\mathcal{T}}) = \mathbb{C}\mathbf{1}$, this implies that $\tilde{\mathcal{T}}$ is irreducible and ω is its unique normal invariant state by virtue of Thm. 1 and Lemma 2 of [11]. We have thus proved that conditions $1, 2$ of Prop. 20 are fulfilled; therefore, $\mathcal{T}$ is irreducible and its unique faithful normal invariant state is $\underbrace{\omega \otimes \ldots \otimes \omega}_{N \text{ times}}$.

6. Characterization of normal invariant states. In this section we analyze a classical situation: assume that there exists a set $\{p_n\}_{n \in N}$, $N \subset \mathbb{N}$, of orthogonal projections which satisfy:

1. $\sum_{n \in N} p_n = \mathbf{1}$,
2. $\mathcal{T}_t(p_n) = p_n$ for all $n \in N$,

3. the restriction of $\mathcal{T}$ to the subalgebra $p_n\mathcal{A}p_n$ is irreducible for all $n \in N$,
4. there exists a normal invariant state ρ_n with $s(\rho_n) \leq p_n$.

Since ρ_n is clearly also $\mathcal{T}_{|_{p_n\mathcal{A}p_n}}$-invariant, $s(\rho_n)$ is a subharmonic projection for the irreducible semigroup $\mathcal{T}_{|_{p_n\mathcal{A}p_n}}$ (see Prop. 1), so that ρ_n is faithful on $p_n\mathcal{A}p_n$ (i.e. $s(\rho_n) = p_n$); moreover, it is the unique normal invariant state on $p_n\mathcal{A}p_n$ by Thm. 1 of [11].

This implies that $\sum_{n\in N} 2^{-n}\rho_n$ is a faithful normal invariant state on $\mathcal{A}$ and then $p_R = \mathbf{1}$, so there exists a $\mathcal{T}$-invariant normal conditional expectation $\mathcal{E}$ onto $\mathcal{F}(\mathcal{T})$ by Thm. 3.

We wish now to determine when it is possible to write every normal invariant state in the form $\sum_{n\in N} \lambda_n\rho_n$ for some positive λ_n such that $\sum_{n\in N} \lambda_n = 1$.

In general, this is not the case, as the following example shows: take $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ separable with $\{e_m\}_{m\geq 1}$ a orthonormal basis, and $\mathcal{T}_t$ the identity map. Then, $\{|e_m\rangle\langle e_m|\}_{m\geq 1}$ is a sequence of orthogonal projections which fulfills the above properties by letting $\rho_m = |e_m\rangle\langle e_m|$, every state is invariant, yet not every normal state can be expressed as $\sum_{m\geq 1} \lambda_m |e_m\rangle\langle e_m|$.

THEOREM 21. *Let $(p_n)_{n\in N}$ be a set of orthogonal projections with* card$(N) \leq \aleph_0$. *If $(p_n)_{n\in N}$ satisfies 1-4, then the following conditions are equivalent:*

1. any normal invariant states on $\mathcal{A}$ has the form $\sum_{n\in N} \lambda_n\rho_n$ for some $\lambda_n \geq 0$ with $\sum_{n\in N} \lambda_n = 1$;
2. $\mathcal{F}(\mathcal{T}) = \overline{\mathrm{span}\{p_n : n \in N\}}$.

Proof. $1 \Rightarrow 2$. Let ω be a normal invariant state on $\mathcal{A}$; since $\omega(a) = \sum_{i,j\in N} \omega(p_i a p_j)$ for all $a \in \mathcal{A}$, and $p_n\omega p_n = \omega(p_n)\rho_n$ by Thm. 1 of [11] (because $p_n\omega p_n$ is a normal invariant functional on $p_n\mathcal{A}p_n$), it is enough to prove that $\mathcal{E}(p_i a p_j) = 0$ for all $i, j \in N$, $i \neq j$, for $\omega = \omega \circ \mathcal{E}$ by Thm. 3.

So, fix $a \in \mathcal{A}$ and $i \neq j$: since $\mathcal{E}(p_i a p_j) = \lim_k x_k$ with $x_k \in \mathrm{span}\{p_n : n \in N\}$, we have $p_l\mathcal{E}(p_i a p_j)p_n = 0$ for all $l \neq n$, and also $p_l\mathcal{E}(p_i a p_j)p_l = \mathcal{E}(p_l p_i a p_j p_l) = 0$ for all $l \in N$, for $p_l = \mathcal{E}(p_l)$ and $i \neq j$. Therefore, $\mathcal{E}(p_i a p_j) = 0$, as claimed.

$2 \Rightarrow 1$. Let $x \in \mathcal{F}(\mathcal{T})$, which is an algebra. Thus, $p_n x p_n \in \mathcal{F}(\mathcal{T})$, so that $p_n x p_n \in \mathcal{F}(\mathcal{T}_{|_{p_n\mathcal{A}p_n}})$; but $\mathcal{F}(\mathcal{T}_{|_{p_n\mathcal{A}p_n}}) = \mathbb{C}p_n$ by virtue of Thm. 1 of [11], so $p_n x p_n = \rho_n(x)p_n$ for all $n \in N$. We want to prove that $p_i x p_j = 0$ for $i, j \in N$, $i \neq j$.

Fix $i \neq j$ and $\omega \in \mathcal{A}_*$ a state: since $(t^{-1}\int_0^t \mathcal{T}_{*s}(\omega)ds)_{t\geq 0}$ is weakly convergent to a normal invariant state by Thm. 2.1 of [13], there exists a sequence $\{\lambda_n\}_{n\in N}$ of positive numbers such that $\sum_{n\in N} \lambda_n = 1$ and

$$\frac{1}{t}\int_0^t \mathcal{T}_{*s}(\omega)(p_i x p_j)ds \to \sum_{n\geq 0} \lambda_n\rho_n(p_i x p_j) = 0.$$

But this means $\omega(p_i x p_j) = 0$, because $p_i x p_j$ also belongs to $\mathcal{F}(\mathcal{T})$, and finally $p_i x p_j = 0$ by the arbitrariness of ω. Therefore,

$$x = \sum_{n\in N} p_n x p_n = \sum_{n\in N} \rho_n(x)p_n \in \overline{\mathrm{span}\{p_n : n \in N\}}. \blacksquare$$

References

[1] W. B. Averson, *Subalgebras of C^*-algebras*, Acta Math. 123 (1969), 141–224.

[2] F. Bagarello, *The stochastic limit in the analysis of the open BCS model*, J. Phys. A 37 (2004), 1537–1548.

[3] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[4] C. Dellacherie and P. A. Meyer, *Probabilités et potentiel. Chapitres XII-XVI*, second edition, Hermann, Paris, 1987.

[5] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969.

[6] D. E. Evans and R. Høegh-Krohn, *Spectral properties of positive maps on C^*-algebras*, J. London Math. Soc. (2) 17 (1978), 345–355.

[7] F. Fagnola, *Quantum Markov semigroups and quantum flows*, Proyecciones 18 (1999), 1–144.

[8] F. Fagnola and R. Rebolledo, *Lectures on the qualitative analysis of quantum Markov semigroups*, in: Quantum Interacting Particle Systems (Trento, 2000), QP-PQ Quantum Probab. White Noise Anal. 14, World Scientific, Singapore, 2002, 1–192.

[9] F. Fagnola and R. Rebolledo, *Subharmonic projections for a quantum semigroups*, J. Math. Phys. 43 (2002), 1074–1082.

[10] F. Fagnola and R. Rebolledo, *Transience and recurrence of quantum Markov semigroups*, Probab. Theory and Relat. Fields 126 (2003), 289–306.

[11] A. Frigerio, *Quantum dynamical semigroups and approach to equilibrium*, Lett. Math. Phys. 2 (1977), 79–87.

[12] A. Frigerio, *Stationary states of quantum dynamical semigroups*, Comm. Math. Phys. 63 (1978), 269–276.

[13] A. Frigerio and M. Verri, *Long-time asymptotic properties of dynamical semigroups on W^*-algebras*, Math. Z. 180 (1982), 275–286.

[14] T. Kato, *Perturbation Theory for Linear Operators*, Corr. printing of the 2nd ed., Springer-Verlag, New York, 1980.

[15] V. Umanità, *Classification and decomposition of quantum Markov semigroups*, Probab. Theory and Relat. Fields, 2004, to appear.