QUANTUM PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 73
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2006

THE VON NEUMANN ALGEBRA ASSOCIATED WITH AN INFINITE NUMBER OF t-FREE NONCOMMUTATIVE GAUSSIAN RANDOM VARIABLES

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Abstract. We show that the von Neumann algebras generated by an infinite number of t-deformed free gaussian operators are factors of type I_∞.

1. Introduction. In [6] we constructed, for each positive real number t, families of non-commutative random variables associated with the central limit measures for t-transformed classical and free convolutions. In this paper we shall study the families related to t-transformed free convolution, in the von Neumann algebras' framework. Let us briefly recall the constructions.

For $t \geq 0$ and a given separable Hilbert space $\mathcal{H}$, (being the complexification of a real Hilbert space $\mathcal{H}_\mathbb{R}$), with the scalar product $\langle\cdot|\cdot\rangle$, we consider the Fock space

$$\mathcal{F}_t(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} \mathcal{H}^{\otimes n}$$

completed with respect to the following scalar product $\langle\cdot|\cdot\rangle_t$:

$$\langle x_1 \otimes x_2 \otimes \ldots \otimes x_n | y_1 \otimes y_2 \otimes \ldots \otimes y_k\rangle_t = \delta_{n,k} \cdot t^{n-1} \cdot \prod_{j=1}^{n} \langle x_j|y_j\rangle, \quad \langle\Omega|\Omega\rangle_t = 1.$$

Now, given a vector $f \in \mathcal{H}_\mathbb{R}$, we define a creation operator $B_t(f)$ and annihilation operator $A_t(f)$ on $\mathcal{F}_t(\mathcal{H})$. For arbitrary $x_1, x_2, \ldots, x_n \in \mathcal{H}$ we put

$$B_t(f) x_1 \otimes x_2 \otimes \ldots \otimes x_n = f \otimes x_1 \otimes x_2 \otimes \ldots \otimes x_n, \quad B_t(f)\Omega = f$$

2000 *Mathematics Subject Classification*: 47C15, 60F05.

Key words and phrases: t-deformation, free convolution, von Neumann factors.

Research partially supported by KBN grant 2P03A00723 and by EU Network "QP-Applications", contract HPRN-CT-2002-00279.

The paper is in final form and no version of it will be published elsewhere.

where $n \geq 1$, and

$$A_t(f)\Omega = 0, \quad A_t(f)x_1 = \langle x_1|f\rangle\Omega,$$
$$A_t(f)x_1 \otimes x_2 \otimes \ldots \otimes x_n = t \cdot \langle x_1|f\rangle x_2 \otimes \ldots \otimes x_n,$$

where $n \geq 2$. Then for every $f \in \mathcal{H}_{\mathbb{R}}$ the operators $A_t(f)$ and $B_t(f)$ are bounded by $\max\{1, \sqrt{t}\} \cdot \|f\|$, and are adjoint to each other and we shall consider the self-adjoint operators $G_t(f) = A_t(f) + B_t(f)$, which are thus bounded by $2 \cdot \max\{1, \sqrt{t}\}\|f\|$.

Definition 1.1. By $\mathcal{M}_t$ we shall denote the von Neumann algebra generated by the set $\{G_t(f) : f \in \mathcal{H}\}$, that is its double commutant in $\mathcal{B}(\mathcal{F}_t(\mathcal{H}))$, the C*-algebra of all bounded operators on $\mathcal{H}$:

$$\mathcal{M}_t := \{G_t : f \in \mathcal{H}\}'' \subset \mathcal{B}(\mathcal{F}_t(\mathcal{H})).$$

For $t = 1$, which is the case of free convolution, it was shown by Voiculescu that $\mathcal{M}_1 = VN(\mathbb{F}_k)$ is the von Neumann type II_1 factor of the free group $\mathbb{F}_k$ on k free generators. We shall show that for $t \neq 1$ the situation is quite different, if the number of operators $G_t(f)$ is infinite.

2. Main result.

Our main result is the following.

Theorem 2.1. *For $0 < t \neq 1$ the von Neumann algebra $\mathcal{M}_t$ is the type I_∞ factor $\mathcal{M}_t = \mathcal{B}(\mathcal{F}_t(\mathcal{H}))$.*

In proving the theorem the crucial role is played by the fact that the orthogonal projection P onto the vacuum Ω is in $\mathcal{M}_t$. This follows from the following:

Lemma 2.2. *Let $S_t := t \cdot I + (1 - t) \cdot P$, where P is the orthogonal projection onto the vacuum Ω and I is the identity operator in $\mathcal{B}(\mathcal{F}_t(\mathcal{H}))$. Let $G_i := G_t(x_i)$, where $\{x_i : i \geq 1\}$ is an orthonormal basis in $\mathcal{H}$, be a sequence of operators in $\mathcal{M}_t$. Then the sequence*

$$K_n := \frac{1}{n}\sum_{i=0}^{n}(G_i)^2$$

converges to S_t in the strong operator topology, when $n \to \infty$.

Proof. Since the operators G_i are all uniformly bounded, it is sufficient to show that $K_n(y) \to S_t(y)$ for any simple tensor of the form $y = x_{j_1} \otimes \ldots \otimes x_{j_m}$, with $m \geq 1$, or $y = \Omega$. It follows directly from the definition of the creation and annihilation operators $B_t(x_i)$ and $A_t(x_i)$ that:

$$K_n\Omega = \Omega + \frac{1}{n}\sum_{i=0}^{n} x_i \otimes x_i \to \Omega,$$

the convergence being in the norm of $\mathcal{F}_t(\mathcal{H})$. On the other hand, in computing the limit of $\frac{1}{n}\sum_{i=0}^{n}(G_i)^2(y)$ we consider only the simple tensors $y \in \mathcal{F}_t(\mathcal{H})$ of the form $y = x_{j_1} \otimes \ldots \otimes x_{j_m}$, $m \geq 1$, in which case $\frac{1}{n}\sum_{i=0}^{j_1}(G_i)^2(y) \to 0$ in norm, and $\frac{1}{n}\sum_{i=j_1+1}^{n}(G_i)^2(y) = \frac{1}{n}\sum_{i=j_1+1}^{n}(ty + x_i \otimes x_i \otimes y) \to ty$ in the norm of $\mathcal{F}_t(\mathcal{H})$. Hence the lemma follows. ∎

It follows from the lemma that $S_t \in \mathcal{M}_t$, hence also $(S_t)^2 - tS_t = (1-t)P \in \mathcal{M}_t$. Now we shall show that the vacuum vector Ω is cyclic for $\mathcal{M}_t$. This will yield that the commutant $\mathcal{M}_t'$ is trivial.

LEMMA 2.3. *The vacuum vector Ω is cyclic for the von Neumann algebra $\mathcal{M}_t$, which means that the linear span of the vectors $\{G_t(f)\Omega : f \in \mathcal{H}_{\mathbb{R}}\}$ is dense in $\mathcal{F}_t(\mathcal{H})$.*

Proof. This follows the well known scheme used in the free case, since for any finite sequence of indices $i_1, i_2, \ldots i_m$ we have the formula

$$y = G_{i_1} G_{i_2} \ldots G_{i_m} \Omega = x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_m} + \sum_{j=0}^{m-1} y_j$$

where y_j is the orthogonal projection of y onto the subspace $\mathcal{H}^{\otimes j}$ of $\mathcal{F}_t(\mathcal{H})$, spanned by tensors of length j (i.e. tensors of the form $x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_j}$). It follows by induction on m that each tensor $x_{i_1} \otimes x_{i_2} \otimes \ldots \otimes x_{i_m}$ can be expressed as a linear combination of vectors of the form $G_{r_1} G_{r_2} \ldots G_{r_s} \Omega$. This proves that Ω is cyclic. ∎

Now a standard argument shows that if the orthogonal projection onto a cyclic vector for a von Neumann algebra belongs to the algebra, then its commutant is trivial.

PROPOSITION 2.4. *The commutant $\mathcal{M}_t'$ of $\mathcal{M}_t$ in $\mathcal{B}(\mathcal{F}_t(\mathcal{H}))$ consists only of multiples of identity.*

Proof. For a given $K \in \mathcal{M}_t'$ we have $K\Omega = KP\Omega = PK\Omega$, so $K\Omega$ is invariant for the orthogonal projection P onto Ω. Thus $K\Omega = c\Omega$ for some constant c.

Now, for a vector $f \in \mathcal{F}_t(\mathcal{H})$ there exists a sequence $G_n \in \mathcal{M}_t$ such that $f = \lim_n G_n \Omega$. Then

$$K(f) = \lim_n K G_n \Omega = \lim_n G_n (K\Omega) = \lim_n G_n (c\Omega) = c \lim_n G_n \Omega = c \cdot f$$

which proves that $K = c \cdot I$. Since K was chosen arbitrary, it follows that $\mathcal{M}_t' = \{c \cdot I\}$ is trivial. ∎

Proof of Theorem 2.1. Since in $\mathcal{M}_t$ there is the orthogonal projection P onto the vector Ω cyclic for $\mathcal{M}_t$, it follows from the above Lemmas and Proposition that $\mathcal{M}_t = ((\mathcal{M}_t)')' = \mathcal{B}(\mathcal{F}_t(\mathcal{H}))$. This proves the theorem. ∎

3. Final remarks

REMARK 3.1. The natural vacuum state φ_t on $\mathcal{M}_t$, defined as $\varphi_t(K) = (K\Omega|\Omega)_t$ for $K \in \mathcal{M}_t$, is not tracial, since, for example, $\varphi_t((G_i)^2 (G_j)^2) = 1$ while $\varphi_t(G_i (G_j)^2 G_i) = t$. Of course, in general there is no trace on $\mathcal{M}_t$ if $\mathcal{M}_t = \mathcal{B}(\mathcal{F}_t(\mathcal{H}))$. Moreover, this state is not faithful on $\mathcal{M}_t$, since for $Y = 1 - P = Y^*$ we have $\varphi_t(Y^* Y) = \varphi_t(1-P) = 0$.

REMARK 3.2. Quanhua Xu [8] showed a general fact, that in the interacting Fock space determined by a sequence $(\lambda_n)_{n=0}^{\infty}$ the vacuum state is tracial if and only if the sequence is constant.

REMARK 3.3. Recently Eric Ricard [7] has given the description of the von Neumann algebras generated by finite number of the t-gaussian operators.

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