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PAIRS OF CONVEX BODIES IN A HYPERSPACE OVER A MINKOWSKI TWO-DIMENSIONAL SPACE JOINED BY A UNIQUE METRIC SEGMENT

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Abstract. Let $(\mathbb{R}^n, \|\cdot\|_{\mathbb{B}})$ be a Minkowski space with a unit ball $\mathbb{B}$ and let $\varrho_H^{\mathbb{B}}$ be the Hausdorff metric induced by $\|\cdot\|_{\mathbb{B}}$ in the hyperspace $\mathcal{K}^n$ of convex bodies (nonempty, compact, convex subsets of $\mathbb{R}^n$). R. Schneider [3] characterized pairs of elements of $\mathcal{K}^n$ which can be joined by unique metric segments with respect to $\varrho_H^{B^n}$ for the Euclidean unit ball B^n. We extend Schneider's theorem to the hyperspace $(\mathcal{K}^2, \varrho_H^{\mathbb{B}})$ over any two-dimensional Minkowski space.

1. Preliminaries and introduction. A *Minkowski space* (see [5]) is a finite dimensional normed linear space $(\mathbb{R}^n, \|\cdot\|)$.

Let $\mathbb{B}$ be the unit ball determined by the norm $\|\cdot\|$:

$$\mathbb{B} := \{x \in \mathbb{R}^n \mid \|x\| \le 1\}.$$

Then $\mathbb{B}$ is a convex body symmetric at 0 with nonempty interior. Conversely, every convex body A symmetric at 0 with nonempty interior determines a norm, $\|\cdot\|_A$, usually referred to as the Minkowski functional:

$$\|x\|_A := \inf\{t \in \mathbb{R}_+ \mid x \in tA\}$$

(see [5], p. 17).

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Let $\mathcal{K}^n$ be the family of nonempty compact convex subsets of $\mathbb{R}^n$. Following R. Schneider (see [3]), we will refer to the elements of $\mathcal{K}^n$ as convex body.

Let $\varrho_H^{\mathbb{B}}$ be the Hausdorff metric in $\mathcal{K}^n$ associated with the metric $\varrho^{\mathbb{B}}$ induced by the norm $\|\cdot\|_{\mathbb{B}}$ (compare [5]):

$$\varrho_H^{\mathbb{B}}(A_1, A_2) := \max\{\inf\{\varepsilon > 0 \mid A_1 \subset A_2 + \varepsilon\mathbb{B}\}, \inf\{\varepsilon > 0 \mid A_2 \subset A_1 + \varepsilon\mathbb{B}\}\}$$

for every $A_1, A_2 \in \mathcal{K}^n$.

In the Euclidean case the Hausdorff metric is denoted by ϱ_H.

Let (X, ϱ) be a metric space. For any $a, b \in X$ a point $c \in X$ such that

$$\varrho(a, c) = \varrho(c, b) = \frac{1}{2}\varrho(a, b)$$

is called a *metric midpoint in* (X, ϱ) of the pair (a, b). A *metric segment* (*in* (X, ϱ)) with endpoints a, b is a subset of X isometric to the interval $[0, \varrho(a, b)]$.

The *affine segment* in $\mathcal{K}^n$ with endpoints $K, L \in \mathcal{K}^n$ is defined by the formula

$$\triangle(K, L) := \{(1 - t)K + tL \mid t \in [0, 1]\}.$$

The *affine midpoint* of the pair (K, L) is the set $\frac{1}{2}(K + L)$.

In the following theorem R. Schneider characterized pairs of convex bodies with unique metric segments joining them in $(\mathcal{K}^n, \varrho_H)$.

THEOREM 1.1 (Schneider [3]). *Let $K, L \in \mathcal{K}^n$ be sets joined by a unique metric segment with respect to the metric ϱ_H determined by the Euclidean metric ϱ. Then either*

(i) $K = L + \lambda B^n$ *or* $L = K + \lambda B^n$ *for some* $\lambda > 0$ *or else*
(ii) $\dim K < n$ *and* $L = K + u$ *for some* $u \in \mathbb{R}^n$ *orthogonal to* $\operatorname{aff} K$.

Let us recall that for any nonempty compact convex sets K, L in $(\mathcal{K}^n, \varrho_H)$ either K, L can be joined by a unique metric segment or there exists an infinite family of metric segments joining K and L. A pair (K, L) has a unique metric segment if and only if it has a unique metric midpoint (see [2] p. 244).

In this paper we give necessary and sufficient conditions for a pair (K, L) of elements of $\mathcal{K}^2$ to have a unique metric segment joining them with respect to $\varrho_H^{\mathbb{B}}$. In Section 2 we present these conditions and give many examples. In Section 3 we prove the sufficiency and in Section 4 the necessity of our conditions.

For $A \in \mathcal{K}^n$ the sets $\operatorname{bd}A, \operatorname{cl}A, \operatorname{int}A$ and $\operatorname{relint}A$ are the boundary, closure, interior and relative interior of A, $\operatorname{conv}A$ is the convex hull of A and $\operatorname{aff}A$ is the smallest affine subspace containing A. For distinct points $a, b \in \mathbb{R}^n$ let $\triangle(a, b)$ be the usual segment with endpoints a, b. For $A_1, A_2 \in \mathcal{K}^n$ we denote $A_1 \vee A_2 := \operatorname{conv}(A_1 \cup A_2)$.

Let $A \subset \mathbb{R}^n$ be a convex set. A *face* of A is a convex subset $F \subset A$ such that each segment $\triangle(x, y) \subset A$ with $F \cap \operatorname{relint}\triangle(x, y) \neq \emptyset$ is contained in F or, equivalently, such that $x, y \in A$ and $(x + y)/2 \in F$ implies $x, y \in F$. If $\{e\}$ is a face of A, then e is called an *extreme point* of A. In other words, e is an extreme point of A if and only if it cannot be written in the form $e = (1 - \lambda)x + \lambda y$ with $x, y \in A$, $x \neq y$ and $\lambda \in (0, 1)$. (see [4]).

Let $A \subset \mathbb{R}^n$ and let $H \subset \mathbb{R}^n$ be a hyperplane. We say that H supports A at x if $x \in A \cap H$ and A is contained in one of the closed half-spaces bounded by H. We denote the half-space containing A by E^+ and the other half-space by E^-.

For $f \in (\mathbb{R}^n)^*$, real linear function on $\mathbb{R}^n$, we denote by

$$H_f A = \{x \in A \mid f(x) = \sup_{y \in A} f(y)\}$$

the support set of $A \in \mathcal{K}^n$ with respect to f.

Let e be an extreme point of a set $A \in \mathcal{K}^2$ with nonempty interior. Then either there exists a unique line supporting A at e or there exists an infinite family $\mathcal{H}(e)$ of lines supporting A at e. In this family there are two limit lines $\overleftarrow{H}(e)$ and $\overrightarrow{H}(e)$ such that $\overleftarrow{H}(e) \cup \overrightarrow{H}(e) = \mathrm{bd}(\bigcup \mathcal{H}(e))$.

The following theorem is well known and we will apply it in Section 4:

THEOREM 1.2. *Let $K, L \in \mathcal{K}^n$, $L \nsubseteq K$ and $K \nsubseteq L$. Then there exists a hyperplane $H \subset \mathbb{R}^n$ supporting all the sets $(1-t)K + tL$, where $t \in [0,1]$.*

2. Necessary and sufficient conditions. In the Euclidean case there are two classes of pairs of sets with a unique metric segment. In the Minkowski case the matter is much more complex. Even in a plane we obtain four classes of pairs of sets. Not all of them relate to classes described by Schneider.

All pairs of elements of $\mathcal{K}^2$ joined by a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ are of the form $\{K, K + \lambda F\}$, where F is a face of the unit ball $\mathbb{B}$ and $\lambda > 0$. Moreover, K and F satisfy certain additional conditions. Since these conditions are quite complicated, we collect them in the following table:

Table of Conditions

	(a)	(b)	(c)
1	$F = \mathbb{B}$	K arbitrary	-
2	$F = \{e\}$	$K = \{x\}$	-
3	$F = \{e\}$	There exist points $a, b, c, d \in \mathbb{R}^2$ such that: i) $\triangle(a,c) \subset K \subset \mathrm{conv}\{a,b,c,d\}$ ii) $\triangle(a,d) \| \triangle(b,c) \| \overleftarrow{H}(e)$ iii) $\triangle(a,b) \| \triangle(d,c) \| \overrightarrow{H}(e)$ iv) there exists $H \in \mathcal{H}(e)$ such that $\triangle(a,c) \| H$	If $\overleftarrow{H}(e) \cap \mathbb{B} \neq \{e\}$ and $\overrightarrow{H}(e) \cap \mathbb{B} \neq \{e\}$, then i) $\triangle(a,d) \cap K = \{a\}$ or $\triangle(a,b) \cap K = \{a\}$ and ii) $\triangle(c,d) \cap K = \{c\}$ or $\triangle(b,c) \cap K = \{c\}$
4	$F = \triangle(e_1, e_2)$	There exist points $a, b, c \in \mathbb{R}^2$ such that: i) $\triangle(a,b) \subset K \subset \mathrm{conv}\{a,b,c\}$ ii) $\triangle(a,b) \| \triangle(e_1, e_2)$ iii) $\triangle(a,c) \| \overrightarrow{H}(e_1)$ iv) $\triangle(b,c) \| \overleftarrow{H}(e_2)$	i) If $\overrightarrow{H}(e_1) \cap \mathbb{B} \neq \{e_1\}$, then $\triangle(a,c) \cap K = \{a\}$ and ii) If $\overleftarrow{H}(e_2) \cap \mathbb{B} \neq \{e_2\}$, then $\triangle(b,c) \cap K = \{b\}$

In the four rows of the table we listed all conditions that have to be met by respective classes of pairs.

Now we shall consider several examples of pairs of convex bodies satisfying the conditions listed in the table. In the following figures, thick lines show such bodies. Thin lines show some elements of unique metric segments. Since the shape of the unit ball is crucial, we also present unit balls $\mathbb{B}$.

EXAMPLE 2.1 (Condition 1). Let $K \subset \mathbb{R}^2$ be a triangle. Figures 1.a and 1.b show pairs $(K, K + \lambda\mathbb{B})$, for some $\lambda > 0$, which satisfy Condition 1.

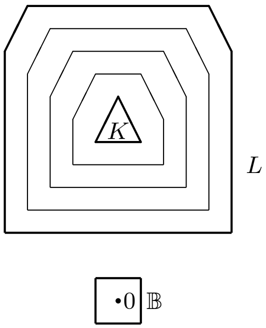

Fig. 1.a

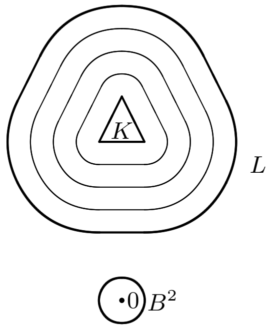

Fig. 1.b

These pairs consist of K and a parallel body of K. Figure 1.b illustrates one of the cases described by Schneider (compare with [1]). If we take a non-Euclidean unit ball $\mathbb{B}$ (see Figure 1.a), then we obtain a different parallel body of K. Example 2.1 corresponds to (i) from Theorem 1.1.

EXAMPLE 2.2 (Condition 2). The set F is a zero-dimensional face of the unit ball $\mathbb{B}$. Pairs of sets satisfying Condition 2 are pairs of singletons, and segments joining them consist of singletons, too. In the Euclidean case (Figure 2.b), all pairs of singletons have a unique metric segment joining them. If $\mathbb{B}$ contains a segment in its boundary, then the pair $(\{x\}, \{x + e\})$ can be joined by a unique metric segment if and only if e is a zero-dimensional face of $\mathbb{B}$ (see Figure 2.a).

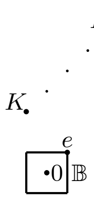

Fig. 2.a

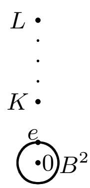

Fig. 2.b

This example corresponds to class (ii) from Theorem 1.1.

EXAMPLE 2.3 (Condition 3). Since Condition 3 is very complex, we divide it into subconditions 3(a), 3(b) and 3(c), which have to be satisfied simultanously. The set F is again a zero-dimensional face of the unit ball $\mathbb{B}$ (Condition 3(a)).

The body K is contained in the parallelogram with vertices a, b, c, d. The sides of the parallelogram are parallel to the limit lines $\overleftarrow{H}(e)$ and $\overrightarrow{H}(e)$ supporting $\mathbb{B}$. The diagonal $\triangle(a, c)$ is parallel to one of the lines supporting $\mathbb{B}$ in e. This diagonal should be contained in K (Condition 3(b)).

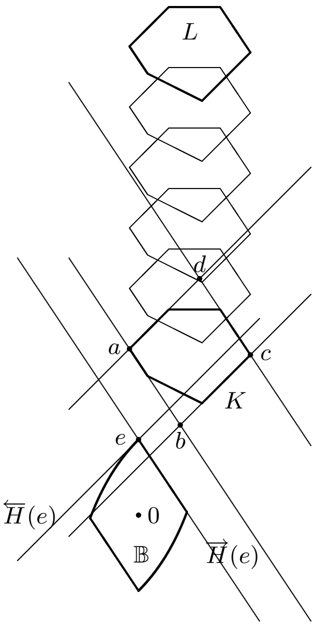

Fig. 3.a

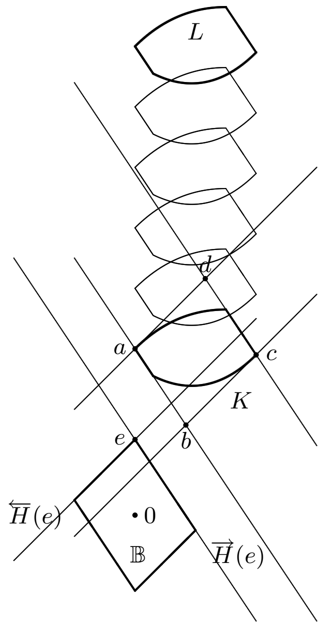

Fig. 3.b

If the point e is an endpoint of at most one segment contained in bd$\mathbb{B}$, then any point of the boundary of the parallelogram $a \vee b \vee c \vee d$ can belong to K (Figure 3.a).

If the point e is the endpoint of two segments contained in bd$\mathbb{B}$, then K is disjoint from relint $\triangle$ (a, b) or from relint $\triangle$ (a, d). Also K is disjoint from relint $\triangle$ (c, b) or from relint $\triangle$ (c, d) (Figure 3.b).

The pair of sets in Figure 3.a does not satisfy the assumption of Condition 3(c) but the pair of sets in Figure 3.b does.

This case does not correspond to any case described by R. Schneider.

Example 2.4 (Condition 3). It can happen that the parallelogram $a \vee b \vee c \vee d$ degenerates to a segment. If $\mathbb{B}$ is a Euclidean ball, then we obtain case (ii) from Schneider's theorem (Figure 4.a).

If the ball $\mathbb{B}$ can be supported at some point e of the boundary by different straight lines, then we can obtain segments like K and L (Figures 4.b and 4.c). These two examples depend on which side of the parallelogram $a \vee b \vee c \vee d$ degenerates to a point. These two examples correspond to case (ii) from Schneider's theorem.

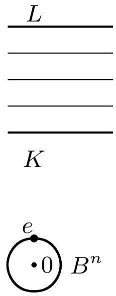

Fig. 4.a

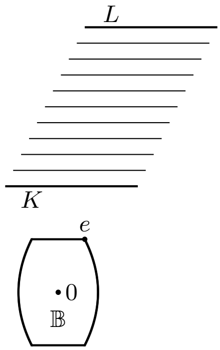

Fig. 4.b

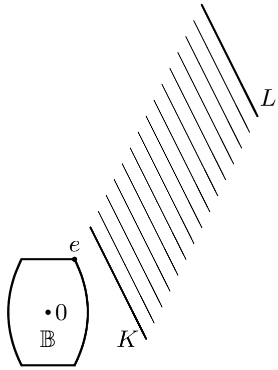

Fig. 4.c

Example 2.5 (Condition 4). This case is complex and we have to consider Conditions 4(a), 4(b) and 4(c) from the Table of Conditions. The set F is a segment $\triangle(e_1, e_2) \subset \mathbb{B}$. Then we have three different lines supporting $\mathbb{B}$ at points e_1, e_2. Condition 4(b) says that K is contained in a triangle $a \vee b \vee c$ whose sides are parallel to the limit lines supporting $\mathbb{B}$ at points e_1, e_2. The side $\triangle(a, b)$ parallel to $\triangle(e_1, e_2)$ must be contained in K (see Figures 5.a and 5.b).

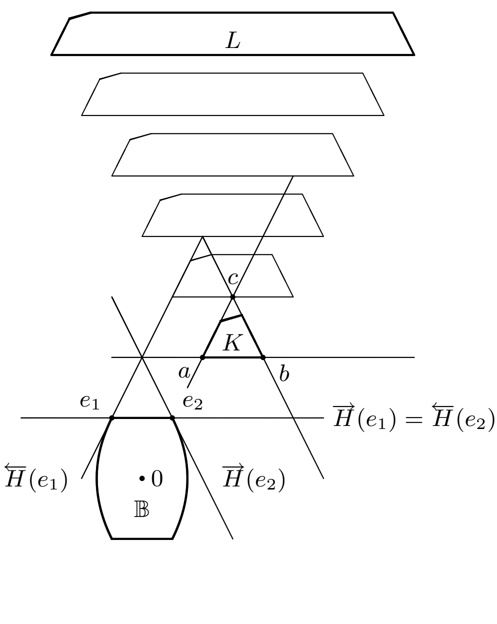

Fig. 5.a

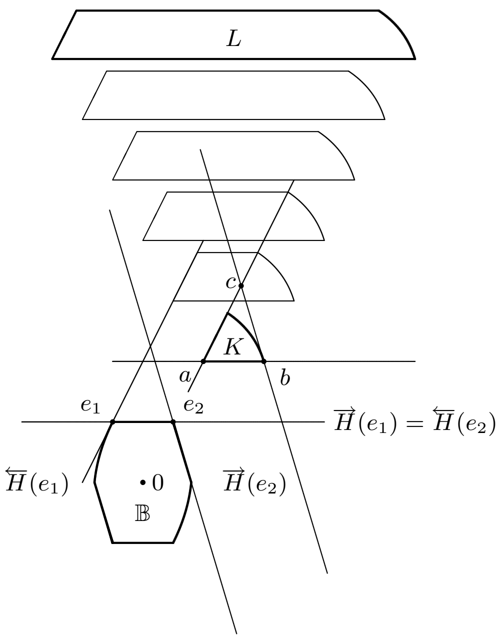

Fig. 5.b

If no segment contained in bd$\mathbb{B}$ other than $\triangle(e_1, e_2)$ contains e_1 or e_2, than any point of the triangle $a \vee b \vee c$ can be contained in K (see Figure 5.a).

If the point e_1 or e_2 contains a segment contained in bd$\mathbb{B}$ other than $\triangle(e_1, e_2)$, then the set K has to be disjoint from relint $\triangle\,(a, c)$ or relint $\triangle\,(b, c)$ (see Figure 5.b).

This example does not correspond to any case of Schneider.

EXAMPLE 2.6 (Condition 4). In Figure 6.a two of three limit lines supporting $\mathbb{B}$ at points e_1, e_2 are identical. Then the triangle $a \vee b \vee c$ reduces to a segment parallel to F. It can also happen that two of three limit lines supporting $\mathbb{B}$ at points e_1, e_2 are parallel. Then $\mathbb{B}$ is a parallelogram and the triangle $a \vee b \vee c$ transforms to a half-strip (see Figure 6.b).

Also this example does not correspond to any case of Schneider.

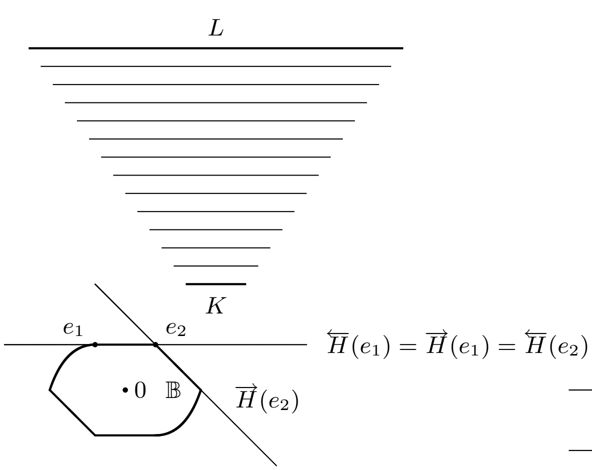

Fig. 6.a

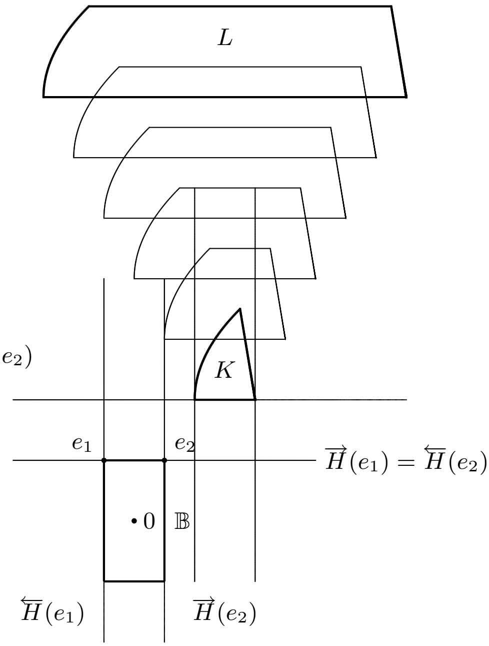

Fig. 6.b

3. Sufficiency. In section 2 we described four types of pairs of elements of $\mathcal{K}^2$; in this section we prove that for every such pair there is a unique metric segment in $(\mathcal{K}^2, \varrho_H^{\mathbb{B}})$ joining the elements of this pair.

PROPOSITION 3.1. *Let $K \in \mathcal{K}^n$ and let $F = \mathbb{B}$; then, for every $\lambda > 0$, the pair of sets $\{K, K + \lambda F\}$ has a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them.*

The proof of Proposition 3.1 is analogous to that for $\mathcal{K}^n$ (see [3]).

PROPOSITION 3.2. *Let $K = \{x\} \in \mathcal{K}^n$ and let $F = \{e\}$ be a face of the unit ball $\mathbb{B}$. Then, for every $\lambda > 0$, the pair of sets $\{K, K + \lambda F\}$ has a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them.*

Proof. It suffices to prove that the pair $\{\{x\}, \{x + \lambda e\}\}$ has a unique metric midpoint with respect to $\varrho_H^{\mathbb{B}}$.

With no loss of generality we may assume that $\lambda = 2$. Then $\varrho_H^{\mathbb{B}}(\{x\}, \{x + 2e\}) = 2$. The singleton $\{x + e\}$ is the affine midpoint of the pair $\{\{x\}, \{x + 2e\}\}$.

The point e is an extreme point of $\mathbb{B}$. Therefore $x+e$ is an exposed point of the balls $(x+\mathbb{B})$ and $(x+2e+\mathbb{B})$. Then

$$\{x+e\} = (x+\mathbb{B}) \cap (x+2e+\mathbb{B}).$$

The singleton $\{x+e\}$ is the greatest metric midpoint of the pair $\{\{x\},\{x+2e\}\}$ in the sense of inclusion (see [1]), so it is the unique metric midpoint of $\{\{x\},\{x+2e\}\}$. ∎

The assumption that $\{e\}$ is a face of $\mathbb{B}$ is essential in Proposition 3.2. It is easy to see that if $\{e\}$ is not a face of $\mathbb{B}$, then in the set $(x+\mathbb{B}) \cap (x+2e+\mathbb{B})$ there exist points other than $x+e$.

Proposition 3.3. *Let $K \in \mathcal{K}^2$ and let $F = \{e\}$ be a face of the unit ball $\mathbb{B}$. If F and K satisfy Conditions* 3(a), 3(b) *and* 3(c) *from the Table of Conditions, then, for every $\lambda > 0$, the pair of sets $\{K, K+\lambda F\}$ has a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them.*

Proof. Let $\lambda = 2$ and assume that K fulfills the assumptions of Proposition 3.3 (see Figures 3.a and 3.b). Let K have nonempty interior. Then the boundary bd$K$ is the union of two arcs joining the points a and c. Then the "upper" arc of bd$K$ is contained in bd$(K+\mathbb{B})-e$ and the "lower" arc is contained in bd$(K+\mathbb{B})+e$. Therefore, $K = (K+e+\mathbb{B}) \cap (K-e+\mathbb{B})$. Hence $K+e = (K+\mathbb{B}) \cap (K+2e+\mathbb{B})$. This implies that any metric center M between K and $K+2e$ is contained in $K+e$.

Let x be an exposed point of K. If $x \neq a, x \neq b$ and x belongs to the "upper" arc of bd$K$, then $x+2e \in K+2e$ and $(x+2e+\mathbb{B}) \cap (K+e) = x+e$. Let $y = x+e$. Then $x+2e \in y+\mathbb{B}$ and $x+2e \notin y_1+\mathbb{B}$ for every $y_1 \in K+e$ such that $y_1 \neq y$. Since $K+2e \subset M+\mathbb{B}$, it follows that $x+e \in M$. In a similar way we can prove that M contains $x+e$ for an exposed point x of the "lower" arc of bd$K$ (x is not equal to a or b). If $\overleftarrow{H}(e) \cap \mathbb{B} = e$, then $(a+2e+\mathbb{B}) \cap (K+e) = a+e$ and $(c+\mathbb{B}) \cap (K+e) = c+e$ and $a+e, c+e \in M$. In a similar way $\overrightarrow{H}(e) \cap B = e$ implies that $a+e, c+e \in M$. If $\triangle(a,d) \cap K = a$ or $\triangle(a,b) \cap K = a$, then $(a+2e+\mathbb{B}) \cap (K+e) = a+e$ or $(a+\mathbb{B}) \cap (K+e) = a+e$. Hence $a+e \in M$. If $\triangle(c,d) \cap K = c$ or $\triangle(b,c) \cap K = c$, then $(c+2e+\mathbb{B}) \cap (K+e) = c+e$ or $(c+\mathbb{B}) \cap (K+e) = c+e$. Hence $c+e \in M$. We have just proved that M contains all exposed points of $K+e$. Therefore, $M = K+e$.

If K has an empty interior (see Figures 4.a, 4.b and 4.c), then $K = \triangle(a,c)$ and we see that $(K+\mathbb{B}) \cap (K+2e+\mathbb{B}) = \triangle(a+e, c+e) = K$ and that both points $a+e$ and $c+e$ have to belong to any metric center M. ∎

Proposition 3.4. *Let $K \in \mathcal{K}^2$ and let $F = \triangle(e_1, e_2)$ be a face of the unit ball $\mathbb{B}$. If F and K satisfy Conditions* 4(a), 4(b) *and* 4(c) *from the Table of Conditions, then, for some $\lambda > 0$, the pair of sets $\{K, K+\lambda F\}$ has a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them.*

Proof. Let $\lambda = 2$ and assume that K fulfills the assumptions of Proposition 3.4 (see Figures 5.a and 5.b). Let K have an nonempty interior and d be a point of $K \setminus \triangle(a,b)$ belonging to the straight line supporting K. Then the boundary bd$K$ is the union of $\triangle(a,b)$ and two arcs, a "left" arc l joining a and d and a "right" arc r joining d and b. Hence $(l+e_1) \cup (d+F) \cup (r+e_2) \subset$bd$(K+\mathbb{B})$ and $\triangle(a+e_1, b+e_2) \subset$ bd$(K+2F+\mathbb{B})$.

Therefore, $K + F = (K + \mathbb{B}) \cap (K + 2F + \mathbb{B})$. This implies that any metric center M of the pair $(K, K + 2F)$ is contained in $K + F$.

Let x be an exposed point of $(K+F)\setminus\triangle(a+e_1, b+e_2)$. If $x \in l+e_1$, then $x+e_1 \in K+2F$ and $(x + e_1 + \mathbb{B}) \cap (K + F) = x$. Hence $x + e_1 \in x + \mathbb{B}$ and $y + e_1 \notin x + \mathbb{B}$ for every $y \in K + F$ and $y \neq x$. Since $K + 2F \subset M + \mathbb{B}$, it follows that $x \in M$. If $x \in r + e_2$, then in a similar way we can prove that $x \in M$.

If $\overleftarrow{H}(e_1) \cap \mathbb{B} = e_1$ or $\triangle(a, c) \cap K = \{a\}$, then $(a + 2e_1 + \mathbb{B}) \cap (K + F) = a + e_1$ Hence $a + e_1 \in M$. In a similar way, if $\overrightarrow{H}(e_2) \cap \mathbb{B} = e_2$ or $\triangle(b, c) \cap K = \{b\}$, then $b + e_2 \in M$. We have just proved that M contains all exposed points of $K + F$. Therefore, $M = K + F$.

If K has an empty interior (see Figures 6.a and 6.b), then $K = \triangle(a, b)$ and we see that $(K + \mathbb{B}) \cap (K + 2F + \mathbb{B}) = \triangle(a + e_1, b + e_2) = K$ and both points $a + e_1$ and $b + e_2$ have to belong to any metric center M. ■

Notice that the following theorem is a simple corollary of Propositions 3.1–4.

THEOREM 3.5. *Let $K \in \mathcal{K}^2$ and let F be a face of the unit ball $\mathbb{B}$. If K and F satisfy one of Conditions 1–4 from the Table of Conditions, then, for every $\lambda > 0$, the pair of sets $\{K, K + \lambda F\}$ has a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them.*

4. Necessity. We are now going to prove that every pair of elements of $\mathcal{K}^2$ with a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them is of one of the four types described in Section 2.

We shall need the following seven lemmas. The first two hold true for any $n \geq 2$.

LEMMA 4.1. *Let $K, L, C \in \mathcal{K}^n$ and let $L \not\subseteq K$ and $K \not\subseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K, L) and let H be the common hyperplane supporting K, L and C* (see Theorem 1.2). *Then there exist $k \in K$, $l \in L$, $c \in C$ and a line $S \subset H$ such that $\{k\} = S \cap K$, $\{l\} = S \cap L$ and $\{c\} = S \cap C$.*

Proof. We may assume that $\rho_H^{\mathbb{B}}(K, L) = 2$. By assumption, $C = (K + \mathbb{B}) \cap (L + \mathbb{B}) = \frac{1}{2}(K + L)$. By Theorem 1.2, $K \cup L \cup C \subset E^+$. Let $k \in K \cap H$, $l \in L \cap H$, $c \in C \cap \mathrm{aff}(k, l)$. Suppose that there exists $c' \neq c$ such that $c' \in C \cap \mathrm{aff}(k, l)$. Then, for any $\varepsilon > 0$, there exist $m, n \in E^-$ such that $m \in k + \varepsilon\mathbb{B}$, $n \in l + \varepsilon\mathbb{B}$ and points m, n, c, c' are coplanar.

Notice that $\triangle(m, c) \cap \triangle(n, c') \neq \varnothing$ or $\triangle(m, c') \cap \triangle(n, c) \neq \varnothing$ for sufficiently small ε. Let $c_1 \in \triangle(m, c) \cap \triangle(n, c')$. Then $c_1 \in (K + \mathbb{B}) \cap (L + \mathbb{B}) \cap \mathrm{int}E^- = C \cap \mathrm{int}E^-$, which contradicts the assumptions.

Analogously, we prove that $\{k\} = K \cap \mathrm{aff}(k, l)$ and $\{l\} = L \cap \mathrm{aff}(k, l)$. Then $S := \mathrm{aff}(k, l)$. ■

LEMMA 4.2. *Let $K, L, C \in \mathcal{K}^n$ and let $L \not\subseteq K$ and $K \not\subseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K, L) and let H be the common hyperplane supporting K, L and C. Let $c \in H \cap C$. Then*

$$\mathrm{dist}(c, K) = \mathrm{dist}(c, L) = \frac{1}{2}\rho_H^{\mathbb{B}}(K, L).$$

Proof. We may assume that $\rho_H^{\mathbb{B}}(K, L) = 2$, $l \neq k$. Certainly, $\mathrm{dist}(c, K) \leq 1$. Suppose that $\mathrm{dist}(c, K) < 1$. Then $c + \varepsilon\mathbb{B} \subset K + \mathbb{B}$ for some $\varepsilon > 0$. By Lemma 4.1, there exist a line $S \subset H$ and points $\{k\} = S \cap K$, $\{l\} = S \cap L$ and $\{c\} = S \cap C$. The segment

$\triangle(c,l) \subset L + \mathbb{B}$. Hence, $\triangle(c,l) \cap (c+\varepsilon B) \subset \triangle(k,l) \cap C$ which is impossible by Lemma 4.1. Then $\operatorname{dist}(c,K) = 1$.

Analogously, we prove that $\operatorname{dist}(c,L) = 1$. ∎

LEMMA 4.3. *Let $K, L, C \in \mathcal{K}^2$. Suppose $L \nsubseteq K$ and $K \nsubseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K,L) and let H be the common line supporting K, L and C. Let $H \cap K = \{k\}$, $H \cap L = \{l\}$, $H \cap C = \{c\}$. Then*

$$\rho^{\mathbb{B}}(c,k) = \rho^{\mathbb{B}}(c,l) = \frac{1}{2}\rho_H^{\mathbb{B}}(K,L).$$

Proof. We may assume that $\rho_H^{\mathbb{B}}(K,L) = 2$ and $l \neq k$. Lemma 4.2 implies $\rho^{\mathbb{B}}(c,k) \geq 1$. Suppose that $\rho^{\mathbb{B}}(c,k) > 1$. By Lemma 4.2, $\operatorname{dist}(c,K) = 1$. Hence $(c+\mathbb{B}) \cap K \neq \emptyset$ and $(c + \operatorname{int}\mathbb{B}) \cap K = \emptyset$. Let $k_1 \in (c+\mathbb{B}) \cap K$. By our assumption, $k_1 \neq k$.

Let H_1 be a line which separates K and $c+\mathbb{B}$ and let f_1 be a linear function on $\mathbb{R}^2$ such that $f_1(H_1)$ is a singleton and $f_1(c) > f_1(H_1)$. Then $k_1 \in (c+\mathbb{B}) \cap K \subset H_1 \cap K = H_{f_1}K$ and $k_1 \in (c+\mathbb{B}) \cap K \subset H_1 \cap (c+\mathbb{B}) = H_{-f_1}(c+\mathbb{B})$. Hence $k_1 - c \in H_{-f_1}\mathbb{B}$ and $c - k_1 \in H_{f_1}\mathbb{B}$. Then $c = k_1 + (c-k_1) \in H_{f_1}K + H_{f_1}\mathbb{B} = H_{f_1}(K+\mathbb{B})$.

The set C is contained in $K+\mathbb{B}$, hence $c \in H_{f_1}C$. By the assumptions of Lemma 4.3, the segment $\triangle(k,l)$ is the unique segment with endpoints belonging to K and L and with the midpoint c. Since $H_{f_1}C = \frac{1}{2}(H_{f_1}K + H_{f_1}L)$, it implies that $k \in H_{f_1}K$ and $l \in H_{f_1}L$. Hence $\triangle(k,k_1) \subset H_{f_1}K$ and $\triangle(c, \frac{1}{2}(k_1+l)) \subset H_{f_1}C$.

Analogously, there exists $l_1 \in (c+\mathbb{B}) \cap L$ such that $l_1 \neq l$ and there exists a line H_2 which separates $c+\mathbb{B}$ and L. Let f_2 be a linear function on $\mathbb{R}^2$ such that $f_2(H_2)$ is a singleton and $f_2(c) > f_2(H_2)$. Then $\triangle(l,l_1) \subset H_{f_2}L$ and $\triangle(c, \frac{1}{2}(k+l_1)) \subset H_{f_2}C$.

Let now $C_1 := \operatorname{cl}(C \setminus \triangle(c, \frac{1}{2}(k_1+l), \frac{1}{2}(k+l_1))$. We shall show that C_1 is another metric midpoint of (K,L). Notice that $\operatorname{dist}(c,H_1) = 1$. Hence $\operatorname{dist}(l, H_1 + c - k) = 1$. Since $f_1(c) = f_1(H_1 + c - k)$, it follows that $H_{f_1}C \subset H_1 + c - k$ and hence $\operatorname{dist}(l,C) \geq 1$. Since $L \subset C + \mathbb{B}$, we have $\operatorname{dist}(l,C) = 1$. Then there exists $c_1 \in C$ such that $\operatorname{dist}(l,C) = \rho^{\mathbb{B}}(l,c_1)$ and $c_1 \in H_1 + c - k$. Since $k_1 + l - c \in H_1 + c - k$ and $\rho^{\mathbb{B}}(k_1+l-c, l) = 1$, it follows that $c_1 \in \triangle(k_1 + l - c, \frac{1}{2}(k_1+l))$, whence $l \in C_1 + \mathbb{B}$ and $c_1 \in C_1$.

Analogously, we prove that there exists $c_2 \in C_1$ such that $\operatorname{dist}(K, c_2) = 1$ and $c_2 \in \triangle(l_1 + k - c, \frac{1}{2}(k+l_1))$. Hence $k \in C_1 + \mathbb{B}$.

Then $L \subset C_1 + \mathbb{B}$ and $K \subset C_1 + \mathbb{B}$ and $L \nsubseteq C_1 + \varepsilon\mathbb{B}$ and $K \nsubseteq C_1 + \varepsilon\mathbb{B}$ for $0 < \varepsilon < 1$. Certainly, $C_1 \subset C = (K+\mathbb{B}) \cap (L+\mathbb{B})$. The set C_1 is another metric midpoint of the pair (K,L). This contradicts the assumption of this lemma. Finally, $\rho^{\mathbb{B}}(c,k) = 1$.

Analogously, we prove that $\rho^{\mathbb{B}}(c,l) = 1$. ∎

LEMMA 4.4. *Let $K, L, C \in \mathcal{K}^2$. Suppose $L \nsubseteq K$ and $K \nsubseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K,L) and let H be the common line supporting K, L and C. Let $H \cap K = \{k\}$, $H \cap L = \{l\}$, $H \cap C = \{c\}$. Then*

$$\operatorname{dist}(k,C) = \operatorname{dist}(l,C) = \frac{1}{2}\rho_H^{\mathbb{B}}(K,L).$$

Proof. Assume that $\rho_H^{\mathbb{B}}(K,L) = 2$. By Lemma 4.2 $\operatorname{dist}(c,L) = 1$. Hence $\operatorname{dist}(k,C) \geq \operatorname{dist}(k, L+\mathbb{B}) \geq \operatorname{dist}(k,L) - 1 = \operatorname{dist}(k,L) + \operatorname{dist}(l,L) - 1 \geq \operatorname{dist}(k+l, L+L) - 1 \geq 2\operatorname{dist}(c,L) - 1 = 1$. On the other hand $\operatorname{dist}(k,C) \leq \rho_H^{\mathbb{B}}(K,C) = 1$. ∎

Lemma 4.5. *Let $K, L, C \in \mathcal{K}^2$. Suppose $L \not\subseteq K$ and $K \not\subseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K, L). For distinct $c_1, c_2 \in (C \cap \operatorname{bd}(K \vee L))$ where $c_i = \frac{1}{2}(k_i + l_i), k_i \in K, l_i \in L$, the line $\operatorname{aff}(c_1, c_2)$ supports the balls $k_i + \mathbb{B}, l_i + \mathbb{B}$ for $i = 1, 2$.*

Proof. Assume that $\rho_H^{\mathbb{B}}(K, L) = 2$. By Lemma 4.4 the segment $\triangle(c_1, c_2)$ is disjoint from $\operatorname{int}(k_1 + \mathbb{B})$ and $\operatorname{int}(l_1 + \mathbb{B})$. Then $\operatorname{int}(k_1 + \mathbb{B}) = \operatorname{int}(k_1 - \mathbb{B}) = (k_1 + l_1) - \operatorname{int}(l_1 + \mathbb{B}) = 2c_1 - \operatorname{int}(l_1 + \mathbb{B})$ is disjoint from $2c_1 - \triangle(c_1, c_2) = \triangle(c_1, 2c_1 - c_2)$. Hence the line $\operatorname{aff}(c_1, c_2)$ supports the ball $k_1 + \mathbb{B}$. ■

Lemma 4.6. *Let $K, L, C \in \mathcal{K}^2$. Let $L \not\subseteq K$ and $K \not\subseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K, L). If $\operatorname{card}(C) > 1$, then $\operatorname{card}(C \cap \operatorname{bd}(K \vee L)) = 2$.*

Proof. Notice that $\operatorname{bd}(K \vee L)$ is homeomorphic to a circle. Each point of $\operatorname{bd}(K \vee L) \setminus (K \cup L)$ lies in a relative interior of a segment contained in $\operatorname{bd}(K \vee L)$ with endpoints belonging to K and L. By Lemmas 4.1 and 4.2, such a segment has the length $\rho_H^{\mathbb{B}}(K, L)$ and its midpoint belongs to C. Hence $\operatorname{card}(C \cap \operatorname{bd}(K \vee L)$ is finite. There are a finite number of segments described above and contained in $\operatorname{bd}(K \vee L)$ with relative interiors disjoint from $K \cup L$. Therefore $\operatorname{card}(C \cap \operatorname{bd}(K \vee L))$ is an even number.

Suppose that $\operatorname{card}(C \cap \operatorname{bd}(K \vee L)) > 2$. Let c_1, c_2, c_3, c_4 be different elements of $C \cap \operatorname{bd}(K \vee L)$ and let $D := \operatorname{conv}(c_1, c_2, c_3, c_4)$ with c_1, c_2, c_3, c_4 subsequent vertices of D. In view of Lemma 4.3, points c_1, c_2, c_3, c_4 belong to four different segments contained in $\operatorname{bd}(K \vee L) \cap C$. Thus, there exist $k_1, k_2, k_3, k_4 \in K$ and $l_1, l_2, l_3, l_4 \in L$ such that $c_i = \frac{1}{2}(k_i + l_i)$ for $i = 1, 2, 3, 4$.

We may assume that $\rho_H^{\mathbb{B}}(K, L) = 2$. By Lemma 4.4 the lines $H = \operatorname{aff}(c_1, c_2)$ and $H_1 = \operatorname{aff}(c_1, c_4)$ support the ball $k_1 + \mathbb{B}$ and the lines H and $H_2 = \operatorname{aff}(c_2, c_3)$ support the ball $k_2 + \mathbb{B}$. Hence lines $H_1 - k_1$ and $H_2 - k_2$ support $\mathbb{B}$ at $c_1 - k_1$ and $c_2 - k_2$, respectively. Moreover, $(H - k_1) \cap \mathbb{B} = \triangle(c_1 - k_1, c_2 - k_2)$. Denote $F = (H - k_1) \cap \mathbb{B}$. The unit ball $\mathbb{B}$ is centrally symmetric, thus $-F \subset \mathbb{B}$. Hence the sum of angles at two adjacent vertices of D is less than or equal to π. Therefore D is a parallelogram and $\mathbb{B}$ is a summand of D.

Hence $C = D$ and $\mathbb{B} = I + J$, where I and J are two segments with midpoint 0 and length greater than or equal to 0. Thus $C = x + \alpha I + \beta J$ for some $x \in \mathbb{R}^2$ and $\alpha, \beta \geq 1$.

Since $\triangle(c_1, c_2)$ and $\triangle(c_3, c_4)$ are parallel sides of D, it follows that the lines $\operatorname{aff}(k_1, k_2)$ and $\operatorname{aff}(k_3, k_4)$ support K and $\operatorname{aff}(l_1, l_2)$, $\operatorname{aff}(l_3, l_4)$ support L. Analogously, $\operatorname{aff}(k_2, k_3)$, $\operatorname{aff}(k_1, k_4)$ support K and $\operatorname{aff}(l_2, l_3)$, $\operatorname{aff}(l_1, l_4)$ support L. Thus $K = x + (\alpha - \varepsilon)I + (\beta + \delta)J$ and $K = x + (\alpha + \varepsilon)I + (\beta - \delta)J$ for some $\varepsilon, \delta > 0$. Consider $C_1 = x + ((\alpha - \varepsilon)I + \beta J) \vee (\alpha I + (\beta - \delta)J)$. Notice that $C_1 + \mathbb{B} = x + (\alpha I + (\beta + 1)J) \vee ((\alpha + 1)I + \beta J)$ contains $K \vee L$. The set C_1 is a metric midpoint of the pair (K, L) different than C. This contradicts the assumption. ■

Lemma 4.7. *Let $K, L, C \in \mathcal{K}^2$ with $L \not\subseteq K$ and $K \not\subseteq L$. Let C be the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K, L), $\rho_H^{\mathbb{B}}(K, L) = 2$ and $\operatorname{card}(C) > 1$. Then there exist two lines H_1 and H_2 supporting the sets K, L, C at the unique points k_1, l_1, c_1 and k_2, l_2, c_2 respectively. Moreover, $(k_i + \mathbb{B}) \cap C = \{c_i\}$ and $(l_i + \mathbb{B}) \cap C = \{c_i\}$ for $i = 1, 2$ and:*

i) If $H_1 \parallel H_2$, then $e := c_1 - l_1 = c_2 - l_2$ is a zero-dimensional face of the unit ball $\mathbb{B}$, $C = L + e$ and $K = L + 2e$. The set C is contained in the parallelogram D bounded by $\overleftarrow{H}(c_1)$, $\overleftarrow{H}(c_1) + (c_2 - c_1)$, $\overrightarrow{H}(c_1)$ and $\overrightarrow{H}(c_1) + (c_2 - c_1)$, where the lines $\overleftarrow{H}(c_1)$ and $\overrightarrow{H}(c_1)$ support the balls $l_1 + \mathbb{B}$ and $k_1 + \mathbb{B}$ at the point c_1 and the lines $\overleftarrow{H}(c_1) + (c_2 - c_1)$ and $\overrightarrow{H}(c_1) + (c_2 - c_1)$ support the balls $l_2 + \mathbb{B}$ and $k_2 + \mathbb{B}$ at the point c_2 (see Figure 7.a).

ii) If $H_1 \nparallel H_2$, then $F := \triangle((c_1 - l_1), (c_2 - l_2))$ is one-dimensional face of the unit ball $\mathbb{B}$, $C = L + F$ and $K = L + 2F$ or $C = K + F$ and $L = K + 2F$. The set C is contained in the triangle D bounded by H, $\overrightarrow{H}(c_1)$ and $\overleftarrow{H}(c_2)$, where $H = \overleftarrow{H}(c_1) = \overrightarrow{H}(c_2)$ (or by H, $\overleftarrow{H}(c_1)$ and $\overrightarrow{H}(c_2)$, where $H = \overrightarrow{H}(c_1) = \overleftarrow{H}(c_2)$). The lines $\overleftarrow{H}(c_1)$ and $\overrightarrow{H}(c_1)$ support the balls $l_1 + \mathbb{B}$ and $k_1 + \mathbb{B}$ at the point c_1 and the lines $\overleftarrow{H}(c_2)$ and $\overrightarrow{H}(c_2)$ support the balls $l_2 + \mathbb{B}$ and $k_2 + \mathbb{B}$ at the point c_2 (see Figure 7.b).

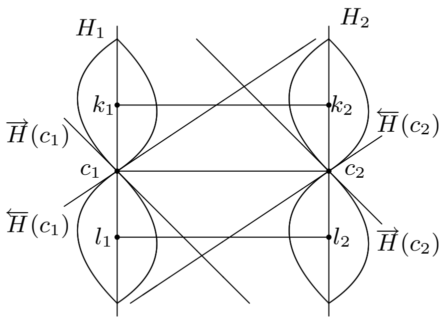

Fig. 7.a

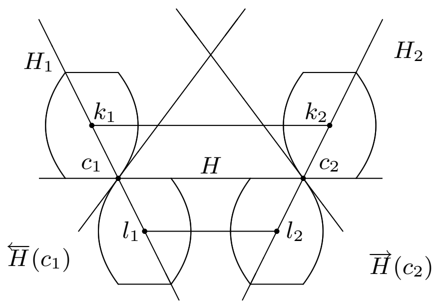

Fig. 7.b

Proof. By Lemma 4.4, $H = \operatorname{aff}(c_1, c_2)$ supports the sets $l_1 + \mathbb{B}$ and $l_2 + \mathbb{B}$. Hence the segment $\triangle(c_1 - l_1, c_2 - l_2)$ is contained in $\operatorname{bd}\mathbb{B}$. Let F be the one-dimensional face of $\mathbb{B}$ containing $\triangle(c_1 - l_1, c_2 - l_2)$. Then F is parallel to $\triangle(c_1, c_2)$, the line H supports C and $H \cap C = \triangle(c_1, c_2)$.

Similarly, $H + l_1 - c_1$ supports L with $(H + l_1 - c_1) \cap L = \triangle(l_1, l_2)$ and $H + k_1 - c_1$ supports K with $(H + k_1 - c_1) \cap K = \triangle(k_1, k_2)$.

Assume that $|l_1 - l_2| < |k_1 - k_2|$. Denote by $E_H^+(L)$ the half-plane bounded by H and containing L. Then $\operatorname{int}E_H^+(L) \cap C = \emptyset$. If $\triangle(c_1 - l_1, c_2 - l_2) \in \operatorname{relint}F$, then $c_i \in \operatorname{relint}((k_i + \mathbb{B}) \cap (l_i + \mathbb{B}))$ for $i = 1, 2$. Hence there exist $c_3, c_4 \in C$ such that $c_3 \in \operatorname{int}E_{H_1}^-(K \vee L)$ and $c_4 \in \operatorname{int}E_{H_2}^-(K \vee L)$, which is impossible. Thus $\triangle(c_1 - l_1, c_2 - l_2) = F$.

Let $\overleftarrow{H}(c_i)$ and $\overrightarrow{H}(c_i)$ support $l_i + \mathbb{B}$ in c_i for $i = 1, 2$, respectively. Then $\overleftarrow{H}(c_1) = \overrightarrow{H}(c_2)$ or $\overrightarrow{H}(c_1) = \overleftarrow{H}(c_2)$. Let $\overleftarrow{H}(c_1) = \overrightarrow{H}(c_2) = H$. Thus C is contained in the triangle D bounded by H, $\overrightarrow{H}(c_1)$ and $\overleftarrow{H}(c_2)$.

If $\overrightarrow{H}(c_1) = H$ or $\overleftarrow{H}(c_2) = H$, then $D = \triangle(c_1, c_2)$.

Assume now that $\overrightarrow{H}(c_1) \neq H$ and $\overleftarrow{H}(c_2) \neq H$. Since $(L + \mathbb{B}) \cap (K + \mathbb{B}) \subset (L + \mathbb{B}) \setminus E_H^+(L) = L + F$, it follows that $C \subset L + F$. Recall that $L \subset C + \mathbb{B}$ and $-F$ is a face of $\mathbb{B}$. Then $\operatorname{aff}(l_1, l_2)$ supports L. Hence $L + F \subset (K + F) \vee ((l_1, l_2) + F) \subset K + \mathbb{B}$. Obviously, $L + F \subset L + \mathbb{B}$. Thus $C = L + F$ and $K = L + 2F$.

Let us now assume that $|l_1 - l_2| > |k_1 - k_2|$. Then, analogously, we prove that $C = K + F$ and $L = K + 2F$.

Let now $|l_1 - l_2| = |k_1 - k_2|$. Then $c_1 - l_1 = c_2 - l_2 = e$. In a similar way we prove that $\{e\}$ is a zero-dimensional face of the unit ball $\mathbb{B}$. Let $\overleftarrow{H}(c_i)$ and $\overrightarrow{H}(c_i)$ support $l_i + \mathbb{B}$ in c_i for $i = 1, 2$, respectively. Since c_1 and c_2 are exposed points of $l_1 + \mathbb{B}$ and $l_2 + \mathbb{B}$, it follows that $\overleftarrow{H}(c_1) \parallel \overleftarrow{H}(c_2)$ and $\overrightarrow{H}(c_1) \parallel \overrightarrow{H}(c_2)$. If $\overleftarrow{H}(c_1) = \overrightarrow{H}(c_2)$, then $C = \triangle(c_1, c_2)$. If $\overleftarrow{H}(c_1) \neq \overrightarrow{H}(c_2)$, then C is contained in the parallelogram D bounded by $\overleftarrow{H}(c_1)$, $\overleftarrow{H}(c_2)$, $\overrightarrow{H}(c_1)$ and $\overrightarrow{H}(c_2)$. Notice that $L + e \subset L + \mathbb{B}$ and $L - e \subset L + \mathbb{B}$. Then $L \subset L + e + \mathbb{B}$. Since e is a face of $\mathbb{B}$, it follows that $L + e \not\subseteq L + \varepsilon\mathbb{B}$ and $L \not\subseteq L + e + \varepsilon\mathbb{B}$ for $0 < \varepsilon < 1$. Recall that $e = c_1 - l_1 = \frac{1}{2}(k_1 - l_1)$. Thus $K \subset L + \mathbb{B} + e$ and $L \subset K + \mathbb{B} + e$. Therefore the set $L + e$ is a metric midpoint of the pair (K, L). Then $C = L + e$ and $K = L + 2e$. ∎

THEOREM 4.8. *Let $K, L \in \mathcal{K}^2$. If K and L have a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them, then $K = L + \lambda F$ or $L = K + \lambda F$ for some $\lambda > 0$ and a face F of the unit ball $\mathbb{B}$ and, moreover, the sets L and F (or K and F, respectively) satisfy one of Conditions 1–4 from the Table of Conditions.*

Proof. Take $K, L \in \mathcal{K}^2$. Suppose $K \subset L$ or $L \subset K$. If (K, L) have a unique metric midpoint with respect to $\varrho_H^{\mathbb{B}}$, then $L = K + \lambda\mathbb{B}$ or $K = L + \lambda\mathbb{B}$ for some $\lambda > 0$, respectively. The proof is analogous to that for Euclidean case (compare [3]). In this case we obtain pairs of elements of $\mathcal{K}^2$ satisfying Condition 1 from the Table of Conditions.

Suppose now $L \not\subseteq K$ and $K \not\subseteq L$. The set $C \in \mathcal{K}^2$ is the unique metric midpoint (with respect to $\varrho_H^{\mathbb{B}}$) of the pair (K, L). Then either $C = \{c\}$ or $\mathrm{card}(C) > 1$.

Let $C = \{c\}$. It is easy to prove that in this case $K = \{k\}$, $L = \{l\}$, and, for $e := c - k = l - c$, the singleton $\{e\}$ is a zero-dimensional face of the unit ball $\mathbb{B}$. In this case we obtain pairs of elements of $\mathcal{K}^2$ satisfying Condition 2 from the Table of Conditions.

Let now $\mathrm{card}(C) > 1$. Then, by Lemma 4.5, there exist two lines H_1 and H_2 supporting the sets K, L, C at unique points k_1, l_1, c_1 and k_2, l_2, c_2 respectively, $(k_i + \mathbb{B}) \cap C = \{c_i\}$ and $(l_i + \mathbb{B}) \cap C = \{c_i\}$ for $i = 1, 2$.

If $H_1 \parallel H_2$, then, for $e := c_1 - l_1 = c_2 - l_2$, the singleton $\{e\}$ is a zero-dimensional face of the unit ball $\mathbb{B}$, $C = L + e$ and $K = L + 2e$. The set C is contained in the parallelogram D bounded by $\overleftarrow{H}(c_1)$, $\overleftarrow{H}(c_1) + (c_2 - c_1)$, $\overrightarrow{H}(c_1)$ and $\overrightarrow{H}(c_1) + (c_2 - c_1)$, where the lines $\overleftarrow{H}(c_1)$ and $\overrightarrow{H}(c_1)$ support the balls $l_1 + \mathbb{B}$ and $k_1 + \mathbb{B}$ at the point c_1 and the lines $\overleftarrow{H}(c_1) + (c_2 - c_1)$ and $\overrightarrow{H}(c_1) + (c_2 - c_1)$ support the balls $l_2 + \mathbb{B}$ and $k_2 + \mathbb{B}$ at the point c_2. The set D can degenerate to a segment (compare Figures 4.a, 4.b, 4.c). In this case we obtain pairs of elements of $\mathcal{K}^2$ satisfying Condition 3 from the Table of Conditions.

If $H_1 \nparallel H_2$, then $F := \triangle((c_1 - l_1), (c_2 - l_2))$ is a one-dimensional face of the unit ball, $C = L + F$ and $K = L + 2F$ or $C = K + F$ and $L = K + 2F$. The set C is contained in the triangle D bounded by H, $\overrightarrow{H}(c_1)$ and $\overleftarrow{H}(c_2)$, where $H = \overleftarrow{H}(c_1) = \overrightarrow{H}(c_2)$. The lines $\overleftarrow{H}(c_1)$ and $\overrightarrow{H}(c_1)$ support the balls $l_1 + \mathbb{B}$ and $k_1 + \mathbb{B}$ at the point c_1 and the lines $\overleftarrow{H}(c_2)$ and $\overrightarrow{H}(c_2)$ support the balls $l_2 + \mathbb{B}$ and $k_2 + \mathbb{B}$ at the point c_2. The set D can degenerate to a segment (compare Figure 6.a) or to an unbounded half-strip (see Figure 6.b). In this case we obtain pairs of elements of $\mathcal{K}^2$ satisfying Condition 4 from the Table of Conditions. ∎

In view of Theorem 3.5 we obtain the following result:

THEOREM 4.9. *Let $K, L \in \mathcal{K}^2$. The following conditions are equivalent:*

i) there is a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining K and L;

ii) $K = L + \lambda F$ or $L = K + \lambda F$ for some $\lambda > 0$ and a face F of the unit ball $\mathbb{B}$. Moreover, the sets L and F (or K and F, respectively) satisfy one of Conditions 1–4 from the Table of Conditions.

5. Remarks. In this paper we give necessary and sufficient conditions for a pair (K, L) of elements of $\mathcal{K}^2$ to have a unique metric segment with respect to $\varrho_H^{\mathbb{B}}$ joining them. Our conjecture is that in higher dimensions all pairs of elements of $\mathcal{K}^n$ with a unique metric segment joining them are of the form $\{K, K + \lambda F\}$, where F is a face of the unit ball $\mathbb{B}$, $\lambda > 0$ and the sets K and F together with the unit ball $\mathbb{B}$ satisfy certain additional conditions. These conditions may be much more complicated than in the planar case.

Restriction of the class of Minkowski spaces to those with a strictly convex or smooth unit ball seems to be a good starting point for studying unique metric segments in higher dimensions. We thank the referee for this suggestion. Our paper dedicated to this problem is now in preparation.

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