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OPTIMAL ISOMETRIES FOR A PAIR OF COMPACT CONVEX SUBSETS OF $\mathbb{R}^n$

IRMINA HERBURT

Department of Mathematics and Information Science, Warsaw University of Technology
Pl. Politechniki 1, 00-661 Warszawa, Poland
E-mail: herbir@mini.pw.edu.pl

MARIA MOSZYŃSKA

Institute of Mathematics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
E-mail: mariamos@mimuw.edu.pl

Abstract. In 1989 R. Arnold proved that for every pair (A, B) of compact convex subsets of $\mathbb{R}^n$ there is an Euclidean isometry optimal with respect to L_2 metric and if f_0 is such an isometry, then the Steiner points of $f_0(A)$ and B coincide. In the present paper we solve related problems for metrics topologically equivalent to the Hausdorff metric, in particular for L_p metrics for all $p \geq 2$ and the symmetric difference metric.

Problems of approximation theory for hyperspaces of convex bodies (with different metrics) and their subspaces have been studied, e.g., in [2], [1], [8], [5], [6], [4].

In particular, the paper [2] concerns isometries optimal for a given pair in $\mathcal{K}^n$ (the family of nonempty, compact, convex subsets of $\mathbb{R}^n$), with respect to the L_2 metric. Let us recall that for any real $p \geq 1$, the L_p metric, ϱ_p, is defined by

$$\varrho_p(A, B) := \left(\int_{S^{n-1}} |h_A(u) - h_B(u)|^p d\sigma(u) \right)^{\frac{1}{p}},$$

where $h_A, h_B : S^{n-1} \to \mathbb{R}$ are the support functions of A, B.

An isometry $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ is *optimal for* (A, B) *with respect to a metric* ϱ in $\mathcal{K}^n$ provided that for every isometry $f : \mathbb{R}^n \to \mathbb{R}^n$

$$\varrho\big(f_0(A), B\big) \leq \varrho\big(f(A), B\big).$$

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In 1989, R. Arnold ([2]) proved the following two statements:

1. *If an isometry* $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ *is optimal for* (A, B) *with respect to* ϱ_2*, then the Steiner points of* $f_0(A)$ *and* B *coincide.*
2. *For every pair* (A, B) *of nonempty compact convex subsets of* $\mathbb{R}^n$ *there exists an isometry* $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ *optimal for* (A, B) *with respect to* ϱ_2*.*

We are interested in the following problems raised at the problem session of the conference "On the border of geometry and topology" held in Będlewo, Poland, in 2002:

Problem 1. *Let* ϱ *be an arbitrary metric in* $\mathcal{K}^n$ *topologically equivalent to* ϱ_2*. Does there exist a selector* $s : \mathcal{K}^n \to \mathbb{R}^n$ *equivariant under the isometries and such that for every* $A, B \in \mathcal{K}^n$*, if an isometry* f_0 *is optimal for* (A, B) *with respect to* ϱ*, then* $s(f_0(A)) = s(B)$*?*

Problem 2. *Let* ϱ *be an arbitrary metric in* $\mathcal{K}^n$ *topologically equivalent to* ϱ_2*. Does every pair in* $\mathcal{K}^n$ *have an isometry optimal with respect to* ϱ*?*

Let us note that Arnold in [2] applies his statement 1 to prove statement 2.

In Section 2 we solve Problem 1: we prove that a required selector does not exist for the Hausdorff metric not only for $\mathcal{K}^n \times \mathcal{K}^n$ but also for a very narrow subclass of it (Theorem 2.4).

In Section 3 we solve Problem 2: we prove that generally the answer to the question is negative (Remark 3.1 and Example 3.2) and we present positive results under additional assumptions (Theorems 3.6 and 3.8; Corollaries 3.7 and 3.9).

1. Preliminaries. In principle, we follow the terminology and notation of [11], except for the definition of a convex body: by a convex body we understand a compact convex subset of $\mathbb{R}^n$ with nonempty interior, while in [11] the notion of convex body is identified with that of nonempty compact convex set.

As usual, $\mathcal{C}^n$, $\mathcal{K}^n$, and $\mathcal{K}^n_0$ are, respectively, the family of all nonempty compact subsets of $\mathbb{R}^n$, the family of convex elements of $\mathcal{C}^n$, and the family of convex bodies; further, $\mathcal{O}^n$ is the family of strictly convex sets, that is, elements of $\mathcal{K}^n$ with no segments on the boundary, and $\mathcal{O}^n_0$ the family of strictly convex bodies.

Let us recall that the Hausdorff metric ϱ_H in $\mathcal{C}^n$ is defined by

$$\varrho_H(A, B) := \max\{\vec{\varrho_H}(A, B), \vec{\varrho_H}(B, A)\},$$

where

$$\vec{\varrho_H}(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\| = \inf\{\alpha > 0 \mid A \subset B + \alpha B^n\}. \tag{1.1}$$

As was proved by R. Vitale in [12] p. 286, on $\mathcal{K}^n$ the Hausdorff metric ϱ_H is topologically equivalent to each L_p metric ϱ_p, and all the corresponding metric spaces are finitely compact, that is, they satisfy the following (equivalent) conditions (see, e.g., [10]):

- every bounded sequence has a convergent subsequence,
- every closed, bounded subset is compact,
- the balls are compact.

Let $A \in \mathcal{K}_0^n$. For any $x \in A$, let $R_A(x)$ be the radius of the smallest ball with centre x containing A and $r_A(x)$ the radius of a biggest ball with centre x contained in A.

The functions $R_A, r_A : A \to \mathbb{R}_+$ can be extended, respectively, to $\overline{R}_A, \overline{r}_A : \mathbb{R}^n \to \mathbb{R}$, such that for every $x \in \mathbb{R}^n$

$$\overline{R}_A(x) = \inf\{\alpha > 0 \mid x + \alpha B^n \supset A\}$$

and for $x \in \mathbb{R}^n \setminus A$

$$\overline{r}_A(x) = -\mathrm{dist}(x, A),$$

where $\mathrm{dist}(x, A) := \inf\{\|x - a\| \mid a \in A\}$ (see [3] or [9]).

Let us recall that a selector of a given family of sets is a function selecting a point from every member of this family (see, e.g., [11]).

Definition 1.1. Let $\mathcal{X}$ be a subfamily of $\mathcal{C}^n$. A selector $s : \mathcal{X} \to \mathbb{R}^n$ is said to be *associated with a metric* ρ *in* $\mathcal{X}$ if for every $A, B \in \mathcal{X}$ and an isometry f_0 optimal for (A, B) with respect to ϱ,

$$s(f_0(A)) = s(B).$$

A selector s is equivariant under a transformation g of $\mathbb{R}^n$ provided that

$$s(g(A)) = g(s(A))$$

(see [11]).

We shall restrict our considerations to selectors equivariant under the Euclidean isometries.

In view of the result of Arnold [2], there is a selector associated with the L_2 metric, ϱ_2, namely the Steiner point map. As is well known, this selector is equivariant under the isometries.

2. Looking for a selector associated with the Hausdorff metric. As was already mentioned, we shall restrict the class of selectors to those equivariant under the isometries.

We are going to prove that Problem 1 has a negative solution even for pairs of strictly convex bodies (Theorem 2.4). We begin with two examples.

Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$. Let lin be the linear hull; in particular, $\mathrm{lin}\, x = \{tx \mid t \in \mathbb{R}\}$ for any $x \in \mathbb{R}^n$.

Example 2.1. Let $X = B^n \cap E^+$, where

$$E^+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

We calculate the Hausdorff distance $\varrho_H(X, B^n + te_n)$ for arbitrary $t \in \mathbb{R}$. By Lemma 2.1 in [5],

$$\varrho_H(X, B^n + te_n) = \max\{\bar{R}_X(te_n) - 1, 1 - \bar{r}_X(te_n)\}. \tag{2.1}$$

If $t \leq 0$, then by easy calculation, $\bar{R}_X(te_n) = 1 + |t| = 1 - t$ and $\bar{r}_X(te_n) = t$, whence

$$\bar{R}_X(te_n) - 1 < 1 - \bar{r}_X(te_n). \tag{2.2}$$

If $t \in [0,1]$, then $\bar{R}_X(te_n) = \sqrt{1+t^2}$ and

$$\bar{r}_X(te_n) = \begin{cases} t & \text{if } t \le \frac{1}{2}, \\ 1-t & \text{if } t \ge \frac{1}{2}; \end{cases}$$

thus, by a simple calculation, condition (2.2) is satisfied.

If $t \ge 1$, then $\bar{R}_X(te_n) = \sqrt{1+t^2}$ and $\bar{r}_X(te_n) = 1-t$; thus, again (2.2) holds. Hence by (2.1),

$$\varrho_H(X, B^n + te_n) = 1 - \bar{r}_X(te_n) = \begin{cases} 1-t & \text{if } t \le \frac{1}{2}, \\ t & \text{if } t \ge \frac{1}{2}. \end{cases}$$

Thus $\inf_{t\in\mathbb{R}} \varrho_H(X, B^n + te_n)$ is attained for $t = \frac{1}{2}$, that is, for $te_n = \hat{c}(X)$, the centre of the inscribed ball of X.

EXAMPLE 2.2. Let X be as in Example 2.1. We calculate the Hausdorff distance $\varrho_H(X, \frac{1}{2}B^n + te_n)$ for arbitrary $t \in \mathbb{R}$.

As before, by Lemma 2.1 in [5],

$$\varrho_H\left(X, \frac{1}{2}B^n + te_n\right) = \max\left\{\bar{R}_X(te_n) - \frac{1}{2}, \frac{1}{2} - \bar{r}_X(te_n)\right\}. \tag{2.3}$$

If $t \le 0$, then $\bar{R}_X(te_n) - \frac{1}{2} = \frac{1}{2} - t = \frac{1}{2} - \bar{r}_X(te_n)$.

If $0 < t \le 1$, then

$$\bar{R}_X(te_n) - \frac{1}{2} = \sqrt{1+t^2} - \frac{1}{2} > \left|\frac{1}{2} - t\right| = \frac{1}{2} - \bar{r}_X(te_n).$$

If $t \ge 1$, then

$$\bar{R}_X(te_n) - \frac{1}{2} = \sqrt{1+t^2} - \frac{1}{2} > t - \frac{1}{2} = \frac{1}{2} - \bar{r}_X(te_n).$$

Hence by (2.3),

$$\varrho_H\left(X, \frac{1}{2}B^n + te_n\right) = \bar{R}_X(te_n) - \frac{1}{2} = \begin{cases} \frac{1}{2} - t & \text{if } t \le 0, \\ \sqrt{1+t^2} - \frac{1}{2} & \text{if } t \ge 0, \end{cases}$$

and thus $\inf_{t\in\mathbb{R}} \varrho_H(X, \frac{1}{2}B^n + te_n)$ is attained for $t = 0$, that is, for $te_n = \check{c}(X)$, the Čebyšev centre of X.

The proof of the following simple lemma is left to the reader.

LEMMA 2.3. *Let ϕ and ϕ_k, for $k \in \mathbb{N}$, be continuous real functions on $\mathbb{R}$ or an interval (bounded or unbounded). If the sequence $(\phi_k)_{k\in\mathbb{N}}$ is uniformly convergent to ϕ and $\phi > 0$, then there is a k_0 such that $\phi_k > 0$ for all $k \ge k_0$.*

THEOREM 2.4. *There is no selector for $\mathcal{O}_0^n$ equivariant under the isometries and associated with the Hausdorff metric.*

Proof. We shall first prove the assertion for a slightly bigger family UIB which consists of all convex bodies with a unique inscribed ball with the maximal radius.

Claim 1. *There is no selector for UIB equivariant under the isometries and associated with ϱ_H.*

Suppose to the contrary that there exists such a selector s. That is, s is equivariant under the isometries and for every $A, B \in$ UIB, if an isometry f_0 satisfies the condition

$$\varrho_H\big(f_0(A), B\big) \leq \varrho_H\big(f(A), B\big) \ \text{ for every isometry } f,$$

then $s(f_0(A)) = s(B)$.

Let X be as in Examples 2.1 and 2.2. Consider first the pair (B^n, X). Then, evidently, for every isometry f_0 the image $f_0(B^n)$ is a translate of B^n.

If the translation by x is optimal for this pair, then $s(X) = s(B^n + x) = x$, because s is associated with ϱ_H and equivariant under reflection at x. Hence $x \in \operatorname{lin} e_n$, because s is equivariant under the reflection at $\operatorname{lin} e_n$ and X is symmetric with respect to this line. Thus $s(X) = te_n$ for some $t \in \mathbb{R}$.

Therefore, it suffices to consider $\inf_t \varrho_H(B^n + te_n, X)$. By Example 2.1, this infimum is attained for the centre $\hat{c}(X)$ of the ball inscribed in X. Thus $s(X) = \hat{c}(X)$.

Consider now the pair $(\frac{1}{2}B^n, X)$. By the same reasoning as above, the infimum of the Hausdorff distance between $\frac{1}{2}B^n + x$ and X is attained for $x \in \operatorname{lin} e_n$. Thus, in view of Example 2.2, it is attained for $x = \check{c}(X)$, the Čebyšev centre of X, which evidently is different from $\hat{c}(X)$, a contradiction.

Claim 2. *There is no selector $s : \mathcal{O}_0^n \to \mathbb{R}^n$ equivariant under the isometries and associated with ϱ_H.*

The idea is to approximate the convex body X by a strictly convex body so close to X that inequalities for big and small radii are preserved.

Let X_k be the intersection of two balls:

$$X_k := B^n \cap \big(ke_n + \sqrt{1+k^2}B^n\big) \tag{2.4}$$

for every natural k. Then the intersection of the boundaries of these balls is $S^{n-1} \cap (ke^n + \sqrt{1+k^2}S^{n-1})$, which is equal to the intersection of S^{n-1} and the hyperplane $\operatorname{lin}(e_1, \dots e_{n-1})$.

Evidently, each X_k is strictly convex, being the intersection of strictly convex bodies.

Notice that

$$\varrho_H(X_k, X) = \sqrt{1+k^2} - k,$$

whence, by easy calculation, $\lim_H X_k = X$. Thus, by Lemma 3.2 in [9], $\bar{R}_{X_k} \to \bar{R}_X$ and $\bar{r}_{X_k} \to \bar{r}_X$, uniformly.

To apply Lemma 2.3, let us now define the functions ϕ_k and ϕ by the formulae

$$\phi_k(t) := 2 - \big(\bar{R}_{X_k}(te_n) + \bar{r}_{X_k}(te_n)\big) \ \text{ and } \ \phi(t) := 2 - \big(\bar{R}_X(te_n) + \bar{r}_X(te_n)\big).$$

In view of Example 2.1, the function $\phi : \mathbb{R} \to \mathbb{R}$ is positive and thus by Lemma 2.3, there is a k_1 such that $\phi_k > 0$ for every $k \geq k_1$. Equivalently, for $k \geq k_1$,

$$\underline{R}_{X_k}(te_n) - 1 < 1 - \underline{r}_{X_k}(te_n) \ \text{for every } t.$$

Thus, by (2.1), for every t,

$$\varrho_H(X_k, te_n + B^n) = 1 - \bar{r}_{X_k}(te_n),$$

whence the minimizer of this distance is a maximizer of $\bar{r}_{X_k}(te_n)$, i.e., it is the centre of the inscribed ball of X_k. Thus $s(X_k) = \hat{c}(X_k)$ for $k \geq k_1$.

Similarly, we define ψ_k and ψ by the formulae

$$\psi_k(t) := \bar{R}_{X_k}(te_n) + \bar{r}_{X_k}(te_n) - 1 \text{ and } \psi(t) := \bar{R}_X(te_n) + \bar{r}_X(te_n) - 1.$$

By Example 2.2, $\psi(t) > 0$ for every $t > 0$ and $\psi(t) = 0$ for $t \leq 0$. Thus we can apply Lemma 2.3 only to the restrictions $\psi|(0,\infty)$ and $\psi_k|(0,\infty)$. We obtain $\psi_k(t) > 0$ for $t > 0$ and sufficiently large k. It remains to show that $\psi_k(t) > 0$ also for $t \leq 0$ and k large enough. To this end, let us notice that for $t \leq 0$

$$\bar{R}_{X_k}(te_n) = \bar{R}_X(te_n) \text{ and } \bar{r}_{X_k}(te_n) > \bar{r}_X(te_n),$$

whence there is a k_2 such that for $k \geq k_2$

$$\psi_k(t) = \bar{r}_{X_k}(te_n) - \bar{r}_X(te_n) > 0 \text{ for } t \leq 0.$$

Let $k_0 = \max\{k_1, k_2\}$. Then for $k \geq k_0$, the set X_k is a strictly convex body such that

- $\inf \varrho_H(X_k, te_n + B^n)$ is attained for $te_n = \hat{c}(X_k)$ and thus $s(X_k) = \hat{c}(X_k)$,
- $\inf \varrho_H(X_k, te_n + \frac{1}{2}B^n)$ is attained for $te_n = \check{c}(X_k)$ and thus $s(X_k) = \check{c}(X_k)$,

a contradiction. ∎

Remark 2.5. In fact we proved that a selector equivariant under isometries and associated with ϱ_H does not exist even for some pairs with one member being a ball and the other being a strictly convex body.

3. Optimal isometries. Let Iso be the group of isometries of $\mathbb{R}^n$. For any $A \subset \mathbb{R}^n$, let

$$\mathcal{F}(A) = \{f(A) \mid f \in \mathrm{Iso}\}. \tag{3.1}$$

Let us note the following obvious fact.

Remark 3.1. Let $\mathcal{X}$ be a nonempty subfamily of $\mathcal{C}^n$ invariant under Iso and let ϱ be a metric on $\mathcal{X}$. Then for every pair (A, B) in $\mathcal{X}$ the following conditions are equivalent:

(i) there exists $f \in \mathrm{Iso}$ optimal for (A, B) with respect to ϱ;

(ii) for every sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathcal{F}(A)$ with $(\varrho(A_k, B))_{k\in\mathbb{N}}$ decreasing, there exists $A_0 \in \mathcal{F}(A)$ such that $\varrho(A_0, B) \leq \varrho(A_k, B)$ for every k.

We are now going to show that in view of Remark 3.1 generally the answer to the question in Problem 1 is negative: there exists a metric ϱ on $\mathcal{K}^n$ topologically equivalent to ϱ_H (and thus to ϱ_2) and a pair (A, B) of convex bodies with no optimal isometry with respect to ϱ.

Example 3.2. We define $\varrho : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R}_+$ by the formula

$$\varrho(A_1, A_2) := \begin{cases} \varrho_H(A_1, A_2) & \text{if } \varrho_H(A_1, A_2) \leq 2, \\ 1 + \frac{3}{1+\varrho_H(A_1,A_2)} & \text{if } \varrho_H(A_1, A_2) > 2. \end{cases}$$

The function ϱ is a metric topologically equivalent to ϱ_H.

Let $A = B^n$ and $B = 4B^n$. Then

$$\inf\{\varrho_H(f(A), B) \mid f \in \mathrm{Iso}\} = \varrho_H(A, B) = 3.$$

Since, evidently, for every $f \in \mathrm{Iso}$ there is $v \in \mathbb{R}^n$ such that $f(A) = A + v$, it follows that $\varrho_H(f(A), B) = 3 + \|v\| > 2$, whence

$$\varrho\big(f(A), B\big) = 1 + \frac{3}{4 + \|v\|} > 1. \tag{3.2}$$

Consider now the sequence $(A + ku)_{k\in\mathbb{N}}$ for some $u \in S^{n-1}$. Evidently, $\varrho(A + ku, B) = 1 + \frac{3}{k+4}$, that is, the sequence $(\varrho(A + ku, B))_{k\in\mathbb{N}}$ is decreasing. However, by (3.2), there is no $f_0 \in \mathrm{Iso}$ with $\varrho(f_0(A), B) \leq 1$. Thus, by Remark 3.1, the pair (A, B) has no optimal isometry with respect to ϱ.

We now pass to positive results. We shall need a particular case of the following lemma.

Lemma 3.3. *Let $A \in \mathcal{C}^n$ and $f_k : A \to \mathbb{R}^n$ continuous for $k \in \mathbb{N}$. If $(f_k)_{k\in\mathbb{N}}$ is uniformly convergent to an f, then*

$$f(A) = \lim_H f_k(A). \tag{3.3}$$

Proof. Condition (3.3) is equivalent to the conjunction of the following two (see [11] Theorem 1.8.7 and Note 3 p. 57):

(i) for every $x \in f(A)$ there exists a convergent sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in f_k(A)$ and $\lim x_k = x$;

(ii) if a sequence $(i_k)_{k\in\mathbb{N}}$ is increasing and $(x_{i_k})_{k\in\mathbb{N}}$ with $x_{i_k} \in f_{i_k}(A)$ is convergent, then $\lim x_{i_k} \in f(A)$.

Since for a compact domain the uniform convergence is equivalent to the continuous convergence (see [7], p. 109), condition (i) is satisfied.

To verify (ii), let $x = \lim x_{i_k}$ and $x_{i_k} = f_{i_k}(a_k)$ for $a_k \in A$. We may assume (passing to a suitable subsequence) that $(a_k)_{k\in\mathbb{N}}$ is convergent to an $a \in A$ and thus (by continuous convergence), $x = f(a) \in f(A)$, which proves (ii). ∎

We shall also use the following simple fact.

Lemma 3.4. *Let F be a nonempty closed subset of a metric space (X, ϱ) and let $a_0 \in X$. If some ball with centre a_0 has a nonempty and compact intersection with F, then there exists a point in F nearest to a_0 with respect to ϱ.*

Proof. Let B_0 be a ball in (X, ϱ) with centre a_0 such that $B_0 \cap F$ is nonempty and compact. Evidently, a possible nearest point cannot be outside of B_0, i.e.,

$$\inf_{x\in F} \varrho(x, a_0) = \inf_{x\in B_0\cap F} \varrho(x, a_0).$$

Since $B_0 \cap F$ is compact and the function $x \mapsto \varrho(x, a_0)$ is continuous, the infimum is attained at some point $x_0 \in B_0 \cap F$. ∎

Proposition 3.5. *For every $A \in \mathcal{C}^n$ the set $\mathcal{F}(A)$ is closed in $(\mathcal{C}^n, \varrho_H)$.*

Proof. Let $(f_k(A))_{k\in\mathbb{N}}$ be a convergent sequence in $(\mathcal{C}^n, \varrho_H)$ for $f_k \in \mathrm{Iso}$. Let $A' = \lim_H f_k(A)$. We have to prove that $A' \in \mathcal{F}(A)$, i.e., there exists an isometry f_0 with $A' = f_0(A)$.

Since $f_k = \bar{f}_k + f_k(0)$ for some $\bar{f}_k \in O(n)$, it follows that for every k

$$f_k(A) = \bar{f}_k(A) + f_k(0).$$

The sequences $(f_k(A))_{k\in\mathbb{N}}$ and $(\bar{f}_k)_{k\in\mathbb{N}}$ are bounded, because the first one is convergent and the second has elements in the compact set $O(n)$. Thus $(f_k(0))_{k\in\mathbb{N}}$ is bounded. Consequently, there is an increasing sequence of indices, $(i_k)_{k\in\mathbb{N}}$, such that $(\bar{f}_{i_k})_{k\in\mathbb{N}}$ is convergent to an $\bar{f}_0 \in O(n)$ and $f_{i_k}(0)$ is convergent to a point $x_0 \in \mathbb{R}^n$. By Lemma 3.3,

$$\bar{f}_0(A) = \lim_H \bar{f}_{i_k}(A).$$

Let us define f_0 by $f_0 := \bar{f}_0 + x_0$. Then

$$\begin{aligned}\varrho_H\big(A', f_0(A)\big) &= \varrho_H\big(A' - x_0, \bar{f}_0(A)\big) = \lim \varrho_H\big(A' - f_{i_k}(0), \bar{f}_{i_k}(A)\big)\\ &= \varrho_H\big(\lim_H f_k(A), \lim_H f_{i_k}(A)\big) = 0,\end{aligned}$$

whence $A' = f_0(A)$. This completes the proof. ∎

THEOREM 3.6. *Let ϱ be a metric on $\mathcal{C}^n$ topologically equivalent to ϱ_H. If the space $(\mathcal{C}^n, \varrho)$ is finitely compact, then for every $A, B \in \mathcal{C}^n$ there exists a Euclidean isometry of $\mathbb{R}^n$ optimal for the pair (A, B) with respect to ϱ.*

Proof. Let $\mathcal{F}(A) = \{f(A) \mid f \in \mathrm{Iso}\}$ (see (3.1)). In view of Proposition 3.5, the set $\mathcal{F}(A)$ is closed in $(\mathcal{C}^n, \varrho)$. Since the space $(\mathcal{C}^n, \varrho)$ is finitely compact, it satisfies the assumptions of Lemma 3.4. This completes the proof. ∎

Since, evidently, Theorem 3.6 will remain valid if $\mathcal{C}^n$ is replaced by $\mathcal{K}^n$, in view of the result by R. Vitale (see Preliminaries), this theorem yields the following.

COROLLARY 3.7. *If ϱ is either the Hausdorff metric on $\mathcal{K}^n$ or any of the L_p metrics, then for every pair (A, B) in $\mathcal{K}^n$ there exists an isometry optimal with respect to ϱ.*

Another positive result is the following.

THEOREM 3.8. *Let $\mathcal{X}$ be a nonempty subfamily of $\mathcal{C}^n$ invariant under* Iso *and let ϱ be a metric on $\mathcal{X}$ topologically equivalent to ϱ_H. If $A, B \in \mathcal{X}$ and there exists an $f_0 \in \mathrm{Iso}$ such that*

$$f_0(A) \cap B \neq \emptyset \quad \text{and} \quad \varrho\big(f_0(A), B\big) \le \varrho(A, B), \tag{3.4}$$

then there exists an isometry optimal for (A, B) with respect to ϱ.

Proof. Consider the set $\mathcal{F}_0(A)$ defined by

$$\mathcal{F}_0(A) := \big\{C \in \mathcal{F}(A) \mid C \cap B \neq \emptyset\big\}.$$

It is easy to see that $\mathcal{F}_0(A)$ is bounded and closed in $(\mathcal{C}^n, \varrho_H)$ and thus it is a compact subset of $\mathcal{C}^n$ (with any of the metrics ϱ_H and ϱ). By (3.4), this set is nonempty and the function $\varrho(\cdot, B)$ attains its infimum in this set. This completes the proof. ∎

We shall now consider the metric δ^S on the space $\mathcal{K}_0^n$ of convex bodies in $\mathbb{R}^n$, the so called *symmetric difference metric* (see [11], p. 58). It is defined by the formula

$$\delta^S(A, B) := V_n(A \dot{-} B),$$

where $A \dot{-} B := (A \cup B) \setminus (A \cap B)$ and V_n is the n-dimensional Lebesgue measure.

This metric is topologically equivalent to the Hausdorff metric, but evidently $(\mathcal{K}_0^n, \delta^S)$ is not finitely compact. Nevertheless, by Theorem 3.8 we obtain the following.

COROLLARY 3.9. *For every pair (A, B) of convex bodies in $\mathbb{R}^n$, there exists an optimal isometry with respect to δ^S.*

REMARK 3.10. In fact, R. Arnold in [2] deals with rigid motions (i.e. orientation preserving isometries). Let us note that in our Proposition 3.5 and consequently in Theorem 3.6 and Corollary 3.7, the group $O(n)$ of linear isometries can be replaced by its arbitrary nonempty closed subset, in particular by the group $SO(n)$ of linear rigid motions.

4. The uniqueness problem. It is easy to see that generally an isometry optimal for a given pair (A, B) with respect to ϱ_H is not unique and even an isometric copy of A nearest to B is not unique. Moreover, it may happen that for two nearest copies A_1, A_2 of A there is no self-isometry f_0 of B onto $f_0(A_1) = A_2$ (see Example 4.2.)

LEMMA 4.1. *Let $A, A' \in \mathcal{K}^n$,* $\operatorname{diam} A' = \operatorname{diam} A =: \alpha$, *and $\varepsilon \geq 0$. Let $\{x, y\}$ be a unique pair of points in A and $\{x', y'\}$ a unique pair of points in A' such that $\|x - y\| = \alpha = \|x' - y'\|$. If*

$$\{x, y\} \neq \{x', y'\}, \tag{4.1}$$

then $\varrho_H(A, A' + \varepsilon B^n) > \varepsilon$.

Proof. Suppose, to the contrary, that $\varrho_H(A, A' + \varepsilon B^n) =: \varepsilon_0 \leq \varepsilon$. Then, by (1.1), $A' + \varepsilon B^n \subset A + \varepsilon_0 B^n$ and thus by the cancellation law,

$$A' + (\varepsilon - \varepsilon_0) B^n \subset A. \tag{4.2}$$

If $\varepsilon_0 < \varepsilon$, then from (4.2) it follows that $\operatorname{diam} A' < \operatorname{diam} A$, a contradiction.

If $\varepsilon_0 = \varepsilon$, then $A = A' + \varepsilon B^n$ and thus $x', y' \in A$. Then, by the assumption of the uniqueness of the pairs of points that attain the common diameter of A and A', it follows that $\{x, y\} = \{x', y'\}$, contrary to (4.1). ■

EXAMPLE 4.2. We adopt the natural convention that any sum over an empty set of indices is equal to 0.

Consider the following three half-ellipsoids:

$$E_0 = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \frac{x_1^2}{16} + \sum_{i=2}^{n-1} x_i^2 + \frac{x_n^2}{4} \leq 1,\ x_n \geq 0\right\},$$

$$E_1 = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \frac{x_1^2}{16} + \sum_{i=2}^{n} x_i^2 \leq 1,\ x_n \leq 0\right\},$$

$$E_2 = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \frac{x_1^2}{16} + \sum_{i=2}^{n-1} x_i^2 + \frac{x_n^2}{9} \leq 1,\ x_n \leq 0\right\}.$$

Let $A = E_0 \cup E_2$, $A' = E_0 \cup E_1$, and $B = A' + 2B^n$ (for $n = 2$ see Fig. 1). It is easy to check that A and B are strictly convex bodies.

Let $L = \operatorname{lin}(e_1, \ldots, e_{n-1})$. If $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ is a self-isometry of B, then $-3e_n$ and $4e_n$ are fixed points of f_0, whence f_0 restricted to $\Delta(-3e_n, 4e_n)$ is the identity. Thus no self-isometry of B maps B onto $\sigma_L(B)$, the image of B under the reflection at L.

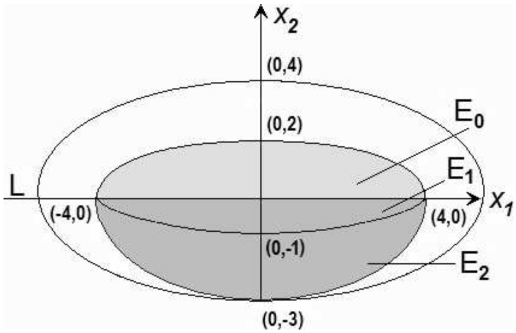

Fig. 1

Evidently, $\varrho_H(A, B) = 2 = \varrho_H(\sigma_L(A), B)$. Moreover, from Lemma 4.1 it follows that for every isometry f, if $\mathrm{id} \neq f \neq \sigma_L$, then $\varrho_H(f(A), B) > 2$; hence, A and $\sigma_L(A)$ are two isometric copies of A nearest to B.

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