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HIGHER ORDER ESTIMATES IN FURTHER DIMENSIONS FOR THE SOLUTIONS OF NAVIER-STOKES EQUATIONS

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The aim of this note is to continue the study of the decay with respect to $t \to \infty$ of higher order norms of strong solutions of the Navier-Stokes equations in the whole space $\mathbb{R}^n$, which was first addressed in a paper of M. E. Schonbek and M. Wiegner [3].

In general (for $n \geq 3$) a smallness assumption is needed for ensuring existence of such a solution. T. Kato [2] (see also [6]) showed that

$$\begin{aligned} u_t - \triangle u + (u\nabla)u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \quad \text{on } (0,\infty) \times \mathbb{R}^n \\ u(0) &= a \end{aligned}$$

has a global solution u, smooth for $t > 0$, provided $a \in L_{n,\sigma}$ with $\|a\|_n$ small enough. Here $L_{n,\sigma}$ denotes the divergence free part of L_n. If moreover additionally $a \in L_2$, the solution has also finite energy for all times and coincides with the Leray-Hopf solution, for which the energy $\|u(t)\|_2^2$ tends to zero (see e.g. [4]). Note that for $n = 3, 4$, all Leray-Hopf solutions with generalized energy inequality fulfil the necessary smallness assumption after some time T_0; see the discussion in [3] or [6].

Suppose now that $\|u(t)\|_2 \leq c\varphi(t)$ with $\varphi(0) \leq 1$ and

$$\varphi(t) \searrow 0 \text{ for } t \nearrow \infty \quad \text{and} \quad \varphi(\gamma t) \leq c_\gamma \varphi(t) \text{ for all } t > 0,\ \gamma < 1 \tag{1}$$

with c_γ independent of t. This is surely true for algebraic decay rates $\varphi(t) = (1+t)^{-\mu}$ with $\mu > 0$, which in turn may be concluded from additional assumptions on the initial value, as e.g. $a \in L_p$ with some p, $1 \leq p < 2$, see e.g. Wiegner [4].

From [7] we then know that for $t > 0$, $\|u(t)\|_\infty \leq c\phi(t)t^{-n/4}$ and $\|D^\alpha u(t)\|_2 \leq c\varphi(t)t^{-1}$ for $|\alpha| = 2$.

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Now the question arises how higher order derivatives behave. Note that even the finiteness of $\|D^\alpha u(t)\|_2$ for $|\alpha| \geq 3$ is not a priori clear. In fact, in [3], M. E. Schonbek and the author had to restrict the dimension to $n \leq 5$, in order to show that $\|D^\alpha u(t)\|_2 \leq ct^{-(|\alpha|/2+\mu)}$ for all α if $\varphi(t) = (1+t)^{-\mu}$.

We are going to prove the same result also in dimension 6 and 7 and at the same time simplify the proof for $n \leq 5$. It seems likely that also pointwise space-time estimates may be derived as in the paper [1] of Amrouche, Girault, M. E. and T. P. Schonbek, which used the results of [3], hence was also restricted to $n \leq 5$.

Theorem. *Let $n \leq 7, a \in L_n \cap L_2$ with $\|a\|_n$ small enough, so that $u(t)$ is a global smooth solution. Let $\|u(t)\|_2 \leq c\varphi(t)$ with φ fulfilling* (1). *Then*

$$\|D^\alpha u(t)\|_2 \leq c\varphi(t)t^{-|\alpha|/2} \quad \textit{for all } \alpha \in \mathbb{N}_0^n.$$

For the proof we start with

Lemma 1. *Let*

$$\|D^2 v\|_2^2 = \sum_{|\alpha|=2} \|D^\alpha v\|_2^2.$$

Then

$$\left\|D^2(u_j u_k)\right\|_2^2 \leq c\|u\|_\infty^2 \|D^2 u\|_2^2.$$

Proof. As

$$\int \left(\frac{\partial u_j}{\partial x_\alpha}\right)^2 \left(\frac{\partial u_k}{\partial x_\beta}\right)^2 dx = -\int u_j \frac{\partial^2 u_j}{\partial x_\alpha^2} \left(\frac{\partial u_k}{\partial x_\beta}\right)^2 dx - 2\int u_j \frac{\partial u_j}{\partial x_\alpha} \frac{\partial u_k}{\partial x_\beta} \frac{\partial^2 u_k}{\partial x_\alpha \partial x_\beta} dx$$

we get

$$\int |\nabla u|^4 \, dx \leq c \int |u| \, |D^2 u| \, |\nabla u|^2 \, dx,$$

hence

$$\int |\nabla u|^4 \, dx \leq \tilde{c} \int |u|^2 |D^2 u|^2 \, dx.$$

Therefore

$$\left\|D^\alpha(u_j u_k)\right\|_2^2 \leq c \int |D^\alpha u|^2 |u|^2 \, dx + c \int |\nabla u_j|^2 |\nabla u_k|^2 \, dx \leq c \int |D^2 u|^2 |u|^2 \, dx,$$

implying the claim. ∎

The next lemma gives a convolution estimate.

Lemma 2. *Let $\widehat{v}(\xi)$ denote the Fourier transform. Then for $q > 0$, $1 \leq p \leq 2$,*

$$\left\||\xi|^q \widehat{u_j \cdot u_k}(\xi)\right\|_r \leq c\|\widehat{u}\|_p \left\|\widehat{u}(\xi)|\xi|^q\right\|_2 \quad \textit{if} \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{2}.$$

Proof. As

$$\widehat{u_j \cdot u_k}(\xi) = \widehat{u_j} * \widehat{u_k},$$

we have

$$|\widehat{u_j \cdot u_k}(\xi)| \leq \int_{\mathbb{R}^n} |\widehat{u}(\mu)| \, |\widehat{u}(\xi - \mu)| \, d\mu.$$

Therefore

$$\big||\xi|^q\widehat{u_ju_k}(\xi)\big| \le c\int |\widehat{u}(\mu)|\,|\mu|^q|\widehat{u}(\xi-\mu)|\,d\mu + c\int |\widehat{u}(\mu)|\,|\widehat{u}(\xi-\mu)|\,|\xi-\mu|^q\,d\mu$$
$$= 2c|\widehat{u}(\xi)| * \big|\widehat{u}(\xi)|\xi|^q\big|$$
$$\Rightarrow \big\||\xi|^q\widehat{u_ju_k}(\xi)\big\|_r \le c\|\widehat{u}\|_p\big\|\widehat{u}(\xi)|\xi|^q\big\|_2$$

by the convolution estimate. ∎

Now we are in the position to prove the theorem.

Proof of the theorem. With the help of the Fourier transformation the solution may be represented as

$$\widehat{u_i}(t,\xi) = \big(\delta_{ij} - \xi_i\xi_j|\xi|^{-2}\big)\Big(e^{-t|\xi|^2/2}\widehat{u_j}(t/2,\xi) - \sum_k \int_{t/2}^t e^{-(t-s)|\xi|^2} i\xi_k\cdot\widehat{u_ju_k}(s,\xi)\,ds\Big)$$

Choose first p with $\max\{1, \frac{2n}{6+n}\} < p < \frac{n}{n-1}$ (this causes the dimension restriction). Then

$$\begin{aligned}\|\widehat{u}(t)\|_p &\le \big\|e^{-t|\xi|^2/2}\widehat{u}(t/2)\big\|_p + \sum_{j,k}\int_{t/2}^t \big\|e^{-(t-s)|\xi|^2}\xi_k\widehat{u_ju_k}(s,\xi)\big\|_p\,ds \\ &\le c\big\|\widehat{u}(t/2)\big\|_2\big\|e^{-t|\xi|^2/2}\big\|_{2p/(2-p)} \\ &\qquad + c\sum_{j,k}\int_{t/2}^t \big\||\xi|^{-1}e^{-(t-s)|\xi|^2}\big\|_{2p/(2-p)}\big\||\xi|^2\widehat{u_ju_k}\big\|_2\,ds \\ &\le c\varphi(t)t^{-n(2-p)/(4p)} + c\int_{t/2}^t (t-s)^{-n(2-p)/(4p)+1/2}\big\|u(s)\big\|_\infty\big\|D^2u(s)\big\|_2\,ds \\ &\le c\varphi(t)t^{-n(2-p)/(4p)} + ct^{3/2-n(2-p)/(4p)}\varphi^2(t)t^{-1-n/4}\end{aligned}$$

if $-\frac{n}{4p}(2-p)+\frac{1}{2} > -1 \Leftrightarrow p > \frac{2n}{6+n}$.

Hence $\|\widehat{u}(t)\|_p \le c\varphi(t)t^{-n(2-p)/(4p)}$.

Now we may estimate after multiplication by $|\xi|^m$:

$$\begin{aligned}\big\||\xi|^m\widehat{u}(t)\big\|_2 &\le \big\||\xi|^m e^{-t|\xi|^2/2}\widehat{u}(t/2)\big\|_2 \\ &\qquad + c\sum_{j,k}\int_{t/2}^t \big\||\xi|^q\widehat{u_ju_k}(\xi)\big\|_r\big\||\xi|^{m+1-q}e^{-(t-s)|\xi|^2}\big\|_{2r/(r-2)}\,ds\end{aligned}$$

with $r = \frac{2p}{2-p} > 2$ and get

$$\begin{aligned}\big\||\xi|^m\widehat{u}(t)\big\|_2 &\le ct^{-m/2}\|u(t/2)\|_2 \\ &\qquad + c\int_{t/2}^t \|\widehat{u}(s)\|_p\big\||\xi|^q\widehat{u}(s)\big\|_2(t-s)^{-(1+\varepsilon)/2-n(r-2)/(4r)}\,ds\end{aligned}$$

with $q = m-\varepsilon$, $\varepsilon > 0$.

Suppose now that

$$(*)\qquad \big\||\xi|^q\widehat{u}(t)\big\|_2 \le c\varphi(t)t^{-q/2}.$$

Then

$$\big\||\xi|^m\widehat{u}(t)\big\|_2 \le c\varphi(t)t^{-m/2} + c\int_{t/2}^{t} \varphi^2(s)s^{-n(2-p)/(4p)}s^{-q/2}(t-s)^{-(1+\varepsilon)/2-n(1-1/p)/2}\,ds$$

as $\frac{r-2}{2r} = \big(\frac{2p}{2-p}-2\big)/\big(\frac{4p}{2-p}\big) = 1-\frac{1}{p}$, and therefore

$$\big\||\xi|^m\widehat{u}(t)\big\|_2 \le c\varphi(t)t^{-m/2} + c\varphi^2(t)t^{-m/2}t^{n/4+1/2-n/2} \le c\varphi(t)t^{-m/2},$$

provided $\frac{1+\varepsilon}{2}+\frac{n}{2}(1-\frac{1}{p}) < 1 \Leftrightarrow \varepsilon + n(1-\frac{1}{p}) < 1$. Such an $\varepsilon > 0$ can be chosen if $p < \frac{n}{n-1}$; hence the estimate $(*)$ can be improved by induction for all $q > 0$, giving $\|D^k u(t)\|_2 \le c_k\varphi(t)t^{-k/2}$ for all $k \in \mathbb{N}_0$. ∎

Remark. By interpolation, for $2 \le p \le \infty$ and $j \in \mathbb{N}_0$ we have the estimate

$$\|D^j u(t)\|_p \le c_{j,p}\varphi(t)^{-j/2-n(1/2-1/p)/2}.$$

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