ON LIPSCHITZ CONTINUITY OF THE SOLUTION MAP 
FOR TWO-DIMENSIONAL WAVE MAPS

PIERO D’ANCONA
Università di Roma “La Sapienza”
Dipartimento di Matematica
Piazzale A. Moro 2, I-00185 Roma, Italy
E-mail: dancona@mat.uniroma1.it

VLADIMIR GEORGIEV
Dipartimento di Matematica
Università degli Studi di Pisa
Via F. Buonarroti 2, 56100 Pisa, Italy
E-mail: georgiev@dm.unipi.it

1. Introduction. The purpose of this paper is to analyze the properties of the solution map

\[(u_0, u_1) \mapsto u(t, x)\]

to the Cauchy problem for the wave map equation

\[\ddot{u} - \Delta u + (|u_t|^2 - |\nabla_x u|^2)u = 0\]

with initial data

\[u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),\]

in the case when \(x \in \mathbb{R}^2\) and the target is the unit sphere \(S^n\) (embedded in \(\mathbb{R}^{n+1}\), \(n \geq 2\)). Thus

\[u : \mathbb{R}_t \times \mathbb{R}_x^2 \rightarrow S^n\]

and we have the additional constraint \(|u| = 1\).

For this problem several results of global well-posedness are available under suitable smallness assumptions on the initial data (see [4], [12], [13], [7]). Moreover, the existence of a global weak solution in \(H^1\) is known ([6], [14], [1]) and the existence and uniqueness

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of smooth solutions under suitable assumptions of symmetry is well-known for the case of geodesically convex two-dimensional targets (see [8], [2]). For the case of target $S^2$ the existence of smooth classical solutions was recently proved by Struwe [10].

A basic question still open concerns the well posedness, even local in time, in the energy space, i.e., for data in $H^1 \times L^2$. This problem is strictly related to the properties of continuity and regularity of the solution map. Indeed, the classical definition of well-posedness implies in particular the continuity of this map; even in a modern sense, we may remark that the standard proofs of existence and uniqueness, which resort to some contraction method, have as a natural consequence the Lipschitz continuity of that map.

To quantify this property, denote by $E(t, u)$ the energy of a solution $u$ at the time $t$:

$$E(t, u) = \| \partial_t u(t, \cdot) \|^2_{L^2(\mathbb{R}^2)} + \| \nabla_x u(t, \cdot) \|^2_{L^2(\mathbb{R}^2)}.$$

Then we may say that the solution map is Lipschitz continuous if we may find a constant $C$ such that for any two solutions $u, v$ the following inequality holds:

$$E(t, u - v) \leq C E(0, u - v), \quad \forall t \in [0, 1]. \tag{1}$$

Note that in this definition the existence of the solution map is not assumed.

The solution map is locally Lipschitz continuous if for any solution $u$ one can find positive constants $\delta, C$ such that for any solution $v$ with $E(0, u - v) \leq \delta$ the inequality (1) holds.

Our goal here is to show, by a suitable counterexample, that the solution map is not locally Lipschitz continuous. More precisely, we prove the following:

**Theorem 1.** There exists a smooth solution $u : \mathbb{R} \times \mathbb{R}^2 \to S^n$ to the wave map equation, such that for any $C > 0$, $\delta > 0$, we can construct a smooth solution $v : \mathbb{R} \times \mathbb{R}^2 \to S^n$ to the wave map equation so that

$$E(0, u - v) \leq \delta$$

and the Lipschitz condition (1) is not satisfied at $t = 1$.

We remark that the solutions used in the counterexample are radially symmetric, hence the symmetry assumption does not improve the regularity of the solution map.

2. Well-posedness of the Cauchy problem for semilinear wave equation.

The linear wave equation

$$\partial_t^2 u - \Delta u = 0 \tag{2}$$

with initial data

$$u(0, x) = u_0(x) \in \dot{H}^1(\mathbb{R}^n), \quad \partial_t u(0, x) = u_1(x) \in L^2(\mathbb{R}^n) \tag{3}$$

satisfies the energy estimate

$$\| \nabla_x u(t) \|_{L^2} + \| \partial_t u(t) \|_{L^2} \leq C \left( \| \nabla_x u_0 \|_{L^2} + \| u_1 \|_{L^2} \right) \tag{4}$$

provided the initial data $u_0, u_1$ belong to the Hilbert space

$$H = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{5}$$
Therefore for any $T > 0$ we have a data-solution map $R$ defined in $H$ with values in $C(I; H)$, $I = [-T, T]$ so that $u(t, x) = R_0(u_0, u_1)$ is a solution to (2) in distribution sense in $(-T, T) \times \mathbb{R}^n$ and satisfies the initial conditions (3).

Moreover, $R_0$ is a bounded (hence continuous) linear operator

$$R_0 : (u_0, u_1) \in H \to C(I; H).$$

A slight generalization of the above definition can be done by taking Banach space $X = X(I) \subseteq C(I; H)$ such that $R_0$ restricted to $H$ is a continuous linear operator

$$R_0 : (u_0, u_1) \in H \to X.$$

Now we can consider the nonlinear Cauchy problem

$$\partial_t^2 u - \Delta u = F(u), \quad t \in [-T, T], \quad x \in \mathbb{R}^n$$

with initial data (3). Here $F$ is a continuous map

$$F : u \in X \to F(u) \in Y$$

and $Y$ is a subset of the space of distributions $D'((T, -T) \times \mathbb{R}^n)$. The classical well-posedness usually is connected with the continuity of the mapping data-solution, $(u_0, u_1) \to u(t)$. More precisely, we shall say that the Cauchy problem (6) is well-posed in $H$ if there exists a Banach spaces $X = X(T) \subset C(I, H)$ and one can find a positive $r > 0$ and a continuous operator

$$R : \{(u_0, u_1) \in H : \|(u_0, u_1)\|_H \leq r\} \to X = X(T),$$

so that $u(t) = R(u_0, u_1)(t)$ is a solution in distribution sense of (6) and satisfies the initial condition (3). The well-posedness of the Cauchy problem for the wave maps in $(t, x) \in \mathbb{R} \times \mathbb{R}$ is studied in [11].

In [3] even weaker regularity of $R$ is assumed, namely the uniform continuity of the mapping data-solution is studied for the case of Schrödinger type equations.

In the case when a standard contraction argument (see [9]) works one can show that the mapping

$$R : (u_0, u_1) \to u(t)$$

is locally Lipschitz continuous. More precisely, one can find a positive $r > 0$ and $C$ so that

$$\|u - \tilde{u}\|_X \leq C\|(u_0, u_1) - (\tilde{u}_0, \tilde{u}_1)\|_H$$

for $(u_0, u_1) \in H$, $(\tilde{u}_0, \tilde{u}_1) \in H$, satisfying

$$\|(u_0, u_1)\|_H + \|(\tilde{u}_0, \tilde{u}_1)\|_H \leq r.$$

3. Stereographic projection for wave maps. The Cauchy problem for wave maps is the semilinear problem

$$u_{tt} - \Delta u + Q(\partial u)u = 0,$$

with initial data

$$u(0, x) = u_0(x) \in \dot{H}^1(\mathbb{R}^2), \quad \partial_t u(0, x) = u_1(x) \in L^2(\mathbb{R}^2),$$
where
\[ Q(\partial u) = |\partial u|^2 - |\nabla_x u|^2. \]
Our first step is the reduction of the vector-valued wave equation (10) to a scalar one. For the purpose, we compose the wave map
\[ u : (t, x) \in \mathbb{R} \times \mathbb{R}^2 \longrightarrow u = u(t, x) \in \mathbb{S}^2 \]
with the stereographic projection
\[ u = (u_1, u_2, u_3) \in \mathbb{S}^2 \longrightarrow z \in \mathbb{C} \cup \infty, \]
where
\[ z = \frac{u_1 + i u_2}{1 + u_3} \]
and the south pole \( S = (0, 0, -1) \) is mapped in \( \infty \). The inverse map is
\[ u_1 = \frac{2 \text{Re} \, z}{1 + |z|^2}, \quad u_2 = \frac{2 \text{Im} \, z}{1 + |z|^2}, \quad u_3 = \frac{1 - |z|^2}{1 + |z|^2}. \]
The metric induced by the projection is \((1 + |z|^2)^2|dz|^2\).

The lines through the origin are geodesics on \( \mathbb{C} \). Hence, we can take a geodesic of type
\[ \gamma : \text{Im} \, z = h(\text{Re} \, z), \quad h(s) = As \]
in \( \mathbb{C} \), where \( A \) is a real constant. This geodesics generates a wave map \( u = u_\gamma \) (see [7]). Indeed, taking
\[ X(t, x) = \text{Re} \, z(t, x), \]
from (14) we get
\[ u_1 = \frac{2X}{1 + X^2 + h^2(X)}, \quad u_2 = \frac{2h(X)}{1 + X^2 + h^2(X)}, \quad u_3 = \frac{1 - X^2 - h^2(X)}{1 + X^2 + h^2(X)}. \]
Substitution of this ansatz into the wave map equation gives the following scalar equation
\[ M(X) \Box X - L(X)Q(\partial X) = 0, \]
where
\[ L(X) = 4h(X)h'(X)(-3X^2 + h^2(X) + 1) \]
\[ M(X) = -X^4 + (1 + h^2(X))^2 - 2X^3h(X)h'(X) - 2X(1 + h^2(X))h(X)h'(X). \]

To verify that the wave map equation is reduced to this scalar equation, we start with the relation
\[ \partial_{x_j} \frac{2X}{1 + X^2 + h^2(X)} = 2\partial_{x_j}X \left( \frac{1 - X^2 + h^2(X) - 2Xh(X)h'(X)}{(1 + X^2 + h^2(X))^2} \right), \]
\[ \partial_{x_j} \frac{2h(X)}{1 + X^2 + h^2(X)} = 2\partial_{x_j}X \left( \frac{-2Xh(X) + (1 + X^2 - h^2(X))h'(X)}{(1 + X^2 + h^2(X))^2} \right) \]
and
\[ \partial_{x_j} \frac{1 - X^2 - h^2(X)}{1 + X^2 + h^2(X)} = -4\partial_{x_j}X \left( \frac{X + h(X)h'(X)}{(1 + X^2 + h^2(X))^2} \right). \]
The above three relations imply

\[ Q(\partial u) = 4Q(\partial X) \frac{1 + (h'(X))^2}{(1 + X^2 + h^2(X))^2} . \]

For the second derivative (note that \( h'' = 0 \)) we have

\[ \frac{2X}{1 + X^2 + h^2(X)} = \frac{2(\partial x_j X)M_1(X) - 4(\partial x_j^2) L_1(X)}{(1 + X^2 + h^2(X))^3} , \]

where

\[ L_1(X) = X(1 + X^2 + h^2(X))(1 + (h'(X))^2) \]

\[ + 2(X + h(X)h'(X))(1 - X^2 + h^2(X) - 2Xh(X)h'(X)) \]

\[ M_1 = (1 - X^2 + h^2(X) - 2Xh(X)h'(X))(1 + X^2 + h^2(X)) . \]

These relations imply

\[ \Box \left( \frac{2X}{1 + X^2 + h^2(X)} \right) = \frac{2(\Box X)M_1(X) - 4Q(\partial X)L_1(X)}{(1 + X^2 + h^2(X))^3} , \]

so combining this identity and (21), we obtain (16).

In the special case \( h(X) = AX \), where \( A \) is a real constant, we obtain

\[ L_1(X) = X(1 + X^2 + h^2(X))(1 + (h'(X))^2) \]

\[ + 2(X + h(X)h'(X))(1 - X^2 + h^2(X) - 2Xh(X)h'(X)) \]

\[ M_1 = (1 - X^2 + h^2(X) - 2Xh(X)h'(X))(1 + X^2 + h^2(X)) . \]

The equation (16) suggests us to take \( X \) so that the equation

\[ \Box X + f(X)Q(\partial X) = 0 \]

be satisfied. Here

\[ f(X) = -\frac{2X(1 + A^2)}{1 + X^2(1 + A^2)} . \]

It is clear that (26) implies (16). This scalar nonlinear wave equation can be transformed into linear wave equation (see [5]) by the aid of the transform

\[ Y = G(X) \equiv \int_0^X e^{F(s)} ds , \quad F(s) = \int_0^s f(\sigma) d\sigma . \]

So using (27), we find \( F(s) = -\ln(1 + B^2 s^2) \), where \( B = \sqrt{1 + A^2} \) and

\[ Y = B^{-1} \arctan(BX) . \]

In conclusion, given any solution of the linear wave equation

\[ \Box Y = 0 \]

the function

\[ X = B^{-1} \tan(BY) \]
is a solution of the scalar nonlinear wave equation (26) and from (15) we see that the function \( u = u_A(t, x) \) defined by

\[
(30) \quad u_1 = \frac{\sin(2BY)}{B}, \quad u_2 = \frac{A \sin(2BY)}{B}, \quad u_3 = \cos(2BY), \quad B = \sqrt{1 + A^2}
\]

is a wave map. The special solutions of (28) we shall use have the form

\[
(31) \quad Y(t, x) = \text{Re} \int_{\mathbb{R}^3} \sin(t|\xi|) e^{i(x \cdot \xi)} \varphi(\xi) \frac{d\xi}{|\xi|}.
\]

With this choice we have

\[
Y(0, x) = 0, \quad \| \partial_t Y(t, \cdot) \|_{L^2} + \| \nabla_x Y(t, \cdot) \|_{L^2} \leq C \| \varphi \|_{L^2}
\]

for any \( t \geq 0 \). These relations and (30) imply

\[
(32) \quad C^{-1} |A_1 - A_2| \| \varphi \|_{L^2}
\]

\[
= \| \partial_t (u_A(0, \cdot) - u_{\tilde{A}}(0, \cdot)) \|_{L^2} + \| \nabla_x (u_A(0, \cdot) - u_{\tilde{A}}(0, \cdot)) \|_{L^2}
\]

\[
\leq C |A_1 - A_2| \| \varphi \|_{L^2}
\]

with some constant \( C \) independent of \( \varphi, A, A_1, A_2 \). Indeed, we have

\[
\partial_t u_A(0) = \partial_t Y(0, x)(2, 2A, 0), \quad \nabla_x u_A(0) = \nabla_x Y(0, x)(2, 2A, 0).
\]

4. The solution map for wave maps is not Lipschitz continuous. Take two real numbers \( A, \tilde{A} \) such that

\[
0 \leq A < \tilde{A},
\]

\( \tilde{A} \) is close enough to \( A \), and consider the wave maps \( u_A \) and \( u_{\tilde{A}} \) constructed in (30). If the solution map is Lipschitz continuous, then the estimate (32) implies that

\[
\| \partial_t (u_A(t, \cdot) - u_{\tilde{A}}(t, \cdot)) \|_{L^2} + \| \nabla_x (u_A(t, \cdot) - u_{\tilde{A}}(t, \cdot)) \|_{L^2} \leq C |A - \tilde{A}| \| \varphi \|_{L^2}.
\]

Dividing by \( |A - \tilde{A}| \) and taking the limit \( \tilde{A} \to A \), we get

\[
\| \partial_t u_A(t, \cdot) \|_{L^2} + \| \nabla_x u_A(t, \cdot) \|_{L^2} \leq C \| \varphi \|_{L^2}.
\]

From (30) we obtain

\[
\partial_A u_1 = \frac{2A}{B^2} Y \cos(2BY) - \frac{A}{B^3} \sin(2BY),
\]

\[
\partial_A u_2 = \frac{2A^2}{B^2} Y \cos(2BY) + \frac{1}{B^3} \sin(2BY),
\]

\[
\partial_A u_3 = -\frac{2A}{B} Y \sin(2BY).
\]

Taking the time derivative, we find the following pointwise estimate

\[
|\partial_t \partial_A u_A(t, x)| \geq C_0(A) |Y(t, x)| |\partial_t Y(t, x)| - C_1(A) |\partial_t Y(t, x)|,
\]

where \( C_0(A) > 0 \) provided \( A > 0 \). For space derivatives we have an analogous estimate

\[
|\nabla_x \partial_A u_A(t, x)| \geq C_0(A) |Y(t, x)| |\nabla_x Y(t, x)| - C_1(A) |\nabla_x Y(t, x)|.
\]
Therefore, we take $A > 0$, say $A = 1$, fix it and then the assumption that the solution map is Lipschitz continuous implies that

$$
C_0(A) \| Y(t, \cdot) \partial_t Y(t, \cdot) \|_{L^2} + C_0(A) \| Y(t, \cdot) \nabla_x Y(t, \cdot) \|_{L^2}
- C_1(A) \| \partial_t Y(t, \cdot) \|_{L^2} - C_1(A) \| \nabla_x Y(t, \cdot) \|_{L^2}
\leq \| \partial_t \partial_t A u_A(t, \cdot) \|_{L^2} + \| \nabla_x \partial_t A u_A(t, \cdot) \|_{L^2} \leq C \| \varphi \|_{L^2}.
$$

From (31) we have the classical energy estimate

$$
\| \partial_t Y(t, \cdot) \|_{L^2} + \| \nabla_x Y(t, \cdot) \|_{L^2} \leq C \| \varphi \|_{L^2}
$$

and we arrive at

$$
(33) \quad \| Y(t, \cdot) \partial_t Y(t, \cdot) \|_{L^2} + \| Y(t, \cdot) \nabla_x Y(t, \cdot) \|_{L^2} \leq C \| \varphi \|_{L^2}.
$$

Recall that this estimate is valid locally, i.e. only for $\| \varphi \|_{L^2} \leq r$ according to (9). It is a standard argument that shows that the estimate (33) with $\| \varphi \|_{L^2} \leq r$ implies the scale invariant estimate

$$
(34) \quad \| Y(t, \cdot) \partial_t Y(t, \cdot) \|_{L^2} + \| Y(t, \cdot) \nabla_x Y(t, \cdot) \|_{L^2} \leq C \| \varphi \|_{L^2}^2
$$

without any upper bound on $\| \varphi \|_{L^2}$. It is clear also that if the estimate (34) is valid for real valued functions $Y$, then the same estimate is valid for complex valued functions $Y$ so we can take

$$
(35) \quad Y(t, x) = \int_{\mathbb{R}^3} \sin(t|\xi|) e^{ix\xi} \varphi(\xi) \frac{d\xi}{|\xi|}.
$$

In the remaining part of this section we shall show that the estimate (34) will lead to a contradiction.

In fact, the estimate (34) will imply

$$
(36) \quad \left| \int \Psi(x) Y(t, x) \partial_t Y(t, x) \, dx \right| \leq C \| \Psi \|_{L^2} \| \varphi \|_{L^2}^2
$$

for any $\Psi \in L^2$. Using the Plancherel identity and (35) we see that this inequality yields

$$
(37) \quad \left| \int \int \hat{\Psi}(\xi - \eta) \cos(t|\xi|) \sin(t|\eta|) \varphi(\xi) \overline{\varphi(\eta)} \, d\xi \frac{d\eta}{|\eta|} \right| \leq C \| \Psi \|_{L^2} \| \varphi \|_{L^2}^2.
$$

Given any even integer $M > 2$, we set (compare with [5])

$$
(38) \quad \varphi_M(\xi) = H(A_M) \frac{1}{|\xi| \ln^{5/8}|\xi|},
$$

where

$$
(39) \quad A_M = \{ \xi \in \mathbb{R}^2 : 2 \leq |\xi| \leq M, \text{dist}(|\xi|, 8\mathbb{Z} + 1) < 1/2 \}
$$

and $H(A)$ denotes the characteristic function of the set $A$. The condition $\text{dist}(|\xi|, 8\mathbb{Z} + 1) < 1/2$ is needed to assure the inequality

$$
(40) \quad \varphi_M(\xi) \sin(t_0|\xi|) \geq C \varphi_M(\xi) \geq 0, \quad \varphi_M(\xi) \cos(t_0|\xi|) \geq C \varphi_M(\xi) \geq 0
$$

with $C > 0$ and $t_0 = \pi/4$. For $\Psi$ we take

$$
(41) \quad \widehat{\Psi}_M(\xi) = H(2 \leq |\xi| \leq M) \frac{1}{|\xi| \ln^{9/16}|\xi|}.
$$
For any $M > 3$ we have the estimates
\begin{equation}
\|\varphi_M\|_{L^2(\mathbb{R}^2)} \leq C, \quad \|\Psi_M\|_{L^2(\mathbb{R}^2)} \leq C
\end{equation}
with some constant $C$ independent of $M > 3$. Further, we take $N \in 8\mathbb{Z} + 1$ and $M \in 16\mathbb{Z} + 1$ so that
\[3 < N < \frac{M}{2} .\]
Using the non-negativity property (40), we find
\[\int \int \hat{\Psi}_M(\xi - \eta) \cos(\pi|\xi|/4) \sin(\pi|\eta|/4) \varphi_M(\xi) \varphi_M(\eta) d\xi \frac{d\eta}{|\eta|} \geq C \int_{3 < |\eta| < N} \int_{|\xi| > 2N} \hat{\Psi}_M(\xi - \eta) \varphi_M(\xi) \varphi_M(\eta) d\xi \frac{d\eta}{|\eta|} .\]
For $|\xi| \geq 3$, $|\eta| \geq 3$ and $|\eta| < |\xi|/2$ we have
\[|\xi - \eta| \sim |\xi|, \quad \ln |\xi - \eta| \sim \ln |\xi| .\]
So the estimate (37) and the definition (41) of $\hat{\Psi}_M$ lead to the estimate
\begin{equation}
\int_{3 < |\eta| < N} \int_{|\xi| > 2N} \frac{\varphi_M(\xi)}{|\xi| \ln^{9/16} 16 |\xi|} \varphi_M(\eta) d\xi \frac{d\eta}{|\eta|} \leq C
\end{equation}
with some constant $C > 0$ independent of $M, N$. Now the definition (38) of $\varphi_M$ implies
\[\int_{3 < |\eta| < N} \varphi_M(\eta) \frac{d\eta}{|\eta|} \sim \sum_{2 \leq j \leq N, j \in \mathbb{Z} + 1} \frac{1}{j \ln^{5/8} 16 j} \sim \ln^{3/8} N.\]
In a similar way
\[\int_{|\xi| > 2N} \frac{\varphi_M(\xi)}{|\xi| \ln^{9/16} 16 |\xi|} d\xi \sim \sum_{2 \leq j \leq M, j \in \mathbb{Z} + 1} \frac{1}{j \ln^{9/16 + 5/8} 16 j} \sim \frac{1}{\ln^{3/16} N}\]
provided $M \geq N^2$. Consequently, the estimate (43) will lead to
\[\ln^{3/16} N \leq C\]
with $C > 0$ independent of $N$. This estimate is an obvious contradiction. This concludes the proof of Theorem 1.

References


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