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REPRESENTATION THEOREM FOR THE SOLUTION OF WEAKLY HYPERBOLIC EQUATIONS WITH FAST OSCILLATING COEFFICIENTS

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Dedicated to Professor Sergio Spagnolo on the occasion of his 60th birthday

Abstract. This work is concerned with the influence of oscillations in weakly hyperbolic operators on the well-posedness of the Cauchy problem. The fundamental solution to the Cauchy problem is constructed for the equations with oscillations in the coefficient very close to the ones destroying the C^∞ well-posedness.

1. Introduction. The subject of this paper concerns with the investigation of the influence of the oscillations in weakly hyperbolic operators on the well-posedness of the Cauchy problem. Since the example constructed by F. Colombini and S. Spagnolo [1], it is well-known that oscillations can break down the well-posedness. Namely, they constructed a second order equation $\partial_t^2 u - a(t)\partial_x^2 u = f(t,x)$, with the smooth coefficient $a \in C^\infty$, for which the Cauchy problem is not C^∞ well-posed (see also [2]). The proof is based on the very delicate investigation of the energy of solutions. For the equation

$$\partial_t^2 u - \exp(-2t^{-\alpha}) b(t^{-1})^2 \partial_x^2 u = 0, \qquad \alpha = \text{const.} > 0\,, \tag{1.1}$$

where $b(s)$ is a non-constant, positive and smooth 1-periodic function on $\mathbb{R}$, the energy method convinces that the Cauchy problem is C^∞ well-posed if $\alpha \geq 1$. S. Tarama [6]

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appeals to the Floquet theory to prove that the problem is C^∞ well-posed if and only if $\alpha \geq 1/2$.

For second order equations with coefficients independent of the spatial variables some sufficient for the well-posedness conditions are given in [7]. The equation under consideration in [7] is the following:

$$D_t^2 u + \sum_{|\alpha|=2} a_{0,\alpha}(t) D_x^\alpha u + \sum_{j+|\alpha|\leq 1} a_{j,\alpha}(t) D_t^j D_x^\alpha u = f. \tag{1.2}$$

It is supposed that the principal symbol can be written in the form

$$\tau^2 + \sum_{|\alpha|=2} a_{0,\alpha}(t)\xi^\alpha = (\tau - \lambda_1(t,\xi))(\tau - \lambda_2(t,\xi)), \tag{1.3}$$

with the real-valued functions $\lambda_l(t,\xi)$ $(l = 1, 2)$ which satisfy the conditions

$$|\lambda_l(t,\xi)| \leq c\lambda(t)|\xi|, \quad l = 1,2, \quad |\lambda_1(t,\xi) - \lambda_2(t,\xi)| \geq \delta\lambda(t)|\xi|, \quad \delta = \text{const.} > 0,$$

for all $t \in [0,T]$, $\xi \in \mathbb{R}^n_\xi$. Here $\lambda \in C^2([0,T])$, $\lambda(0) = \lambda'(0) = 0$, $\lambda'(t) > 0$ for $t > 0$. Thus, at $t = 0$ the operator has multiple characteristics. Furthermore, it is assumed in [7] that the following inequalities are satisfied:

$$|D_t^k \operatorname{Re} a_{0,\alpha}(t)| \leq C\lambda^2(t)\Big(\frac{|\log\lambda(t)|}{\Lambda(t)}\Big)^{2-|\alpha|}\Big(\frac{\lambda(t)\,|\log\lambda(t)|}{\Lambda(t)}\Big)^k, \quad k = 0,1,2, \tag{1.4}$$

$$|D_t^k \operatorname{Im} a_{0,\alpha}(t)| \leq C\,\frac{\lambda^2(t)}{\Lambda(t)}\Big(\frac{\lambda(t)\,|\log\lambda(t)|}{\Lambda(t)}\Big)^k, \quad k = 0,1, \tag{1.5}$$

for all $t \in (0,T]$, $0 < |\alpha| \leq 2$. Then in [7] it is proved that for equation (1.2) the Cauchy problem is C^∞ well-posed. The conditions (1.4) and (1.5) couple together an oscillation with the degeneracy of the principal part.

For the equation (1.1) one can set $\lambda_1(t,\xi) = -\lambda_2(t,\xi) = \exp(-t^{-\alpha})b(t^{-1})|\xi|$, and $\lambda(t) = \exp(-t^{-\alpha})$. Then the *critical value* $\alpha = 1/2$ of (1.1) is reflected in (1.4) by the term $\Big(\frac{\lambda(t)|\log\lambda(t)|}{\Lambda(t)}\Big)^k$ containing $|\log\lambda(t)|$. Indeed, to satisfy that condition with $k = 1$ we have to require $0 < \text{const.} \leq t^2\lambda(t)\,|\log\lambda(t)|\,/\Lambda(t)$ for all $t \in (0,T]$, which is equivalent to $\alpha \geq 1/2$. In [7] such equations are called *equations with fast oscillating coefficients*, while the equations with coefficients satisfying estimates with $\Big(\frac{\lambda(t)}{\Lambda(t)}\Big)^k$ (that corresponds to $\alpha \geq 1$) one can call possessing *slowly oscillating coefficients*. All other cases (corresponding to $\alpha < 1/2$) can be regarded as *very fast oscillating*. Such classification is useful as well in completely other problem of L_p–L_q decay (as $t \to \infty$) estimates for strictly hyperbolic equations (see [4]), where anew oscillations can have destructive consequences.

On the other hand after an investigation of the well-posedness the next interesting question is a construction of the fundamental solution (or of the parametrix) and a description of the propagation of singularities in the framework of the micro-local analysis. For the operators with slow oscillations such construction can be found in [8]. The goal of the present note is to fill up the gap for the equations with oscillations in the coefficients very close to the ones destroying the well-posedness. Thus in this paper we consider the critical case of equations with fast oscillating coefficients.

Let $p = p(t)$ be a smooth function $p \in C^\infty(0,T]$ $(0 < T < 1)$, satisfying

$$\begin{aligned} &p'(t) \le -\frac{\gamma}{t} \; ({}^\exists\gamma > 1), && \forall t \in (0,T], \\ &0 \le p''(t) \le {}^\exists C|p(t)p'(t)|^2, && \forall t \in (0,T]. \end{aligned}$$

Let $a(t)$ be a non-negative function such that

$$a(t) = \lambda(t)b(t),$$

where

$$\lambda(t) = \begin{cases} e^{-p(t)} & \text{for } 0 < t \le T, \\ 0 & \text{for } t = 0, \end{cases}$$

belongs to $C^1[0,T]$ and $b(t)$ is a uniformly positive smooth function satisfying

$$(1.6) \quad b \in C^\infty(0,T] \quad \text{and} \quad |\partial_t^h b(t)| \le {}^\exists C_h |p(t)p'(t)|^h \text{ for } h = 1, 2, \ldots, \; 0 < t \le T.$$

The function $p(t)$ implies the speed of degeneracy while the function $b(t)$ describes the oscillations. Typical examples are the following:

$$a_1(t) = \begin{cases} \exp(-t^{-\alpha})\tilde{b}(t^{-1}) & \text{for } 0 < t \le T, \\ 0 & \text{for } t = 0, \end{cases} \qquad a_2(t) = \begin{cases} t^\gamma \; \tilde{b}\big((-\log t)^\beta\big) & \text{for } 0 < t \le T, \\ 0 & \text{for } t = 0, \end{cases}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 1$ and $\tilde{b}(s)$ is a non-constant, uniformly positive, smooth and 1-periodic function on $(0,\infty)$.

In this paper we shall consider

$$(1.7) \qquad \begin{cases} \partial_t^2 u - a(t)^2 \partial_x^2 u = 0 & \text{in } [0,T] \times \mathbb{R}_x, \\ u(0,x) = u_0(x), \; \partial_t u(0,x) = u_1(x) & \text{in } \mathbb{R}_x. \end{cases}$$

By $H^s(\mathbb{R}_x)$ we denote the Sobolev space equipped with the norm

$$\|u\|_s := \Big(\int_{\mathbb{R}} \sum_{k \le s} |\partial_x^k u(x)|^2 \, dx\Big)^{1/2}.$$

Further $H^\infty(\mathbb{R}_x) := \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}_x)$.

Theorem 1.1.

a) *Assume that $b(t)$ satisfies* (1.6). *Then for every u_0, $u_1 \in H^\infty(\mathbb{R}_x)$, the Cauchy problem* (1.7) *has a unique solution $u \in C^2\big([0,T], H^\infty(\mathbb{R}_x)\big)$ represented as follows*:

$$u(t,x) = \sum_{l=0,1} \sum_{m=1,2} \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i[(x-y)\cdot\xi + \varphi_m(t,\xi)]} a_{lm}(t,\xi) u_l(y) \, dy \, d\xi,$$

where $\varphi_m(t,\xi) = (-1)^m \int_0^t a(\tau)\,d\tau \cdot \xi$ for $m = 1,2$, while there exist $C_{h\alpha} > 0$, $M > 0$ and $0 \le \rho_1 < \rho_2 \le 1$ such that

$$(1.8) \quad \sup_{\substack{0 \le t \le T, \\ |\xi| \ge 1}} |D_t^h D_\xi^\alpha a_{lm}(t,\xi)| \le C_{h\alpha} \langle\xi\rangle^{M + \rho_1 h - \rho_2 \alpha} \quad \textit{for } \alpha \ge 0, \quad h, l = 0, 1, \quad m = 1, 2.$$

b) *The representation is valid for every u_0, $u_1 \in H^M(\mathbb{R}_x)$, and*

$$\mathrm{WF}\big(u(t)\big) \subset \Big\{(x,\xi) = \Big(y \pm \int_0^t a(\tau)\,d\tau, \eta\Big) : (y,\eta) \in \mathrm{WF}(u_0) \cup \mathrm{WF}(u_1)\Big\}.$$

c) *The problem possesses the finite propagation speed property that for smooth coefficient* $a^2 \in C^\infty([0,T])$ *together with* a) *leads to* C^∞ *well-posedness, that is, for every* u_0, $u_1 \in C^\infty(\mathbb{R}_x)$, *the Cauchy problem* (1.7) *has a unique solution* $u \in C^\infty\big([0,T] \times \mathbb{R}_x\big)$.

Remark 1.2. In particular when $a(t) \equiv a_1(t)$ (resp. $a_2(t)$), the condition (1.6) corresponds to $\alpha \geq 1/2$ (resp. $\beta \leq 2$). For $\alpha < 1/2$ according to [6] the Cauchy problem is not C^∞ well-posed. The results of a) and c) are optimal. While, the result b) can be obtained from the general theory of Fourier integral operators.

Remark 1.3. If $\lambda(t)$ vanishes of infinite order, the Cauchy problem (1.7) has a unique solution $u \in C^\infty\big([0,T] \times \mathbb{R}_x\big)$ and (1.8) holds for any $h \geq 0$.

2. Notation and classes of symbols. In this paper we often use the cut-off functions $\chi(s)$ and $\psi(s)$ such that

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \qquad \chi'(s) \leq 0 \quad \text{and} \quad \psi(s) = 1 - \chi(s).$$

We define $\Lambda(t) = \int_0^t \lambda(\tau)\,d\tau$ and $\Lambda^*(t) = \frac{\Lambda(t)}{p(t)}$. Let $N > 0$ and $\langle\xi\rangle = \sqrt{e^2 + |\xi|^2}$ $(\geq e)$.

Definition 2.1. The functions $t_N(\xi)$ and $\tilde{t}_N(\xi)$ are (unique) roots of $\Lambda^*(t)\langle\xi\rangle = 2N\log\langle\xi\rangle$ and $\Lambda^*(t)\langle\xi\rangle = 4N\log\langle\xi\rangle$, respectively, i.e., $\Lambda^*(t_N(\xi))\langle\xi\rangle = 2N\log\langle\xi\rangle$ and $\Lambda^*(\tilde{t}_N(\xi))\langle\xi\rangle = 4N\log\langle\xi\rangle$.

Definition 2.2. We define the *hyperbolic zone*

$$Z_N(t,\xi) = \big\{(t,\xi) \in [0,T] \times \mathbb{R}_\xi : \Lambda^*(t)\langle\xi\rangle \geq 2N\log\langle\xi\rangle \text{ and } |\xi| \geq 1\big\}.$$

Definition 2.3. Let m_1, m_2 and m_3 be real numbers. We define the spaces of the symbols

$$S_N(m_1,m_2,m_3) = \Big\{a(t,\xi) \in C^\infty : \sup_{(t,\xi)\in Z_N} \frac{\langle\xi\rangle^{|\alpha|-m_1}|D_t^h D_\xi^\alpha a(t,\xi)|}{\lambda(t)^{m_2}|p(t)p'(t)|^{m_3+h}} \leq C_{h\alpha}\Big\}, \tag{2.1}$$

$$S_N^{-\infty}(m_1,m_2,m_3) = \bigcap_{k=0}^{\infty} S_N(m_1-k, m_2-k, m_3+k). \tag{2.2}$$

Remark 2.4. The following properties are known (see [7]).

(i) $S_N(m_1,m_2,m_3) \supset S_N(m_1-k,m_2-k,m_3+k)$ for $k \geq 0$.

(ii) If $a \in S_N(m_1,m_2,m_3)$, then $D_\xi^\alpha a \in S_N(m_1-|\alpha|,m_2,m_3)$.

(iii) If $a \in S_N(m_1,m_2,m_3)$ and $b \in S_N(\tilde{m}_1,\tilde{m}_2,\tilde{m}_3)$ (resp. $S_N^{-\infty}(\tilde{m}_1,\tilde{m}_2,\tilde{m}_3)$), then $ab \in S_N(m_1+\tilde{m}_1, m_2+\tilde{m}_2, m_3+\tilde{m}_3)$ (resp. $S_N^{-\infty}(m_1+\tilde{m}_1, m_2+\tilde{m}_2, m_3+\tilde{m}_3)$).

Proposition 2.5 ([7]). *Suppose that for all* $k = 1,2,\ldots$, $a_k \in S_N(m_1-k+1, m_2-k+1, m_3+k-1)$, $a_k(t,\xi) = 0$ *for* $0 \leq t \leq t_N(\xi)$, $|\xi| \geq 1$. *Then there exists a symbol* $a(t,\xi) \in S_N(m_1,m_2,m_3)$ *such that*

$$\operatorname{supp} a \subset Z_N(t,\xi) \text{ and } a \sim a_1 + a_2 + \ldots \mod S_N^{-\infty}(m_1,m_2,m_3)$$

in the sense that $a - a_1 - \ldots - a_k \in S_N(m_1-k, m_2-k, m_3+k)$ *for all* $k = 0,1,\ldots$.

3. Reduction to a first order diagonal system. Without loss of generality we may suppose that $b(t)$ has the form

$$b(t) \equiv 1 + c(t),$$

where $c(t)$ is a smooth function satisfying

$$|c(t)| \le 1/2,\ c \in C^\infty(0,T] \quad \text{and} \quad |\partial_t^h c(t)| \le C_h |p(t)p'(t)|^h \ ({}^\exists C_h > 0).$$

By Fourier transform the Cauchy problem (1.7) is changed into

$$\begin{cases} \partial_t^2 v + a(t,\xi)^2 v = 0 & \text{in } [0,T] \times \mathbb{R}_\xi, \\ v(0,\xi) = v_0(\xi), \quad \partial_t v(0,\xi) = v_1(\xi) & \text{in } \mathbb{R}_\xi, \end{cases} \tag{3.1}$$

where $a(t,\xi) = \lambda(t)\{1 + c(t)\}|\xi| \big(= a(t)|\xi|\big)$.

Definition 3.1. The functions $\varepsilon = \varepsilon(\xi)$ and $\tilde\varepsilon = \tilde\varepsilon(\xi)$ are (unique) roots of the equations $\lambda(t)\langle\xi\rangle = 1$ and $\lambda(t)\langle\xi\rangle = 2$, respectively, i.e., $\lambda(\varepsilon)\langle\xi\rangle = 1$ and $\lambda(\tilde\varepsilon)\langle\xi\rangle = 2$.

Definition 3.2. The functions $\delta = \delta(\xi)$ and $\tilde\delta = \tilde\delta(\xi)$ are (unique) roots of the equations $\Lambda(t)\langle\xi\rangle = N\log\langle\xi\rangle$ and $\Lambda(t)\langle\xi\rangle = 2N\log\langle\xi\rangle$, respectively, i.e., $\Lambda(\delta)\langle\xi\rangle = N\log\langle\xi\rangle$ and $\Lambda(\tilde\delta)\langle\xi\rangle = 2N\log\langle\xi\rangle$.

Lemma 3.3. *For sufficiently large $N > 0$, the following relation holds*:

$$0 < \varepsilon < \tilde\varepsilon < \delta < \tilde\delta \le t_N(\xi) < \tilde t_N(\xi) \quad \text{for} \quad \xi \in \mathbb{R}_\xi.$$

Now we approximate $a(t,\xi)$ with two functions defined by the following formulas.

$$a^*(t,\xi) = \chi\big(\lambda(t)\langle\xi\rangle\big) + \psi\big(\lambda(t)\langle\xi\rangle\big)\lambda(t)\Big\{1 + \psi\Big(\frac{\Lambda(t)\langle\xi\rangle}{N\log\langle\xi\rangle}\Big)c(t)\Big\}\langle\xi\rangle,$$

$$a^{**}(t,\xi) = \frac{\lambda(\tilde\delta)\langle\xi\rangle}{2}\chi\Big(\frac{\Lambda(t)\langle\xi\rangle}{N\log\langle\xi\rangle}\Big) + \psi\Big(\frac{\Lambda(t)\langle\xi\rangle}{N\log\langle\xi\rangle}\Big)\lambda(t)\Big\{1 + \psi\Big(\frac{\Lambda(t)\langle\xi\rangle}{N\log\langle\xi\rangle}\Big)c(t)\Big\}\langle\xi\rangle.$$

Noting that $|c(s)| \le 1/2$ and considering the supports of χ and ψ, we get

Lemma 3.4. *For any $t \in [0,T]$ and $\xi \in \mathbb{R}_\xi$, $a^*(t,\xi)$ and $a^{**}(t,\xi)$ are positive, more precisely*

$$a^*(t,\xi) \ge \max\Big\{\frac{\lambda(t)\langle\xi\rangle}{2}, \frac{1}{2}\Big\}, \qquad a^{**}(t,\xi) \ge \max\Big\{\frac{\lambda(t)\langle\xi\rangle}{2}, \frac{\lambda(\delta)\langle\xi\rangle}{2}\Big\},$$

$$|D_t^h D_\xi^\alpha a^*(t,\xi)| \le C_{h\alpha}\langle\xi\rangle^{-|\alpha|}\big(\max\{|p'(\varepsilon)|, |p'(\delta)|p(\delta)\}\big)^h a^*(t,\xi),$$

$$|D_t^h D_\xi^\alpha a^{**}(t,\xi)| \le C_{h\alpha}\langle\xi\rangle^{-|\alpha|}\big(|p'(\delta)|p(\delta)\big)^h a^{**}(t,\xi).$$

Putting $W = \begin{pmatrix} a^{**}(t,\xi)^{1/2} v \\ \partial_t\{a^{**}(t,\xi)^{1/2} v\} \end{pmatrix}$ and $W_0 := W\big|_{t=0} = \Big\{\frac{\lambda(\tilde\delta)\langle\xi\rangle}{2}\Big\}^{1/2}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ and multiplying both sides of (3.1) by $a^{**}(t,\xi)^{1/2}$, we find that the Cauchy problem (3.1) is equivalent to the one for the system,

$$\begin{cases} \partial_t W = \begin{pmatrix} 0 & 1 \\ -a(t,\xi)^2 - r(t,\xi) & \frac{\partial_t a^{**}(t,\xi)}{a^{**}(t,\xi)} \end{pmatrix} W, \\ W(0) = W_0, \end{cases}$$

where $r(t,\xi) = \dfrac{3}{4}\Big(\dfrac{\partial_t a^{**}(t,\xi)}{a^{**}(t,\xi)}\Big)^2 - \dfrac{1}{2}\dfrac{\partial_t^2 a^{**}(t,\xi)}{a^{**}(t,\xi)}$.

Further we make one step of the diagonalization by putting

$$W^* = \begin{pmatrix} 1 & 1 \\ -ia^*(t,\xi) & ia^*(t,\xi) \end{pmatrix}^{-1} W \quad \text{and} \quad W_0^* := W^*\big|_{t=0} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} W_0,$$

and multiplying both sides of the system by $\begin{pmatrix} 1 & 1 \\ -ia^*(t,\xi) & ia^*(t,\xi) \end{pmatrix}^{-1}$, where $a^*(t,\xi)^{-1}$ belongs to $S_N(-1,-1,0)$. Then we obtain

$$\begin{cases} \partial_t W^* = D(t,\xi)W^* + B(t,\xi)W^*, \\ W^*(0) = W_0^*, \end{cases} \tag{3.2}$$

where $D = ia(t,\xi)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$\begin{aligned} B = {} & \frac{1}{2}\Big\{\frac{\partial_t a^{**}(t,\xi)}{a^{**}(t,\xi)} - \frac{\partial_t a^*(t,\xi)}{a^*(t,\xi)}\Big\}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ & + i\big\{a^*(t,\xi) - a(t,\xi)\big\}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a(t,\xi)^2 - a^*(t,\xi)^2 + r(t,\xi)}{2ia^*(t,\xi)}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \end{aligned}$$

Definition 3.5. We define recursively the sequence $\big\{B^{(k)}(t,\xi)\big\}_{k\ge 1}$ as

$$\begin{cases} B^{(1)}(t,\xi) = B(t,\xi) - \chi\Big(\dfrac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)B(t,\xi), \\ B^{(k)}(t,\xi) = \tilde{B}^{(k)}(t,\xi) - \chi\Big(\dfrac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)B(t,\xi) \quad \text{for } k = 2,3,\ldots, \end{cases}$$

where

$$\begin{aligned} \tilde{B}^{(k)}(t,\xi) = {} & \Big(I + \sum_{j=1}^{k-1} H^{(j)}(t,\xi)\Big)\Big(\partial_t - D(t,\xi) - \sum_{j=1}^{k-1} F^{(j)}(t,\xi)\Big) \\ & - \Big(\partial_t - D(t,\xi) - B(t,\xi)\Big)\Big(I + \sum_{j=1}^{k-1} H^{(j)}(t,\xi)\Big), \end{aligned}$$

$$F^{(k)}(t,\xi) = \operatorname{diag} B^{(k)}(t,\xi), \quad H^{(k)}(t,\xi) = \frac{1}{2ia(t,\xi)}\begin{pmatrix} 0 & B_{12}^{(k)}(t,\xi) \\ -B_{21}^{(k)}(t,\xi) & 0 \end{pmatrix}.$$

When $0 \le t \le t_N(\xi)$, we see that $B^{(1)}(t,\xi) = 0$ and recursively $B^{(k)}(t,\xi) = 0$ for $k = 1,2,\ldots$. Thus, we obtain $H^{(k)}(t,\xi) = 0$ for $k = 1,2,\ldots$. Moreover we can derive for $k = 2,3,\ldots$

$$\begin{aligned} B^{(k)} = {} & \sum_{j=1}^{k-1}[D,H^{(j)}] - \sum_{j=1}^{k-1}\partial_t H^{(j)} - \Big(I + \sum_{j=1}^{k-1}H^{(j)}\Big)\sum_{j=1}^{k-1}F^{(j)} + B\Big(I + \sum_{j=1}^{k-1}H^{(j)}\Big) - \chi B \\ = {} & \big\{B^{(k-1)} + \big[D,H^{(k-1)}\big] - F^{(k-1)}\big\} + BH^{(k-1)} - \partial_t H^{(k-1)} \\ & - \sum_{j=1}^{k-1}H^{(k-1)}F^{(j)} - \sum_{j=1}^{k-2}H^{(j)}F^{(k-1)}. \end{aligned}$$

Noting that $B^{(k-1)} + [D,H^{(k-1)}] - F^{(k-1)} \equiv 0$, we also obtain recursively the following

LEMMA 3.6. *For sufficiently large $N > 0$,*

$$B^{(k)} \in S_N(-k,-k,k+1), \quad H^{(k)} \in S_N(-k-1,-k-1,k+1) \quad \textit{for } k = 1,2,\dots,$$

$$F^{(k)} \in \begin{cases} S_N(-1,-1,2) & \textit{for } k=1, \\ S_N(-k-1,-k-1,k+2) & \textit{for } k=2,3,\dots \end{cases}$$

By Proposition 2.5 there exists $H(t,\xi) = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix} \in S_N(-2,-2,2)$ such that $\operatorname{supp} H \subset Z_N(t,\xi)$ and $H \sim H^{(1)} + H^{(2)} + \dots \mod S_N^{-\infty}(-2,-2,2)$, and for sufficiently large $N > 0$, there exists $H^*(t,\xi) \in S_N(-2,-2,2)$ defined by

$$H^*(t,\xi) \equiv \frac{1}{1 - H_{12}H_{21}} \begin{pmatrix} H_{12}H_{21} & -H_{12} \\ -H_{21} & H_{12}H_{21} \end{pmatrix} = \bigl(I + H(t,\xi)\bigr)^{-1} - I.$$

Finally, putting

$$W^{**} = \bigl(I + H^*(t,\xi)\bigr)W^* \quad \text{and} \quad W_0^{**} := W^{**}\big|_{t=0} = \bigl(I + H^*(0,\xi)\bigr)W_0^* = W_0^*,$$

and multiplying both sides of (3.2) by $\bigl(I + H^*(t,\xi)\bigr)$, we obtain

$$\tag{3.3} \begin{cases} \partial_t W^{**} = D(t,\xi)W^{**} + \operatorname{diag}\bigl\{B^*(t,\xi)\bigr\}W^{**} + \chi\Bigl(\dfrac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Bigr)B(t,\xi)W^{**} \\ \qquad\qquad + \chi\Bigl(\dfrac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Bigr)B^{**}(t,\xi)W^{**} + R(t,\xi)W^{**}, \\ W^{**}(0) = W_0^{**}, \end{cases}$$

where $R(t,\xi) \subset S_N^{-\infty}(-1,-1,2)$, $B^*(t,\xi) = B^{(1)} + BH \in S_N(-1,-1,2)$ and $B^{**}(t,\xi) = H^*B \in S_N(-3,-3,4)$.

4. Some estimates. In this section we show some estimates which will be used to construct the fundamental solution of (3.3). For simplicity we define $\chi^*(t) = \chi\bigl(\lambda(t)\langle\xi\rangle\bigr)$, $\chi^{**}(t) = \chi\Bigl(\frac{\Lambda(t)\langle\xi\rangle}{N\log\langle\xi\rangle}\Bigr)$ and $\psi^*(t) = 1 - \chi^*(t)$, $\psi^{**}(t) = 1 - \chi^{**}(t)$.

Noting the support of $\chi\Bigl(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Bigr)$, we obtain

$$\begin{aligned} \int_0^T \chi\Bigl(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Bigr)B(t,\xi)\,dt &\le \frac{1}{2}\int_0^{\tilde t_N(\xi)} \Bigl|\frac{\partial_t a^{**}}{a^{**}} - \frac{\partial_t a^*}{a^*}\Bigr|\,dt + \int_0^{\tilde t_N(\xi)} |a^* - a|\,dt \\ &\quad + \frac{1}{2}\int_0^{\tilde t_N(\xi)} \frac{|a^2 - a^{*2}|}{a^*}\,dt + \frac{1}{2}\int_0^{\tilde t_N(\xi)} \frac{|r|}{a^*}\,dt \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since $\partial_t a^*(t,\xi) = 0$ for $0 \le t \le \varepsilon$, $\partial_t a^{**}(t,\xi) = 0$ for $0 \le t \le \delta$ and $a^*(t,\xi) = a^{**}(t,\xi)$ for $\tilde\delta \le t\bigl(\le \tilde t_N(\xi)\bigr)$, we deduce that

$$I_1 = \frac{1}{2}\int_\varepsilon^{\tilde\delta} \Bigl|\frac{\partial_t a^{**}}{a^{**}} - \frac{\partial_t a^*}{a^*}\Bigr|\,dt \le \frac{1}{2}\int_\varepsilon^{\tilde\delta} \Bigl|\frac{\partial_t a^*}{a^*}\Bigr|\,dt + \frac{1}{2}\int_\delta^{\tilde\delta} \Bigl|\frac{\partial_t a^{**}}{a^{**}}\Bigr|\,dt \equiv I_1^* + I_1^{**}.$$

Noting the supports of $\chi^{*\prime}$, $\psi^{*\prime}$ and $\psi^{**\prime}$, by (1.6) and Lemma 3.4 we obtain

$$I_1^* \le C\int_\varepsilon^{\tilde\delta} \Bigl\{\frac{|\chi^{*\prime}|}{a^*} + \frac{|\psi^{*\prime}|\lambda\langle\xi\rangle}{a^*} + \frac{\psi^*|\lambda'|\langle\xi\rangle}{a^*} + \frac{|\psi^{**\prime}|\lambda\langle\xi\rangle}{a^*} + \frac{\psi^{**}\lambda|c'|\langle\xi\rangle}{a^*}\Bigr\}\,dt$$

$$\leq C\Big\{\int_\varepsilon^{\tilde\varepsilon} |\chi^{*\prime}|\,dt + \int_\varepsilon^{\tilde\varepsilon} \lambda'\langle\xi\rangle\,dt + \int_\varepsilon^{\tilde\delta} \frac{\lambda'}{\lambda}\,dt + \int_\delta^{\tilde\delta} \frac{\lambda\langle\xi\rangle}{N\log\langle\xi\rangle}\,dt + \int_\delta^{\tilde\delta} |pp'|\,dt\Big\}$$

$$\leq C\Big\{1 + 1 + \int_\varepsilon^{\tilde\delta} |p'|dt + 1 + p(\delta)\int_\delta^{\tilde\delta} |p'|\,dt\Big\} \leq Cp(\varepsilon).$$

Similarly we also obtain $I_1^{**} \leq Cp(\varepsilon)$. Thus we get $I_1 \leq I_1^* + I_1^{**} \leq Cp(\varepsilon)$. Since $a(t,\xi) = a^*(t,\xi)$ for $\tilde\delta \leq t\big(\leq \tilde t_N(\xi)\big)$, we get $I_2 = \int_0^{\tilde\delta} |a^* - a|\,dt \leq C\Lambda(\tilde\delta)\langle\xi\rangle \leq C_N\log\langle\xi\rangle$. Similarly, by Lemma 3.4 we get $I_3 \leq C_N\log\langle\xi\rangle$. Since $\partial_t a^{**}(t,\xi) = 0$, $\partial_t^2 a^{**}(t,\xi) = 0$ for $0 \leq t \leq \delta$, we deduce that

$$I_4 \leq C\int_\delta^{\tilde t_N(\xi)} \frac{1}{a^*}\Big|\frac{\partial_t a^{**}}{a^{**}}\Big|^2\,dt + C\int_\delta^{\tilde t_N(\xi)} \frac{1}{a^*}\Big|\frac{\partial_t^2 a^{**}}{a^{**}}\Big|\,dt \equiv I_4^* + I_4^{**}.$$

The last estimate corresponds to the estimate in the *oscillation's subzone* of [5]. Noting the supports of $\chi^{**\prime}$ and $\psi^{**\prime}$, by (1.6) and Lemma 3.4 we obtain

$$I_4^* \leq C\int_\delta^{\tilde t_N} \Big\{\frac{|\lambda(\tilde\delta)\langle\xi\rangle\chi^{**\prime}|^2}{a^*a^{**2}} + \frac{|\psi^{**\prime}|^2\lambda^2\langle\xi\rangle^2}{a^*a^{**2}} + \frac{\psi^{**}|\lambda'|^2\langle\xi\rangle^2}{a^*a^{**2}} + \frac{\psi^{**}\lambda^2|c'|^2\langle\xi\rangle^2}{a^*a^{**2}}\Big\}\,dt$$

$$\leq C\Big\{\int_\delta^{\tilde\delta} \frac{\lambda\langle\xi\rangle}{(N\log\langle\xi\rangle)^2}\,dt + \int_\delta^{\tilde t_N} \Big(\frac{\lambda'}{\lambda}\Big)^2\frac{1}{\lambda\langle\xi\rangle}\,dt + \int_\delta^{\tilde t_N} \frac{|pp'|^2}{\lambda\langle\xi\rangle}dt\Big\} \leq C\int_\delta^{\tilde t_N} \frac{p^2|p'|^2}{\lambda\langle\xi\rangle}\,dt.$$

Similarly we also obtain $I_4^{**} \leq C\int_\delta^{\tilde t_N} \frac{p^2|p'|^2}{\lambda\langle\xi\rangle}\,dt$. Thus $I_4 \leq I_4^* + I_4^{**} \leq C\int_\delta^{\tilde t_N(\xi)} \frac{p^2|p'|^2}{\lambda\langle\xi\rangle}\,dt \leq C\frac{p(\delta)^2|p'(\delta)|}{\langle\xi\rangle}\int_\delta^{\tilde t_N} \big(-p'(t)\big)e^{p(t)}\,dt \leq Cp(\delta)^2|p'(\delta)|e^{p(\delta)}\langle\xi\rangle^{-1}$.

Combining I_1–I_4 and noting that $|p'(\delta)|e^{p(\delta)} \leq \frac{\langle\xi\rangle}{N\log\langle\xi\rangle}$, we have for $\xi \in \mathbb{R}_\xi$

$$\int_0^T \chi\Big(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)B\,dt \leq Cp(\varepsilon) + C_N\log\langle\xi\rangle + Cp(\delta)^2|p'(\delta)|e^{p(\delta)}\langle\xi\rangle^{-1} \leq C_N\log\langle\xi\rangle. \tag{4.1}$$

Noting the supports of $\chi\Big(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)$ and observing that $B^{**}(t,\xi) \in S_N(-3,-3,4)$, we obtain for $\xi \in \mathbb{R}_\xi$

$$\int_0^T \chi\Big(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)B^{**}\,dt \leq C\int_{t_N}^{\tilde t_N} \frac{p(t)^4|p'(t)|^4}{\langle\xi\rangle^3\lambda(t)^3}\,dt = C\int_{t_N}^{\tilde t_N} \frac{p(t)|p'(t)|}{\big(\Lambda^*(t)\langle\xi\rangle\big)^3}\,dt \leq C. \tag{4.2}$$

Since $R(t,\xi) \in S_N^{-\infty}(-1,-1,2) \in S_N(-3,-3,4)$, similarly we have for $\xi \in \mathbb{R}_\xi$

$$\int_0^T R\,dt \leq C\int_{t_N}^T \frac{p(t)^{k+2}|p'(t)|^{k+2}}{\langle\xi\rangle^{k+1}\lambda(t)^{k+1}}\,dt = C\int_{t_N}^T \frac{p(t)|p'(t)|}{\big(\Lambda^*(t)\langle\xi\rangle\big)^{k+1}}\,dt. \tag{4.3}$$

5. The representation formula. We define the diagonal matrix function

$$E(t,\xi) = \exp\Big\{\int_0^t D(\tau,\xi)\,d\tau\Big\} = \begin{pmatrix} \exp\{-i\int_0^t a(\tau,\xi)\,d\tau\} & 0 \\ 0 & \exp\{i\int_0^t a(\tau,\xi)\,d\tau\}\end{pmatrix}.$$

For every $\Psi(\xi)$ the vector function $V(t,\xi) = E(t,\xi)\Psi(\xi)$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t V = D(t,\xi)V \quad \text{in } [0,T]\times\mathbb{R}_\xi, \\ V(0,\xi) = \Psi(\xi). \end{cases} \tag{5.1}$$

Now we put

$$G(t,\xi) \equiv E^{-1}\Big\{\mathrm{diag}\{B^*\} + \chi\Big(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)B + \chi\Big(\frac{\Lambda^*(t)\langle\xi\rangle}{2N\log\langle\xi\rangle}\Big)B^{**} + R\Big\}E.$$

Then the matrix function

$$K(t,\xi) = \sum_{j=1}^{\infty}\int_0^t G(t_1,\xi)\,dt_1\int_0^{t_1} G(t_2,\xi)\,dt_2\cdots\int_0^{t_{j-1}} G(t_j,\xi)\,dt_j$$

is the solution of the Cauchy problem

$$\begin{cases} \partial_t K(t,\xi) = G(t,\xi)K(t,\xi) + G(t,\xi) \quad \text{in } [0,T]\times\mathbb{R}_\xi, \\ K(0,\xi) = 0. \end{cases} \tag{5.2}$$

Moreover by (4.1), (4.2) and (4.3) we get the following result.

PROPOSITION 5.1. *There exist $M_N > 0$ and a strictly increasing function $\phi(s)$ satisfying $\lim_{s\to+\infty}\phi(s)/s = 0$ such that*

$$\sup_{0\le t\le T,\,|\xi|\ge1}\big|D_t^h D_\xi^\alpha K\big| \le C_{Nh\alpha}\langle\xi\rangle^{M_N}\phi(\langle\xi\rangle)^h\Big\{\frac{(\log\langle\xi\rangle)^2}{\langle\xi\rangle}\Big\}^\alpha \quad \textit{for } h=0,1,\quad \alpha\ge0. \tag{5.3}$$

REMARK 5.2. If we take $\phi(s) = \lambda\Big(\Lambda^{*-1}\big(\frac{4N\log s}{s}\big)\Big)s$ which is a strictly increasing function and satisfies $\lim_{s\to+\infty}\phi(s)/s = 0$, then (5.3) holds. In case that $a(t)\equiv a_1(t)$ (resp. $a_2(t)$) (see Section 1), we can set $\phi(s) = C(\log s)^{3+1/\alpha}$ (resp. $C(\log s)^2 s^{1/\gamma}$).

By (5.1) and (5.2) we have

$$\begin{aligned} \partial_t\big\{E(t,\xi)\big(I+K(t,\xi)\big)\Psi(\xi)\big\} &= \partial_t(E\Psi) + (\partial_t E)K\Psi + E(\partial_t K)\Psi \\ &= D(E\Psi) + DK\Psi + E(GK+G)\Psi \\ &= D\big\{E(I+K)\Psi\big\} + EGE^{-1}\big\{E(I+K)\Psi\big\}. \end{aligned}$$

This means that the matrix function $E(t,\xi)\big(I+K(t,\xi)\big)$ is the fundamental solution of the Cauchy problem (3.3). Thus we have the following statement.

THEOREM 5.3. *Assume that $b(t)$ satisfies* (1.6). *Then the solution $v(t,\xi)$ to the Cauchy problem* (3.1) *can be represented as*

$$v(t,\xi) = \Big\{\frac{\lambda(\tilde\delta)\langle\xi\rangle}{8a^{**}}\Big\}^{1/2}\big(H_{21}(t,\xi)+1, H_{12}(t,\xi)+1\big)E(t,\xi)\big(I+K(t,\xi)\big)\begin{pmatrix} v_0+iv_1 \\ v_0-iv_1\end{pmatrix},$$

and

$$\sup_{0\le t\le T,\ |\xi|\ge1}|D_t^h D_\xi^\alpha v(t,\xi)| \le C\langle\xi\rangle^{M_N}\phi(\langle\xi\rangle)^h\Big\{\frac{(\log\langle\xi\rangle)^2}{\langle\xi\rangle}\Big\}^\alpha|v_l(\xi)| \quad \textit{for}\quad \alpha\ge0,\ h,l=0,1.$$

PROOF. From the definitions we obtain $W_0^{**} = \dfrac{\lambda(\tilde\delta)^{1/2}\langle\xi\rangle^{1/2}}{2\sqrt2}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}v_0\\ v_1\end{pmatrix}$,

$$W^{**} = \big(I+H^*\big)\begin{pmatrix}1 & 1\\ -ia^* & ia^*\end{pmatrix}^{-1}\begin{pmatrix} a^{**1/2} & 0 \\ \frac{\partial_t a^{**1/2}}{2a^{**1/2}} & a^{**1/2}\end{pmatrix}\begin{pmatrix} v\\ \partial_t v\end{pmatrix}.$$

Therefore we get

$$\begin{pmatrix} v(t,\xi)\\ \partial_t v(t,\xi)\end{pmatrix} = \begin{pmatrix} a^{**}(t,\xi)^{1/2} & 0\\ \frac{\partial_t a^{**}(t,\xi)^{1/2}}{2a^{**}(t,\xi)^{1/2}} & a^{**}(t,\xi)^{1/2}\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ -ia^*(t,\xi) & ia^*(t,\xi)\end{pmatrix}\big(I+H(t,\xi)\big)$$

$$\times E(t,\xi)\big(I+K(t,\xi)\big)\frac{\lambda(\tilde{\delta})^{1/2}\langle\xi\rangle^{1/2}}{2\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}v_0(\xi)\\ v_1(\xi)\end{pmatrix}.$$

Hence we obtain the representation formula of $v(t,\xi)$. Moreover by Lemma 3.4, Proposition 5.1 we get for $0 \le t \le T$

$$\begin{aligned}|v(t,\xi)| &\le C\left|\frac{\lambda(\tilde{\delta})\langle\xi\rangle}{2a^{**}(t,\xi)}\right|^{1/2}\Big(\max\big\{|H_{12}|,|H_{12}|\big\}+1\Big)|E|\big(1+|K|\big)\big(|\hat{u}_0|+|\hat{u}_1|\big)\\ &\le C\left|\frac{\lambda(\tilde{\delta})\langle\xi\rangle}{2\frac{\lambda(\delta)\langle\xi\rangle}{2}}\right|^{1/2}\cdot(1+1)\cdot 1\cdot\big(1+C\langle\xi\rangle^{M_N}\big)\big(|\hat{u}_0|+|\hat{u}_1|\big)\le C\langle\xi\rangle^{M_N}\big(|\hat{u}_0|+|\hat{u}_1|\big).\end{aligned}$$

Similarly, Lemma 3.4 and Proposition 5.1 give the estimate of derivatives. ∎

Finally, by Theorem 5.3 we can conclude the proof of Theorem 1.1 with

$$a_{l1}=\begin{cases} i^l\Big\{\frac{\lambda(\tilde{\delta})\langle\xi\rangle}{8a^{**}}\Big\}^{1/2}\big(H_{21}(t,\xi)+1\big)\big(I+K(t,\xi)\big) & \text{if } \xi\ge 0,\\ (-i)^l\Big\{\frac{\lambda(\tilde{\delta})\langle\xi\rangle}{8a^{**}}\Big\}^{1/2}\big(H_{12}(t,\xi)+1\big)\big(I+K(t,\xi)\big) & \text{if } \xi< 0,\end{cases}\quad \text{for}\quad l=1,2,$$

$$a_{l2}=\begin{cases} (-i)^l\Big\{\frac{\lambda(\tilde{\delta})\langle\xi\rangle}{8a^{**}}\Big\}^{1/2}\big(H_{12}(t,\xi)+1\big)\big(I+K(t,\xi)\big) & \text{if } \xi\ge 0,\\ i^l\Big\{\frac{\lambda(\tilde{\delta})\langle\xi\rangle}{8a^{**}}\Big\}^{1/2}\big(H_{21}(t,\xi)+1\big)\big(I+K(t,\xi)\big) & \text{if } \xi< 0,\end{cases}\quad \text{for}\quad l=1,2.$$

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References

[1] F. Colombini and S. Spagnolo, *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞*, Acta Math. 148 (1982), 243–253.

[2] F. Colombini and S. Spagnolo, *Hyperbolic equations with coefficients rapidly oscillating in time: a result of nonstability*, J. Differential Equations 52 (1984), 24–38.

[3] K. Kajitani, *The well posed Cauchy problem for hyperbolic operators*, Exposé au Séminaire de Vaillant du 8 février 1989, 1–16.

[4] M. Reissig and K. Yagdjian, *One application of Floquet's theory to L_p–L_q estimates for hyperbolic equations with very fast oscillations*, Math. Methods Appl. Sci. 22 (1999), 937–951.

[5] M. Reissig and K. Yagdjian, *L_p–L_q decay estimates for hyperbolic equations with oscillations in coefficients*, Chinese Ann. Math. Ser. B 21 (2000), 153–164.

[6] S. Tarama, *On the second order hyperbolic equations degenerating in the infinite order. Example*, Math. Japon. 42 (1995), 523–533.

[7] K. Yagdjian, *The Cauchy Problem for Hyperbolic Operators. Multiple Characteristics. Micro-local Approach*, Math. Top. 12, Akademie Verlag, Berlin, 1997.

[8] K. Yagdjian, *Representation theorem for the solutions of equations with the turning point of infinite order*, Ann. Mat. Pura Appl. (4) 173 (1997), 13–30.