Abstract. This work is concerned with the influence of oscillations in weakly hyperbolic operators on the well-posedness of the Cauchy problem. The fundamental solution to the Cauchy problem is constructed for the equations with oscillations in the coefficient very close to the ones destroying the $C^\infty$ well-posedness.

1. Introduction. The subject of this paper concerns with the investigation of the influence of the oscillations in weakly hyperbolic operators on the well-posedness of the Cauchy problem. Since the example constructed by F. Colombini and S. Spagnolo [1], it is well-known that oscillations can break down the well-posedness. Namely, they constructed a second order equation $\partial_t^2 u - a(t)\partial_x^2 u = f(t, x)$, with the smooth coefficient $a \in C^\infty$, for which the Cauchy problem is not $C^1$ well-posed (see also [2]). The proof is based on the very delicate investigation of the energy of solutions. For the equation

$$\partial_t^2 u - \exp(-2t^{-\alpha})b(t^{-1})^2\partial_x^2 u = 0, \quad \alpha = \text{const.} > 0,$$

where $b(s)$ is a non-constant, positive and smooth 1-periodic function on $\mathbb{R}$, the energy method convinces that the Cauchy problem is $C^\infty$ well-posed if $\alpha \geq 1$. S. Tarama [6]

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appeals to the Floquet theory to prove that the problem is $C^\infty$ well-posed if and only if $\alpha \geq 1/2$.

For second order equations with coefficients independent of the spatial variables some sufficient for the well-posedness conditions are given in [7]. The equation under consideration in [7] is the following:

\begin{equation}
D_t^2 u + \sum_{|\alpha|=2} a_{0,\alpha}(t) D_x^\alpha u + \sum_{j+|\alpha|\leq 1} a_{j,\alpha}(t) D_t^j D_x^\alpha u = f.
\end{equation}

It is supposed that the principal symbol can be written in the form

\begin{equation}
\tau^2 + \sum_{|\alpha|=2} a_{0,\alpha}(t) \xi^\alpha = (\tau - \lambda_1(t, \xi))(\tau - \lambda_2(t, \xi)),
\end{equation}

with the real-valued functions $\lambda_l(t, \xi) \ (l = 1, 2)$ which satisfy the conditions

$$|\lambda_l(t, \xi)| \leq c \lambda(t)|\xi|, \quad l = 1, 2, \quad |\lambda_1(t, \xi) - \lambda_2(t, \xi)| \geq \delta \lambda(t)|\xi|, \quad \delta = \text{const.} > 0,$$

for all $t \in [0, T], \xi \in \mathbb{R}^n_\xi$. Here $\lambda \in C^2([0, T]), \lambda(0) = \lambda'(0) = 0, \lambda'(t) > 0$ for $t > 0$. Thus, at $t = 0$ the operator has multiple characteristics. Furthermore, it is assumed in [7] that the following inequalities are satisfied:

\begin{equation}
|D_t^k \text{Re} \ a_{0,\alpha}(t)| \leq C \lambda^2(t) \left( \frac{\log \lambda(t)}{\Lambda(t)} \right)^{2-|\alpha|} \left( \frac{\lambda(t) \log \lambda(t)}{\Lambda(t)} \right)^k, \quad k = 0, 1, 2,
\end{equation}

\begin{equation}
|D_t^k \text{Im} \ a_{0,\alpha}(t)| \leq C \frac{\lambda^2(t)}{\Lambda(t)} \left( \frac{\lambda(t) \log \lambda(t)}{\Lambda(t)} \right)^k, \quad k = 0, 1,
\end{equation}

for all $t \in (0, T], 0 < |\alpha| \leq 2$. Then in [7] it is proved that for equation (1.2) the Cauchy problem is $C^\infty$ well-posed. The conditions (1.4) and (1.5) couple together an oscillation with the degeneracy of the principal part.

For the equation (1.1) one can set $\lambda_1(t, \xi) = -\lambda_2(t, \xi) = \exp(-t^{-\alpha})b(t^{-1})|\xi|$, and $\lambda(t) = \exp(-t^{-\alpha})$. Then the critical value $\alpha = 1/2$ of (1.1) is reflected in (1.4) by the term \( \frac{\lambda(t) \log \lambda(t)}{\Lambda(t)} \) containing $|\log \lambda(t)|$. Indeed, to satisfy that condition with $k = 1$ we have to require $0 < \text{const.} \leq t^2 \lambda(t)|\log \lambda(t)| / \Lambda(t)$ for all $t \in (0, T)$, which is equivalent to $\alpha \geq 1/2$. In [7] such equations are called \textit{equations with fast oscillating coefficients}, while the equations with coefficients satisfying estimates with \( \frac{\lambda(t) \log \lambda(t)}{\Lambda(t)} \) \( k \) (that corresponds to $\alpha \geq 1$) one can call possessing \textit{slowly oscillating coefficients}. All other cases (corresponding to $\alpha < 1/2$) can be regarded as \textit{very fast oscillating}. Such classification is useful as well in completely other problem of $L_p-L_q$ decay (as $t \to \infty$) estimates for strictly hyperbolic equations (see [4]), where new oscillations can have destructive consequences.

On the other hand after an investigation of the well-posedness the next interesting question is a construction of the fundamental solution (or of the parametrix) and a description of the propagation of singularities in the framework of the micro-local analysis. For the operators with slow oscillations such construction can be found in [8]. The goal of the present note is to fill up the gap for the equations with oscillations in the coefficients very close to the ones destroying the well-posedness. Thus in this paper we consider the critical case of equations with fast oscillating coefficients.
Let $p = p(t)$ be a smooth function $p \in C^{\infty}(0,T]$ ($0 < T < 1$), satisfying
\[ p'(t) \leq -\frac{\gamma}{t} \quad (\exists \gamma > 1), \quad \forall t \in (0,T], \]
\[ 0 \leq p''(t) \leq 3C|p(t)p'(t)|^2, \quad \forall t \in (0,T]. \]
Let $a(t)$ be a non-negative function such that
\[ a(t) = \lambda(t)b(t), \]
where
\[ \lambda(t) = \begin{cases} 
  e^{-p(t)} & \text{for } 0 < t \leq T, \\
  0 & \text{for } t = 0, 
\end{cases} \]
belongs to $C^1[0,T]$ and $b(t)$ is a uniformly positive smooth function satisfying
\[ b \in C^{\infty}(0,T] \quad \text{and} \quad |\partial_t^h b(t)| \leq 3C_h|p(t)p'(t)|^h \quad \text{for} \quad h = 1, 2, \ldots, 0 < t \leq T. \]
The function $p(t)$ implies the speed of degeneracy while the function $b(t)$ describes the oscillations. Typical examples are the following:
\[ a_1(t) = \begin{cases} 
  \exp(-t^{-\alpha})\tilde{b}(t^{-1}) & \text{for } 0 < t \leq T, \\
  0 & \text{for } t = 0, 
\end{cases} \quad a_2(t) = \begin{cases} 
  t^\gamma \tilde{b}((-\log t)^\beta) & \text{for } 0 < t \leq T, \\
  0 & \text{for } t = 0, 
\end{cases} \]
where $\alpha > 0$, $\beta > 0$, $\gamma > 1$ and $\tilde{b}(s)$ is a non-constant, uniformly positive, smooth and 1-periodic function on $(0, \infty)$.

In this paper we shall consider
\[ \begin{cases} 
  \partial_t^2 u - a(t)^2 \partial_x^2 u = 0 & \text{in } [0,T] \times \mathbb{R}, \\
  u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) & \text{in } \mathbb{R}. 
\end{cases} \tag{1.7} \]
By $H^s(\mathbb{R})$ we denote the Sobolev space equipped with the norm
\[ ||u||_s := \left( \int_{\mathbb{R}} \sum_{k \leq s} |\partial_x^k u(x)|^2 \, dx \right)^{1/2}. \]
Further $H^\infty(\mathbb{R}) := \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R})$.

**Theorem 1.1.**

a) Assume that $b(t)$ satisfies (1.6). Then for every $u_0, u_1 \in H^\infty(\mathbb{R})$, the Cauchy problem (1.7) has a unique solution $u \in C^2([0,T], H^\infty(\mathbb{R}))$ represented as follows:
\[ u(t,x) = \sum_{l=0,1} \sum_{m=1,2} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i[(x-y)\cdot \xi + \varphi_m(t,\xi)]} a_{lm}(t,\xi) u_l(y) \, dy \, d\xi, \]
where $\varphi_m(t,\xi) = (-1)^m \int_0^t a(\tau) \, d\tau \cdot \xi$ for $m = 1, 2$, while there exist $C_{h\alpha} > 0$, $M > 0$ and $0 \leq \rho_1 < \rho_2 \leq 1$ such that
\[ \sup_{0 \leq t \leq T, \quad |\xi| \geq 1} |D^h_\xi D^\alpha_\xi a_{lm}(t,\xi)| \leq C_{h\alpha,\xi} |\xi|^{M + \rho_1 h - \rho_2 \alpha} \quad \text{for} \quad \alpha \geq 0, \quad h, l = 0, 1, \quad m = 1, 2. \tag{1.8} \]

b) The representation is valid for every $u_0, u_1 \in H^M(\mathbb{R})$, and
\[ \text{WF}(u(t)) \subset \{ (x,\xi) = (y \pm \int_0^t a(\tau) \, d\tau, \eta) : (y,\eta) \in \text{WF}(u_0) \cup \text{WF}(u_1) \}. \]
c) The problem possesses the finite propagation speed property that for smooth coefficient \( a^2 \in C^\infty([0,T]) \) together with a) leads to \( C^\infty \) well-posedness, that is, for every \( u_0, u_1 \in C^\infty(\mathbb{R}_x) \), the Cauchy problem (1.7) has a unique solution \( u \in C^\infty([0,T] \times \mathbb{R}_x) \).

Remark 1.2. In particular when \( a(t) \equiv a_1(t) \) (resp. \( a_2(t) \)), the condition (1.6) corresponds to \( \alpha \geq 1/2 \) (resp. \( \beta \leq 2 \)). For \( \alpha < 1/2 \) according to [6] the Cauchy problem is not \( C^\infty \) well-posed. The results of a) and c) are optimal. While, the result b) can be obtained from the general theory of Fourier integral operators.

Remark 1.3. If \( \lambda(t) \) vanishes of infinite order, the Cauchy problem (1.7) has a unique solution \( u \in C^\infty([0,T] \times \mathbb{R}_x) \) and (1.8) holds for any \( h \geq 0 \).

2. Notation and classes of symbols. In this paper we often use the cut-off functions \( \chi(s) \) and \( \psi(s) \) such that

\[
\chi(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \quad \chi'(s) \leq 0 \quad \text{and} \quad \psi(s) = 1 - \chi(s).
\]

We define \( \Lambda(t) = \int_0^t \lambda(\tau) \, d\tau \) and \( \Lambda^*(t) = \frac{\Lambda(t)}{p(t)} \). Let \( N > 0 \) and \( \langle \xi \rangle = \sqrt{\epsilon^2 + |\xi|^2} \) (\( \geq e \)).

Definition 2.1. The functions \( t_N(\xi) \) and \( \tilde{t}_N(\xi) \) are (unique) roots of \( \Lambda^*(\xi) = 2N \log(\xi) \) and \( \Lambda^*(\xi) = 4N \log(\xi) \), respectively, i.e., \( \Lambda^*(t_N(\xi)) = 2N \log(\xi) \) and \( \Lambda^*(\tilde{t}_N(\xi)) = 4N \log(\xi) \).

Definition 2.2. We define the hyperbolic zone

\[
Z_N(t, \xi) = \{ (t, \xi) \in [0, T] \times \mathbb{R}_\xi : \Lambda^*(\xi) \geq 2N \log(\xi) \text{ and } |\xi| \geq 1 \}.
\]

Definition 2.3. Let \( m_1, m_2 \) and \( m_3 \) be real numbers. We define the spaces of the symbols

\[
S_N(m_1, m_2, m_3) = \left\{ a(t, \xi) \in C^\infty : \sup_{(t, \xi) \in Z_N} \frac{\langle \xi \rangle |\alpha| - m_1 |D_t^h D_{\xi}^a a(t, \xi)|}{\lambda(t)^{m_2} p(t) p'(t)|m_3+h|} \leq C_h \alpha \right\},
\]

\[
S_N^{-\infty}(m_1, m_2, m_3) = \bigcap_{k=0}^{\infty} S_N(m_1 - k, m_2 - k, m_3 + k).
\]

Remark 2.4. The following properties are known (see [7]).

(i) \( S_N(m_1, m_2, m_3) \supset S_N(m_1 - k, m_2 - k, m_3 + k) \) for \( k \geq 0 \).

(ii) If \( a \in S_N(m_1, m_2, m_3) \), then \( D_{\xi}^a a \in S_N(m_1 - |\alpha|, m_2, m_3) \).

(iii) If \( a \in S_N(m_1, m_2, m_3) \) and \( b \in S_N(m_1, \tilde{m}_2, \tilde{m}_3) \) (resp. \( S_N^{-\infty}(m_1, \tilde{m}_2, \tilde{m}_3) \)), then \( \lambda a b \in S_N(m_1 + \tilde{m}_1, m_2 + \tilde{m}_2, m_3 + \tilde{m}_3) \) (resp. \( S_N^{-\infty}(m_1 + \tilde{m}_1, m_2 + \tilde{m}_2, m_3 + \tilde{m}_3) \)).

Proposition 2.5 ([7]). Suppose that for all \( k = 1, 2, \ldots, a_k \in S_N(m_1 - k + 1, m_2 - k + 1, m_3 + k - 1) \), \( a_k(t, \xi) = 0 \) for \( 0 \leq t \leq t_N(\xi) \), \( |\xi| \geq 1 \). Then there exists a symbol \( a(t, \xi) \in S_N(m_1, m_2, m_3) \) such that

\[
\text{supp } a \subset Z_N(t, \xi) \text{ and } a \sim a_1 + a_2 + \ldots \text{ mod } S_N^{-\infty}(m_1, m_2, m_3)
\]

in the sense that \( a - a_1 - \ldots - a_k \in S_N(m_1 - k, m_2 - k, m_3 + k) \) for all \( k = 0, 1, \ldots \).
3. Reduction to a first order diagonal system. Without loss of generality we may suppose that $b(t)$ has the form

$$b(t) \equiv 1 + c(t),$$

where $c(t)$ is a smooth function satisfying

$$|c(t)| \leq 1/2, \ c \in C^\infty(0,T] \quad \text{and} \quad |\partial_t^h c(t)| \leq C_h |p(t)p'(t)|^h \ (2C_h > 0).$$

By Fourier transform the Cauchy problem (1.7) is changed into

$$
\begin{align*}
\partial_t^2 v + a(t,\xi)^2 v = 0 & \quad \text{in } [0,T] \times \mathbb{R}_x, \\
v(0,\xi) = v_0(\xi), \quad \partial_t v(0,\xi) = v_1(\xi) & \quad \text{in } \mathbb{R}_x,
\end{align*}
$$

where $a(t,\xi) = \lambda(t)\{1 + c(t)\} |\xi| = a(t)|\xi|).

**Definition 3.1.** The functions $\varepsilon = \varepsilon(\xi)$ and $\tilde{\varepsilon} = \tilde{\varepsilon}(\xi)$ are (unique) roots of the equations $\lambda(t)\langle \xi \rangle = 1$ and $\lambda(t)\langle \xi \rangle = 2$, respectively, i.e., $\lambda(\varepsilon)\langle \xi \rangle = 1$ and $\lambda(\tilde{\varepsilon})\langle \xi \rangle = 2$.

**Definition 3.2.** The functions $\delta = \delta(\xi)$ and $\tilde{\delta} = \tilde{\delta}(\xi)$ are (unique) roots of the equations $\Lambda(t)\langle \xi \rangle = N \log\langle \xi \rangle$ and $\Lambda(t)\langle \xi \rangle = 2N \log\langle \xi \rangle$, respectively, i.e., $\Lambda(\delta)\langle \xi \rangle = N \log\langle \xi \rangle$ and $\Lambda(\tilde{\delta})\langle \xi \rangle = 2N \log\langle \xi \rangle$.

**Lemma 3.3.** For sufficiently large $N > 0$, the following relation holds:

$$0 < \varepsilon < \tilde{\varepsilon} < \delta < \tilde{\delta} \leq t_N(\xi) < \tilde{t}_N(\xi) \quad \text{for } \xi \in \mathbb{R}_x.$$

Now we approximate $a(t,\xi)$ with two functions defined by the following formulas.

$$
\begin{align*}
a^*(t,\xi) &= \frac{\Lambda(\varepsilon)\langle \xi \rangle}{\Lambda(t)\langle \xi \rangle} \chi\left(1 + \psi\left(\frac{\Lambda(t)\langle \xi \rangle}{N \log\langle \xi \rangle}\right)c(t)\right)\langle \xi \rangle, \\
a^{**}(t,\xi) &= \frac{\Lambda(\delta)\langle \xi \rangle}{\Lambda(t)\langle \xi \rangle} \chi\left(1 + \psi\left(\frac{\Lambda(t)\langle \xi \rangle}{N \log\langle \xi \rangle}\right)c(t)\right)\langle \xi \rangle.
\end{align*}
$$

Noting that $|c(s)| \leq 1/2$ and considering the supports of $\chi$ and $\psi$, we get

**Lemma 3.4.** For any $t \in [0,T]$ and $\xi \in \mathbb{R}_x$, $a^*(t,\xi)$ and $a^{**}(t,\xi)$ are positive, more precisely

$$
\begin{align*}
a^*(t,\xi) &\geq \max\left\{\frac{\Lambda(t)\langle \xi \rangle}{2}, \frac{1}{2}\right\}, \quad a^{**}(t,\xi) \geq \max\left\{\frac{\Lambda(t)\langle \xi \rangle}{2}, \frac{\Lambda(\delta)\langle \xi \rangle}{2}\right\}, \\
|D_t^h D_\xi^\delta a^*(t,\xi)| &\leq C_{ho}\langle \xi \rangle^{-|\alpha|} \left(\max\{|p'(\varepsilon)|, |p'(\delta)|p(\delta)\}\right)^h a^*(t,\xi), \\
|D_t^h D_\xi^\delta a^{**}(t,\xi)| &\leq C_{ho}\langle \xi \rangle^{-|\alpha|} \left(|p'(\delta)|p(\delta)\right)^h a^{**}(t,\xi).
\end{align*}
$$

Putting $W = \left(\frac{a^{**}(t,\xi)^{1/2}v}{\partial_t \{a^{**}(t,\xi)^{1/2}v\}}\right)$ and $W_0 := \left. W \right|_{t=0} = \left(\frac{\Lambda(\delta)\langle \xi \rangle}{2}\right)^{1/2} \left(\frac{v_0}{v_1}\right)$ and multiplying both sides of (3.1) by $a^{**}(t,\xi)^{1/2}$, we find that the Cauchy problem (3.1) is equivalent to the one for the system

$$
\begin{align*}
\partial_t W &= \begin{pmatrix}
0 & 1 \\
-a(t,\xi)^2 - r(t,\xi) & \partial_t a^{**}(t,\xi)
\end{pmatrix} W, \\
W(0) &= W_0,
\end{align*}
$$

where $r(t,\xi) = \frac{3}{4} \left(\frac{\partial_t a^{**}(t,\xi)}{a^{**}(t,\xi)}\right)^2 - \frac{1}{2} \frac{\partial_t^2 a^{**}(t,\xi)}{a^{**}(t,\xi)}$. 

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Further we make one step of the diagonalization by putting
\[
W^* = \begin{pmatrix} 1 & 1 \\ -ia^*(t, \xi) & ia^*(t, \xi) \end{pmatrix}^{-1} W \quad \text{and} \quad W^*_0 := W^*|_{t=0} = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} W_0,
\]
and multiplying both sides of the system by \(\begin{pmatrix} 1 & 1 \\ -ia^*(t, \xi) & ia^*(t, \xi) \end{pmatrix}^{-1}\), where \(a^*(t, \xi)^{-1}\) belongs to \(S_N(-1, -1, 0)\). Then we obtain
\[
\begin{aligned}
\begin{cases}
\partial_t W^* = D(t, \xi) W^* + B(t, \xi) W^*, \\
W^*(0) = W^*_0,
\end{cases}
\end{aligned}
\]
where \(D = ia(t, \xi) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\),
\[
B = \frac{1}{2} \left\{ \frac{\partial_t a^*(t, \xi)}{a^*(t, \xi)} - \frac{\partial_t a(t, \xi)}{a^*(t, \xi)} \right\} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + i \left\{ a^*(t, \xi) - a(t, \xi) \right\} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{a(t, \xi)^2 - a^*(t, \xi)^2 + r(t, \xi)}{2ia^*(t, \xi)} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\]

**Definition 3.5.** We define recursively the sequence \(\{B^{(k)}(t, \xi)\}_{k \geq 1}\) as
\[
\begin{aligned}
\begin{cases}
B^{(1)}(t, \xi) = B(t, \xi) - \chi \left( \frac{\Lambda^*(t) |\xi\rangle}{2N \log(\xi)} \right) B(t, \xi), \\
B^{(k)}(t, \xi) = \tilde{B}^{(k)}(t, \xi) - \chi \left( \frac{\Lambda^*(t) |\xi\rangle}{2N \log(\xi)} \right) B(t, \xi) \quad \text{for} \ k = 2, 3, \ldots,
\end{cases}
\end{aligned}
\]
where
\[
\tilde{B}^{(k)}(t, \xi) = \left( I + \sum_{j=1}^{k-1} H^{(j)}(t, \xi) \right) \left( \partial_t - D(t, \xi) - \sum_{j=1}^{k-1} F^{(j)}(t, \xi) \right) - \left( \partial_t - D(t, \xi) - B(t, \xi) \right) \left( I + \sum_{j=1}^{k-1} H^{(j)}(t, \xi) \right),
\]
\[
F^{(k)}(t, \xi) = \text{diag} B^{(k)}(t, \xi), \quad H^{(k)}(t, \xi) = \frac{1}{2ia(t, \xi)} \begin{pmatrix} 0 & B^{(k)}_{12}(t, \xi) \\ -B^{(k)}_{12}(t, \xi) & 0 \end{pmatrix}.
\]

When \(0 \leq t \leq t_N(\xi)\), we see that \(B^{(1)}(t, \xi) = 0\) and recursively \(B^{(k)}(t, \xi) = 0\) for \(k = 1, 2, \ldots\). Thus, we obtain \(H^{(k)}(t, \xi) = 0\) for \(k = 1, 2, \ldots\). Moreover we can derive for \(k = 2, 3, \ldots\)
\[
B^{(k)} = \sum_{j=1}^{k-1} [D, H^{(j)}] - \sum_{j=1}^{k-1} \partial_t H^{(j)} - \left( I + \sum_{j=1}^{k-1} H^{(j)} \right) \sum_{j=1}^{k-1} F^{(j)} + B \left( I + \sum_{j=1}^{k-1} H^{(j)} \right) - \chi B
\]
\[
= \{ B^{(k-1)} + [D, H^{(k-1)}] - F^{(k-1)} \} + BH^{(k-1)} - \partial_t H^{(k-1)}
- \sum_{j=1}^{k-1} H^{(k-1)} F^{(j)} - \sum_{j=1}^{k-2} H^{(j)} F^{(k-1)}.
\]
Noting that \(B^{(k-1)} + [D, H^{(k-1)}] - F^{(k-1)} \equiv 0\), we also obtain recursively the following
LEMMA 3.6. For sufficiently large $N > 0$,

$$B^{(k)} \in S_N(-k, -k, k + 1), \quad H^{(k)} \in S_N(-k - 1, -k - 1, k + 1) \quad \text{for } k = 1, 2, \ldots,$$

$$F^{(k)} \in \begin{cases} S_N(-1, -1, 2) & \text{for } k = 1, \\ S_N(-k - 1, -k - 1, k + 2) & \text{for } k = 2, 3, \ldots \end{cases}$$

By Proposition 2.5 there exists $H(t, \xi) = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix} \in S_N(-2, -2, 2)$ such that $\supp H \subset Z_N(t, \xi)$ and $H \sim H^{(1)} + H^{(2)} + \ldots \mod S_N^{\infty}(-2, -2, 2)$, and for sufficiently large $N > 0$, there exists $H^*(t, \xi) \in S_N(-2, -2, 2)$ defined by

$$H^*(t, \xi) = \frac{1}{1 - H_{12}H_{21}} \begin{pmatrix} H_{12}H_{21} + H_{12} & -H_{12} \\ -H_{21} & H_{12}H_{21} \end{pmatrix} = (I + H(t, \xi))^{-1} - I.$$

Finally, putting

$$W^{**} = (I + H^*(t, \xi))W^* \quad \text{and} \quad W^{**}_0 := W^{**}|_{t = 0} = (I + H^*(0, \xi))W^*_0 = W^*_0,$$

and multiplying both sides of (3.2) by $(I + H^*(t, \xi))$, we obtain

$$\begin{align*}
\partial_t W^{**} &= D(t, \xi)W^{**} + \text{diag}\{B^*(t, \xi)\}W^{**} + \chi\left(\frac{\Lambda^*(t)(\xi)}{2N\log(\xi)}\right)B(t, \xi)W^{**} \\
&\quad + \chi\left(\frac{\Lambda^*(t)(\xi)}{2N\log(\xi)}\right)B^{**}(t, \xi)W^{**} + R(t, \xi)W^{**},
\end{align*}$$

(3.3)

where $R(t, \xi) \subset S_N^{\infty}(-1, -1, 2)$, $B^*(t, \xi) = B^{(1)} + BH \in S_N(-1, -1, 2)$ and $B^{**}(t, \xi) = H^*B \in S_N(-3, -3, 4)$.

4. Some estimates. In this section we show some estimates which will be used to construct the fundamental solution of (3.3). For simplicity we define $\chi^*(t) = \chi(\lambda(\xi))$, $\chi^{**}(t) = \chi\left(\frac{\Lambda^*(t)(\xi)}{2N\log(\xi)}\right)$ and $\psi^*(t) = 1 - \chi^*(t)$, $\psi^{**}(t) = 1 - \chi^{**}(t)$.

Noting the support of $\chi\left(\frac{\Lambda^*(t)(\xi)}{2N\log(\xi)}\right)$, we obtain

$$\int_0^T \chi\left(\frac{\Lambda^*(t)(\xi)}{2N\log(\xi)}\right)B(t, \xi) \, dt \leq \frac{1}{2} \int_0^{\hat{t}_N(\xi)} \left| \partial_t a^{**} - \partial_t a^* \right| \, dt + \int_0^{\hat{t}_N(\xi)} |a^* - a| \, dt$$

$$+ \frac{1}{2} \int_0^{\hat{t}_N(\xi)} \frac{|a^{2} - a^{*2}|}{a^*} \, dt + \frac{1}{2} \int_0^{\hat{t}_N(\xi)} \frac{|v|}{a^*} \, dt$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$
Similarly we also obtain $I_1^{**} \leq C p(\varepsilon)$. Thus we get $I_1 \leq I_1^* + I_1^{**} \leq C p(\varepsilon)$. Since $a(t, \xi) = a^*(t, \xi)$ for $\delta \leq t \leq t_N(\xi)$, we get $I_2 = \int_\delta^\delta |a^* - a| \, dt \leq C A(\delta) |\langle \xi \rangle | \leq C N |\log |\xi| |$. Similarly, by Lemma 3.4 we get $I_3 \leq C_N |\log |\xi| |$. Since $\partial_t a^*(t, \xi) = 0$, $\partial^2_t a^*(t, \xi) = 0$ for $0 \leq t \leq \delta$, we deduce that

$$I_4 \leq C \int_\delta^{t_N(\xi)} \frac{1}{a^*}|\partial_t a^*|^2 \, dt + C \int_\delta^{t_N(\xi)} \frac{1}{a^*}|\partial^2_t a^*|^2 \, dt \leq I_1^* + I_1^{**}.$$

The last estimate corresponds to the estimate in the oscillation’s subzone of [5]. Noting the supports of $\chi^{**}$ and $\psi^{**}$, by (1.6) and Lemma 3.4 we obtain

$$I_4^* \leq C \int_\delta^{t_N(\xi)} \frac{|\lambda(\delta) | \langle \xi \rangle | \chi^{**} |^2}{a^* a^{**}} + \frac{|\psi^{**}|^2 |\lambda^2(\xi)|^2}{a^* a^{**}} + \frac{\psi^{**}|\lambda^2(\xi)|^2}{a^*} + \frac{\psi^{**} |\lambda^2(\xi)| |\psi^{**} |}{a^* a^{**}} \, dt \leq C \int_\delta^{t_N(\xi)} \frac{p(t)^2 |p(t)|^2}{\lambda(\xi)} \, dt.$$

Similarly we also obtain $I_4^{**} \leq C \int_\delta^{t_N(\xi)} \frac{p(t)^2 |p(t)|}{\lambda(\xi)} \, dt$. Thus $I_4 \leq I_4^* + I_4^{**} \leq C \int_\delta^{t_N(\xi)} \frac{p(t)^2}{\lambda(\xi)} \, dt \leq C \frac{p(\delta)^2}{\langle \xi \rangle^3} \int_\delta^{t_N(\xi)} (-p(t)) e^{p(t)} \, dt \leq C p(\delta)^2 |p(\delta)| e^{p(\delta)} |\langle \xi \rangle |^{-1}.$

Combining $I_1 - I_4$ and noting that $|p(\delta)| e^{p(\delta)} \leq \frac{\langle \xi \rangle^3}{N |\log |\xi| |}$, we have for $\xi \in \mathbb{R}_\xi$

$$\int_0^T \chi \left( \frac{\lambda^*(t) |\langle \xi \rangle |}{2N |\log |\xi| |} \right) B \, dt \leq C p(\varepsilon) + C_N |\log |\xi| | + C p(\delta)^2 |p(\delta)| e^{p(\delta)} |\langle \xi \rangle |^{-1} \leq C_N |\log |\xi| |.

Noting the supports of $\chi \left( \frac{\lambda^*(t) |\langle \xi \rangle |}{2N |\log |\xi| |} \right)$ and observing that $B^{**}(t, \xi) \in S_N(-3, -3, 4)$, we obtain for $\xi \in \mathbb{R}_\xi$

$$\int_0^T \chi \left( \frac{\lambda^*(t) |\langle \xi \rangle |}{2N |\log |\xi| |} \right) B^{**} \, dt \leq C \int_{t_N}^{t_N} \frac{p(t)^4 |p(t)|^4}{\lambda(t)^3} \, dt = C \int_{t_N}^{t_N} \frac{p(t)^4 |p(t)|}{\lambda^*(t) |\langle \xi \rangle |^3} \, dt \leq C.$$

Since $R(t, \xi) \in S_{-\infty}(-1, -1, 2) \subset S_N(-3, -3, 4)$, similarly we have for $\xi \in \mathbb{R}_\xi$

$$\int_0^T R \, dt \leq C \int_{t_N}^{t_N} \frac{p(t)^4 |p(t)|^4}{\lambda(t)^3} \, dt = C \int_{t_N}^{t_N} \frac{p(t)^4 |p(t)|}{\lambda^*(t) |\langle \xi \rangle |^3} \, dt \leq C.$$

5. **The representation formula.** We define the diagonal matrix function

$$E(t, \xi) = \exp \left\{ \int_0^t D(\tau, \xi) \, d\tau \right\} = \begin{pmatrix} \exp \left\{ -i \int_0^t a(\tau, \xi) \, d\tau \right\} & 0 \\ 0 & \exp \left\{ i \int_0^t a(\tau, \xi) \, d\tau \right\} \end{pmatrix}.$$

For every $\Psi(\xi)$ the vector function $V(t, \xi) = E(t, \xi) \Psi(\xi)$ is a solution of the Cauchy problem

$$\begin{cases} \partial_t V = D(t, \xi)V \quad \text{in } [0, T] \times \mathbb{R}_\xi, \\
V(0, \xi) = \Psi(\xi). \end{cases}$$
Now we put
\[
G(t, \xi) \equiv E^{-1}\left\{ \text{diag}\{B^*\} + \chi \left( \frac{\Lambda^*(t)\langle \xi \rangle}{2N \log\langle \xi \rangle} \right) B + \chi \left( \frac{\Lambda^*(t)\langle \xi \rangle}{2N \log\langle \xi \rangle} \right) B^* + R \right\} E.
\]
Then the matrix function
\[
K(t, \xi) = \sum_{j=1}^{\infty} \int_{0}^{t} G(t_1, \xi) \, dt_1 \int_{0}^{t_1} G(t_2, \xi) \, dt_2 \cdots \int_{0}^{t_{j-1}} G(t_j, \xi) \, dt_j
\]
is the solution of the Cauchy problem
\[
\begin{cases}
\partial_t K(t, \xi) = G(t, \xi)K(t, \xi) + G(t, \xi) & \text{in } [0, T] \times \mathbb{R}, \\
K(0, \xi) = 0.
\end{cases}
\]
Moreover by (4.1), (4.2) and (4.3) we get the following result.

**Proposition 5.1.** There exist \(M_N > 0\) and a strictly increasing function \(\phi(s)\) satisfying \(\lim_{s \to +\infty} \phi(s)/s = 0\) such that
\[
\sup_{0 \leq t \leq T, |\xi| \geq 1} |D_t^h D_\xi^\alpha K| \leq C_{N, h, \alpha}(\langle \xi \rangle)^{M_N} \phi(\langle \xi \rangle) \left( \frac{(\log \langle \xi \rangle)^2}{\langle \xi \rangle} \right)^\alpha \quad \text{for } h = 0, 1, \quad \alpha \geq 0.
\]

**Remark 5.2.** If we take \(\phi(s) = \lambda \left( \Lambda^{-1}(\frac{4N \log s}{s}) \right) s\) which is a strictly increasing function and satisfies \(\lim_{s \to +\infty} \phi(s)/s = 0\), then (5.3) holds. In case that \(a(t) \equiv a_1(t)\) (resp. \(a_2(t)\)) (see Section 1), we can set \(\phi(s) = C(\log s)^{3+1/\alpha}\) (resp. \(C(\log s)^2 s^{1/\gamma}\)).

By (5.1) and (5.2) we have
\[
\partial_t \{E(t, \xi)(I + K(t, \xi))\Psi(\xi)\} = \partial_t (E\Psi) + (\partial_t E)K\Psi + E(\partial_t K)\Psi
\]
\[
= D(E\Psi) + DK\Psi + E(GK + G)\Psi
\]
\[
= D \{E(I + K)\Psi\} + EGE^{-1} \{E(I + K)\Psi\}.
\]
This means that the matrix function \(E(t, \xi)(I + K(t, \xi))\) is the fundamental solution of the Cauchy problem (3.3). Thus we have the following statement.

**Theorem 5.3.** Assume that \(b(t)\) satisfies (1.6). Then the solution \(v(t, \xi)\) to the Cauchy problem (3.1) can be represented as
\[
v(t, \xi) = \left\{ \frac{(\delta(x))^2}{8a^{**}} \right\}^{1/2} \left( H_{21}(t, \xi) + 1, H_{12}(t, \xi) + 1 \right) E(t, \xi)(I + K(t, \xi)) \left( \begin{array}{c} v_0 + iv_1 \\ v_0 - iv_1 \end{array} \right),
\]
and
\[
\sup_{0 \leq t \leq T, |\xi| \geq 1} |D_t^h D_\xi^\alpha v(t, \xi)| \leq C\langle \xi \rangle^{M_N} \phi(\langle \xi \rangle)^h \left( \frac{(\log \langle \xi \rangle)^2}{\langle \xi \rangle} \right)^\alpha |v_l(\xi)| \quad \text{for } \alpha \geq 0, \quad h, l = 0, 1.
\]

**Proof.** From the definitions we obtain \(W^{**} = \frac{\lambda(\delta)^{1/2}(\langle \xi \rangle)^{1/2}}{2\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right) \left( \begin{array}{c} v_0 \\ v_1 \end{array} \right)\)

\[
W^{**} = (I + H^*) \left( \begin{array}{cc} 1 & 1 \\ -ia^* & ia^* \end{array} \right)^{-1} \left( \begin{array}{cc} a^{*1/2} & 0 \\ \partial_t a^{**1/2} & a^{**1/2} \end{array} \right) \left( \begin{array}{c} v \\ \partial_t v \end{array} \right).
\]
Therefore we get
\[
\left( \begin{array}{c} v(t, \xi) \\ \partial_t v(t, \xi) \end{array} \right) = \left( \begin{array}{cc} a^{**}(t, \xi)^{1/2} & 0 \\ \partial_t a^{**}(t, \xi)^{1/2} a^{**}(t, \xi)^{1/2} \end{array} \right)^{-1} \left( \begin{array}{cc} 1 & 1 \\ -ia^*(t, \xi) & ia^*(t, \xi) \end{array} \right) \left( \begin{array}{c} \v(t, \xi) \\ \partial_t \v(t, \xi) \end{array} \right)(I + H(t, \xi))
\]
Hence we obtain the representation formula of \( v(t, \xi) \). Moreover by Lemma 3.4, Proposition 5.1 give the estimate of derivatives.

Similarly, Lemma 3.4 and Proposition 5.1 give the estimate of derivatives.

Finally, by Theorem 5.3 we can conclude the proof of Theorem 1.1 with

\[
\begin{align*}
\alpha_{l1} &= \begin{cases} 
  i \left\{ \frac{\lambda(\xi)}{8a^{**}} \right\}^{1/2} (H_{21}(t, \xi) + 1) (I + K(t, \xi)) & \text{if } \xi \geq 0, \\
  (-i) \left\{ \frac{\lambda(\xi)}{8a^{**}} \right\}^{1/2} (H_{12}(t, \xi) + 1) (I + K(t, \xi)) & \text{if } \xi < 0,
\end{cases} \\
\alpha_{l2} &= \begin{cases} 
  (-i) \left\{ \frac{\lambda(\xi)}{8a^{**}} \right\}^{1/2} (H_{12}(t, \xi) + 1) (I + K(t, \xi)) & \text{if } \xi \geq 0, \\
  i \left\{ \frac{\lambda(\xi)}{8a^{**}} \right\}^{1/2} (H_{21}(t, \xi) + 1) (I + K(t, \xi)) & \text{if } \xi < 0,
\end{cases}
\end{align*}
\]

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