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C^∞-WELL POSEDNESS OF THE CAUCHY PROBLEM FOR QUASI-LINEAR HYPERBOLIC EQUATIONS WITH COEFFICIENTS NON-LIPSCHITZ IN TIME AND SMOOTH IN SPACE

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Abstract. In this paper we prove the C^∞-well posedness of the Cauchy problem for quasi-linear hyperbolic equations of second order with coefficients non-Lipschitz in $t \in [0,T]$ and smooth in $x \in \mathbb{R}^n$.

0. Introduction. In this paper we consider Cauchy problems for quasi-linear hyperbolic equations with coefficients non-Lipschitz in the time variable and smooth in spatial variables. Our goal is to prove C^∞-well posedness for the Cauchy problem

$$\text{(CP1)} \qquad \begin{cases} P_u[u] = u_{tt} - \displaystyle\sum_{j,k=1}^{n} a_{jk}(t,x;u,u_t,\nabla u)u_{x_jx_k} + \rho(t,x;u,u_t,\nabla u) = 0 \\ \hfill \text{for } (t,x) \in (0,T)\times\mathbb{R}^n, \\ u = \varphi(x),\ u_t = \psi(x) \hfill \text{at } t = 0. \end{cases} \tag{0.1}$$

The paper [2] is devoted to the study of Cauchy problems for second order hyperbolic equations with coefficients depending on the time variable of the form

$$u_{tt} - \sum_{j,k=1}^{n} a_{jk}(t)u_{x_jx_k} + \sum_{j=1}^{n} b_j(t)u_{x_j} + c(t)u = 0, \tag{0.2}$$

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where $a_{jk}(t) = a_{kj}(t)$ are non-Lipschitz coefficients in the following sense:

$$|a_t(t;\eta)| \le \frac{C}{t}|\eta|^2, \quad a(t;\eta) := \sum_{j,k=1}^{n} a_{jk}(t)\eta_j\eta_k \tag{0.3}$$

for all $\eta = (\eta, \dots, \eta_n) \in \mathbb{R}^n$. The authors proved that the Cauchy problem is C^∞-well posed, where its solution possesses the property of regularity loss of derivatives in x. On the other hand, the condition (0.3) can be weakened to the optimal condition

$$|a_t(t,\eta)| \le \frac{C}{t}\log\Big(\frac{1}{t}\Big)|\eta|^2, \ t \in (0,T], \tag{0.4}$$

to guarantee C^∞-well posedness for the Cauchy problem (0.2). This is shown in [3] for the model Cauchy problem

$$u_{tt} - a(t)u_{xx} + b(t)u_x = 0, \qquad u(0,x) = \varphi(x), \ u_t(0,x) = \psi(x)$$

(see also Remark 3.3 from [3]). A more general model with optimal non-Lipschitz condition of C^∞- and Gevrey-type is studied in [5]. Recently, [1] considered the linear Cauchy problem of the same type as (0.2) with coefficients depending on time and spatial variables. There the elliptic term satisfies a condition like (0.3). The C^∞-well posedness of the Cauchy problem was proved by using pseudo-differential operators based on an argument used in [2]. Finally, in the recent paper [7] the question for C^∞-well posedness was studied for the Cauchy problem

$$u_{tt} - \sum_{k,l=1}^{n} a_{k,l}(t,x)u_{x_k x_l} = f(t,x) \qquad \text{in} \quad [0,T]\times\mathbb{R}^n$$
$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),$$

under the main assumption

$$|D_t^k D_x^\beta a_{k,l}(t,x)| \le C_{k,\beta}\Big(\frac{1}{t}\Big(\ln\frac{1}{t}\Big)^\gamma\Big)^k$$

for all k, β and $(t,x) \in (0,T]\times\mathbb{R}^n$, where T is sufficiently small and $\gamma \ge 0$. A C^∞-well posedness result was proved after construction of parametrix and the proof of existence of a cone of dependence.

In this paper we consider the general quasi-linear Cauchy problem (CP1). Our approach is quite different from those from [2] and [1]. Actually, according to [9], making use of solutions of a family of nonlinear ordinary differential equations associated with (0.1), we reduce (CP1) to some Cauchy problem with special asymptotic behaviour in t on the right-hand side. During this procedure we only lose regularity in x without any loss of regularity in t. Then, by a standard way, we can derive the time local existence of smooth solutions.

Finally, by proving the domain of dependence property we obtain C^∞-well posedness of (CP1).

To explain our assumptions we define multi-indices

$$\xi = (\xi_0, \xi_1, \dots, \xi_{n+1}), \ \eta = (\eta_1, \dots, \eta_n), \ \alpha = (\alpha_1, \dots, \alpha_n), \ \beta = (\beta_0, \beta_1, \dots, \beta_{n+1}).$$

We make the following assumptions with $D_x = (\partial_{x_1}, \dots, \partial_{x_n})$ and $D_\xi = (\partial_{\xi_0}, \dots, \partial_{\xi_{n+1}})$:

(A-I) (*strict hyperbolicity*)

There exists a positive constant C_0 such that

$$\sum_{j,k=1}^{n} a_{jk}(t,x;\xi)\eta_j\eta_k \geq C_0|\eta|^2, \quad \eta \in \mathbb{R}^n \setminus \{0\},$$

$$a_{jk}(t,x;\xi) = a_{kj}(t,x;\xi), \quad j,k = 1,\dots,n,\ j \neq k,$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\xi \in \mathbb{R}^{n+2}$.

(A-II) (*regularity properties*)

Let K be an open ball in $\mathbb{R}^{n+2}$. Then

i) $a_{jk} \in C^1\big((0,T];\mathcal{B}^\infty(\mathbb{R}^n \times K)\big)$,

ii) $\rho \in C\big((0,T];\mathcal{B}^\infty(\mathbb{R}^n \times K)\big)$,

$\sup\limits_{\xi\in K} |D_x^\alpha \rho(t,x;\xi)| \in L^2\big((0,T)\times\mathbb{R}^n\big)$ for $|\alpha| \geq 0$.

(A-III) (*asymptotic behaviour near* $t = 0$)

We assume

i) $|\partial_t a_{jk}(t,x;\xi)| \leq C_{jk}/t$, $|D_x^\alpha D_\xi^\beta a_{jk}(t,x;\xi)| \leq C_{jk\alpha\beta}/t^r$ with a fixed $0 \leq r < 1$ and for all multi-indices α and β,

ii) $|D_x^\alpha D_\xi^\beta \rho(t,x;\xi)| \leq C_{\alpha\beta}/t^q$, with a fixed $0 \leq q < 1$ and for all multi-indices α and β with $|\alpha| \geq 0$ and $|\beta| > 0$.

Throughout this paper we use the following notation.

By $H^m(\mathbb{R}^n)$ with a non-negative integer m we denote the usual Sobolev space with the norm $\|\cdot\|_m$. Sometimes we denote $(\cdot,\cdot)_{L^2(\mathbb{R}^n)}$, $\|\cdot\|_{L^2(\mathbb{R}^n)}$ by $(\cdot,\cdot)$, $\|\cdot\|$ respectively. For any integer $m \geq 0$ we put

$$\|h(t,x;\xi)\|_{(m)}(t) = \sum_{|\alpha|+|\beta|\leq m} \sup_{(x,\xi)\in\mathbb{R}^n\times K} \big|D_x^\alpha D_\xi^\beta h(t,x;\xi)\big|.$$

We define for $0 < t \leq T$ the function space

$$W^{(m)}\big(\mathbb{R}^n;K\big)(0,T] = \big\{h(t,x;\xi); \|h\|_{(m)}(t) < \infty\big\}.$$

With a positive parameter κ and with $L_0 := \left[\frac{n}{2}\right] + 2$ we introduce

$$\Pi_\kappa = \Big\{ g = g(t,x) \in \bigcap_{j=0}^{1} C^j\big([0,T];H^{L_0-j}(\mathbb{R}^n)\big) : \|g\|_{L_0} + \|g_t\|_{L_0-1} < \kappa \Big\}.$$

From now on, κ is taken sufficiently small that $(g, g_t, \nabla g) \in K$ for $g \in \Pi_\kappa$. Moreover, we introduce the energies

$$E[v](t) = \|\partial_t v\|^2 + \sum_{j,k=1}^{n} (a_{jk}v_{x_j}, v_{x_k}),$$

$$E_m[v](t) = \|\partial_t v\|_m^2 + \sum_{|\alpha|\leq m} \sum_{j,k=1}^{n} (a_{jk}D_x^\alpha v_{x_j}, D_x^\alpha v_{x_k}),$$

for any integer $m \geq 0$. Finally, we sometimes use the notation $h_{(\alpha)}(t,x;\xi) = D_x^\alpha h(t,x;\xi)$, $h^{(\beta)}(t,x;\xi) = D_\xi^\beta h(t,x;\xi)$ and $\Lambda u = (u, u_t, \nabla u)$.

The main results of our paper are given in the following theorems.

THEOREM 0.1 (H^∞-well posedness). *Assume that the assumptions* (A-I) *to* (A-III) *are satisfied and* $(\varphi(x),\psi(x)) \in H^\infty(\mathbb{R}^n)\times H^\infty(\mathbb{R}^n)$ *fulfil* $\big(\varphi(x),\psi(x),\nabla\varphi(x)\big)\in K$. *Then there exist a constant* $T^*>0$ *and a unique solution* $u:=u(t,x)\in C^1\big([0,T^*];H^\infty(\mathbb{R}^n)\big)$ *of* (CP1) *for* $T=T^*$. *Moreover, the solution possesses the domain of dependence property.*

THEOREM 0.2 (C^∞-well posedness). *Assume that* (A-I) *to* (A-III) *hold and that* $(\varphi(x),\psi(x)) \in C^\infty(\mathbb{R}^n)\times C^\infty(\mathbb{R}^n)$ *satisfy* $(\varphi(x),\psi(x),\nabla\varphi(x))\in K$. *Then* (CP1) *is* C^∞*-well posed, this means, there exists a unique solution in* $C^1\big([0,T^*];C^\infty(\mathbb{R}^n)\big)$, *where the solution possesses the domain of dependence property.*

1. Reduction scheme. In this section we reduce (CP1), this problem implies the finite loss of derivatives for its solutions, by a finite family (Qj), $j=0,1,\dots,l$, of Cauchy problems for nonlinear ordinary differential equations to an auxiliary Cauchy problem (CP3), which is of strict hyperbolic type, see [8]. We follow ideas according to [9]. Let us consider the next Cauchy problems in a strip $[0,T]\times\mathbb{R}^n$:

$$\text{(Q0)}\qquad \begin{cases} u^{(0)}_{tt}+\rho\big(t,x;u^{(0)},u^{(0)}_t,\nabla\varphi\big)=0, & \\ u^{(0)}=\varphi(x),\ u^{(0)}_t=\psi(x) & \text{at } t=0, \end{cases} \tag{1.1}$$

$$\text{(Q1)}\qquad \begin{cases} u^{(1)}_{tt}+\rho\big(t,x;u^{(0)}+u^{(1)},(u^{(0)}+u^{(1)})_t,\nabla u^{(0)}\big) & \\ \qquad =\sum\limits_{j,k=1}^{n} a_{jk}(t,x;\Lambda u^{(0)})u^{(0)}_{x_jx_k}+\rho\big(t,x;u^{(0)},u^{(0)}_t,\nabla\varphi\big), & \\ u^{(1)}=u^{(1)}_t=0 & \text{at } t=0, \end{cases} \tag{1.2}$$

$$\text{(Q2)}\quad \begin{cases} u^{(2)}_{tt}+\rho\big(t,x;u^{(0)}+u^{(1)}+u^{(2)},(u^{(0)}+u^{(1)}+u^{(2)})_t,\nabla(u^{(0)}+u^{(1)})\big) & \\ \quad =\sum\limits_{j,k=1}^{n}\Big(a_{jk}(t,x;\Lambda(u^{(0)}+u^{(1)}))(u^{(0)}+u^{(1)})_{x_jx_k}-a_{jk}(t,x;\Lambda u^{(0)})u^{(0)}_{x_jx_k}\Big) & \\ \qquad\qquad +\rho\big(t,x;u^{(0)}+u^{(1)},(u^{(0)}+u^{(1)})_t,\nabla u^{(0)}\big), & \\ u^{(2)}=u^{(2)}_t=0 & \text{at } t=0, \end{cases} \tag{1.3}$$

and in general

$$\text{(Q}\ell\text{)}\quad \begin{cases} u^{(\ell)}_{tt}+\rho\big(t,x;U^{(\ell)},U^{(\ell)}_t,\nabla U^{(\ell-1)}\big) & \\ \quad =\sum\limits_{j,k=1}^{n}\Big(a_{jk}(t,x;\Lambda U^{(\ell-1)})U^{(\ell-1)}_{x_jx_k}-a_{jk}(t,x;\Lambda U^{(\ell-2)})U^{(\ell-2)}_{x_jx_k}\Big) & \\ \qquad\qquad\qquad +\rho\Big(t,x;U^{(\ell-1)},U^{(\ell-1)}_t,\nabla U^{(\ell-2)}\Big), & \\ u^{(\ell)}=u^{(\ell)}_t=0 & \text{at } t=0, \end{cases} \tag{1.4}$$

for $\ell\ge 2$, where we use the abbreviations $U^{(\ell)}:=U^{(\ell)}(t,x)=\sum\limits_{s=0}^{\ell}u^{(s)}(t,x)$.

Setting $u(t,x)=U^{(\ell)}(t,x)+v(t,x)$, we obtain the following results.

Lemma 1.1. *If* $u^{(s)} \in C^1\big([0,T];H^\infty(\mathbb{R}^n)\big)$ *solves the Cauchy problem* (Qs), $0 \le s \le \ell$, *and* $v \in C^1\big([0,T];H^\infty(\mathbb{R}^n)\big)$ *solves the Cauchy problem*

$$\text{(CP2)} \qquad \begin{cases} v_{tt} - \sum\limits_{j,k=1}^n a_{jk}\big(t,x;\Lambda(U^{(\ell)}+v)\big)v_{x_jx_k} + \rho\big(t,x;\Lambda(U^{(\ell)}+v)\big) \\ \qquad\qquad + U^{(\ell)}_{tt} - \sum\limits_{j,k=1}^n a_{jk}\big(t,x;\Lambda(U^{(\ell)}+v)\big)U^{(\ell)}_{x_jx_k} = 0, \\ v(0,x) = v_t(0,x) = 0, \end{cases} \tag{1.5}$$

then $u := U^{(\ell)} + v \in C^1\big([0,T];H^\infty(\mathbb{R}^n)\big)$ *solves* (CP1).

Proof. It is easily seen that (1.5) can be written in the form

$$(v+U^{(\ell)})_{tt} - \sum_{j,k=1}^n a_{jk}\big(t,x;\Lambda(U^{(\ell)}+v)\big)(v+U^{(\ell)})_{x_jx_k} + \rho\big(t,x;\Lambda(U^{(\ell)}+v)\big) = 0,$$

and that $U^{(\ell)}|_{t=0} = \varphi(x)$ and $U^{(\ell)}_t|_{t=0} = \psi(x)$. Hence the proof is complete. ∎

Lemma 1.2. *The Cauchy problem* (CP2) *with* v *and* $u^{(s)} \in C^1\big([0,T];H^\infty(\mathbb{R}^n)\big)$, $0 \le s \le \ell$, *is equivalent to the following Cauchy problem for* $w = v$:

$$\text{(CP3)} \qquad \begin{cases} Q_w[v] = v_{tt} - \sum\limits_{j,k=1}^n b_{jk}(t,x;\Lambda w)v_{x_jx_k} + \sum\limits_{j=1}^n b_j(t,x;\Lambda w)v_{x_j} \\ \qquad\qquad + b_0(t,x;\Lambda w)v_t + b(t,x;\Lambda w)v = f_l(t,x;\Lambda w), \\ v(0,x) = v_t(0,x) = 0, \end{cases} \tag{1.6}$$

where the coefficients b_{jk}, b_j, b_0 *and* b *satisfy the following conditions*:

(B-I) (*strict hyperbolicity*)

$$\sum_{j,k=1}^n b_{jk}(t,x;\xi)\eta_j\eta_k \ge C_0|\eta|^2, \quad \eta \in \mathbb{R}^n \setminus \{0\},$$

$$b_{jk}(t,x;\xi) = b_{kj}(t,x;\xi), \quad j,k = 1,\dots,n,\ j \ne k,$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\xi \in \mathbb{R}^{n+2}$ *and with* C_0 *from* (A-I).

(B-II) (*regularity properties*)

The regularity behaviour of coefficients can be described in the following way:

i) $b_{jk} \in C^1\big((0,T];\mathcal{B}^\infty(\mathbb{R}^n \times K)\big)$,

ii) $h \in C\big((0,T];\mathcal{B}^\infty(\mathbb{R}^n \times K)\big)$ *for* $h = b_j$, b_0, *and* b.

(B-III) (*asymptotic behaviour near* $t = 0$)

The asymptotic behaviour of coefficients near $t = 0$ *can be described in the following way*:

i) $|\partial_t b_{jk}(t,x;\xi)| \le C_{jk}/t$, $|D_x^\alpha D_\xi^\beta b_{jk}(t,x;\xi)| \le C_{jk\alpha\beta}/t^r$, *for* $0 \le r < 1$ *and for all multi-indices* α, β,

ii) $|D_x^\alpha D_\xi^\beta h(t,x;\xi)| \le C'_{\alpha\beta}/t^{\max\{q,r\}}$, *for* $h = b_j$, b_0, b *and for all multi-indices* α *and* β.

Proof. It follows from (1.1)–(1.4) that

$$U^{(\ell)}_{tt} = \sum_{j,k=1}^n a_{jk}(t,x;\Lambda U^{(\ell-1)})U^{(\ell-1)}_{x_jx_k} - \rho\big(t,x;U^{(\ell)},U^{(\ell)}_t,\nabla U^{(\ell-1)}\big).$$

We substitute $U_{tt}^{(\ell)}$ in (1.5) by the right-hand side of the above equality. Then we have to study the following four terms to get the properties (B-I) to (B-III):

i) $\rho\big(t,x;\Lambda(U^{(\ell)}+v)\big)-\rho\big(t,x;\Lambda U^{(\ell)}\big)$,

ii) $\rho\big(t,x;\Lambda U^{(\ell)}\big)-\rho\big(t,x;U^{(\ell)},U_t^{(\ell)},\nabla U^{(\ell-1)}\big)$,

iii) $\sum\limits_{j,k=1}^{n}\big(a_{jk}(t,x;\Lambda(U^{(\ell)}+v))-a_{jk}(t,x;\Lambda U^{(\ell-1)})\big)U_{x_jx_k}^{(\ell)}$,

iv) $\sum\limits_{j,k=1}^{n} a_{jk}(t,x;\Lambda U^{(\ell-1)})u_{x_jx_k}^{(\ell)}$.

By the mean value theorem it is seen that b_j, b_0 and b are determined by i) and iii) and that f_l is determined by ii)–iv). This helps us to understand that the assumptions (A-I) to (A-III) are transferred to (B-I) to (B-III). Hence the proof is complete. ■

Now we cite some auxiliary results.

LEMMA 1.3. *Assume that $q=q(t,x;\xi)\in W^{(M)}(\mathbb{R}^n;K)(0,T]$ and that v belongs to $\bigcap\limits_{j=0}^{1} C^j\big([0,T];H^{M+1-j}(\mathbb{R}^n)\big)\cap\Pi_\kappa$ and $[\frac{n}{2}]+1\le M$. Then there exists a constant C, which depends on $\sum\limits_{j=0}^{1}\|\partial_t^j v\|_{L_0-j}$, such that for every fixed $t\in(0,T]$*

$$\|q(t,x;\Lambda v)\|_M\le C\|q\|_{(M)}\Big(\sum_{j=0}^{1}\|\partial_t^j v\|_{M+1-j}+1\Big). \tag{1.7}$$

COROLLARY 1.1. *Suppose that the functions $v=v(t,x)$ and $z=z(t,x)$ belong to $\bigcap\limits_{j=0}^{1} C^j\big([0,T];H^{M+1-j}(\mathbb{R}^n)\big)$ and satisfy $v+\tau z\in\Pi_\kappa$ for $\tau\in[0,1]$. Moreover, let $q=q(t,x;\xi)\in W^{(M)}(\mathbb{R}^n;K)(0,T]$. If $[\frac{n}{2}]+1\le M$, the following estimate is true with a constant C depending on $\|q\|_{(M+1)}\sum\limits_{j=0}^{1}\big(\|\partial_t^j z\|_{L_0-j}+\|\partial_t^j v\|_{M+1-j}\big)$:*

$$\big\|q(t,x;\Lambda(v+z))-q(t,x;\Lambda v)\big\|_M\le C\sum_{j=0}^{1}\|\partial_t^j z\|_{M+1-j}. \tag{1.8}$$

LEMMA 1.4 (Nersesian, cf. [7, Lemma A.2]). *Let us be given the differential inequality*

$$y'(t)\le K(t)y(t)+f(t)$$

for $t\in(0,T)$, where the functions $K=K(t)$ and $f=f(t)$ belong to $C(0,T)$, $T>0$. Under the assumptions

- $\int_0^\varepsilon K(\tau)\,d\tau=\infty$, $\int_\varepsilon^T K(\tau)\,d\tau<\infty$ *for all $\varepsilon\in(0,T)$,*
- $\lim\limits_{\varepsilon\to 0}\int_\varepsilon^t \exp\big(\int_s^t K(\tau)\,d\tau\big)f(s)\,ds$ *exists for all $\varepsilon\le t\le T$,*
- $y(\varepsilon)\exp\big(\int_\varepsilon^t K(\tau)\,d\tau\big)=o(\varepsilon)$,

every solution of the differential inequality belonging to $C[0,T]\cap C^1(0,T)$ satisfies

$$y(t)\le\int_0^t \exp\Big(\int_s^t K(\tau)\,d\tau\Big)f(s)\,ds.$$

Let us now investigate the asymptotic behaviour of the solutions $u^{(0)},\dots,u^{(\ell)}$ to the Cauchy problems (Q0),...,(Qℓ), respectively.

Proposition 1.1. *If $(\varphi(x),\psi(x))\in H^\infty(\mathbb{R}^n)\times H^\infty(\mathbb{R}^n)$, $(\varphi(x),\psi(x),\nabla\varphi(x))\in K$, and l is a fixed nonnegative integer, then there exists a solution $u^{(s)}:=u^{(s)}(t,x)$ of the problem* (Qs) *and a joint life span $[0,T_1]$ such that $U^{(s)}\in C^1\big([0,T_1];H^\infty(\mathbb{R}^n)\big)\cap\Pi_\kappa$ for all $0\le s\le l$. Moreover, the solutions $u^{(s)}$ satisfy for $1\le s$ with $\nu=\min\{1-q,1-r\}$ the estimates*

$$\|\partial_t^j u^{(s)}\|_m^2\le C_m t^{s\nu+1-j},\quad j=0,1. \tag{1.9}$$

Proof. The time local solvability of the problem (Qs), $s=0,1,\dots,l$, is well-known in $[0,T_1]\times\mathbb{R}^n$ for an appropriate constant $T_1>0$. The regularity $u^{(s)}\in C^1([0,T];H^\infty(R^n))$ follows from the assumptions (A-II) and (A-III) and the nonlinear ordinary differential equation in (Qs), $0\le s\le l$, by using (A-II)ii). Let us derive the above estimates.

First, we deal with the problem (Q0). By the standard energy method we have

$$\partial_t\big\|u_t^{(0)}\big\|^2\le 2\Big|\Big(\rho\big(t,x;u^{(0)},u_t^{(0)},\nabla\varphi\big),u_t^{(0)}\Big)\Big|\le\|\rho\|^2+\big\|u_t^{(0)}\big\|^2. \tag{1.10}$$

Integrating over $(0,t)$, $t\in[0,T_1]$, we have for sufficiently small T_1

$$\big\|u_t^{(0)}\big\|^2(t)\le C\int_0^t\|\rho\|^2(\tau)\,d\tau+\|\psi\|^2. \tag{1.11}$$

From (1.11) it follows immediately that

$$\big\|u^{(0)}\big\|^2(t)\le C\Big(t\int_0^t\|\rho\|^2(\tau)\,d\tau+t\|\psi\|^2+\|\varphi\|^2\Big). \tag{1.12}$$

Differentiating m times with respect to x both sides of the equation from (1.1), $m\ge[\frac{n}{2}]+1$, we obtain, by following the same procedure from (1.10) to (1.12), by using Lemma 1.3 and taking into account property (A-II)ii), the inequality

$$\sum_{j=0}^1\big\|\partial_t^j u^{(0)}\big\|_m^2(t)\le C_m\Big(t\int_0^{T_1}\big(\|\rho\|_{(m)}(\tau)+\|\rho\|_m^2(\tau)\big)\,d\tau+(1+t)\|\psi\|_m^2+\|\varphi\|_m^2\Big). \tag{1.13}$$

In fact, for $|\alpha|\le m$ we have

$$\begin{aligned}\partial_t\sum_{|\alpha|\le m}\big\|D_x^\alpha u_t^{(0)}\big\|^2&\le 2\Big|\Big(\sum_{|\alpha|\le m}D_x^\alpha\rho,\ D_x^\alpha u_t^{(0)}\Big)\Big|\\&\le C_m\Big(\|\rho\|_{(m)}\Big(\sum_{j=0}^1\|\partial_t^j u^{(0)}\|_m^2+\|\varphi\|_{m+1}^2+1\Big)+\big(\|\rho\|_m^2+\|u_t^{(0)}\|_m^2\big)\Big).\end{aligned}$$

Multiplying both sides by $\exp\big(-C_m'\int_0^t\|\rho\|_{(m)}(\tau)\,d\tau\big)$ with a sufficiently large positive C_m' we have from (1.11) and (1.12) in the same manner

$$\begin{aligned}\sum_{j=0}^1\big\|\partial_t^j u^{(0)}\big\|_m^2\le C_m\Big(t\exp\Big(C_m'\int_0^{T_1}\|\rho\|_{(m)}(\tau)\,d\tau\Big)\int_0^{T_1}\big(\|\rho\|_{(m)}+\|\rho\|_m^2\big)(\tau)\,d\tau\\+(1+t)\|\psi\|_m^2+\|\varphi\|_m^2\Big).\end{aligned}$$

Since $\|\rho\|_{(m)} \in L^1(0,T_1)$, we arrive at (1.13). From (1.13) it is clear that a sufficiently small T_1 guarantees that $(u^{(0)}, u_t^{(0)}, \nabla u^{(0)}) \subset K$.

Now we consider problem (Q1). We apply in (1.2) the mean value theorem to

$$\rho\big(t,x;U^{(1)},U_t^{(1)},\nabla u^{(0)}\big) - \rho\big(t,x;u^{(0)},u_t^{(0)},\nabla\varphi\big).$$

Then using Lemma 1.3 we obtain for the solution of (1.2) the estimate

$$\big\|u_t^{(1)}\big\|^2(t) \le C\Big(\int_0^t \|\rho\|_{(1)}(\tau)\,d\tau + \sum_{j,k=1}^n \int_0^t \|a_{jk}\|_{(0)}(\tau)\,d\tau\Big) \times \Big(1 + \sup_{t\in[0,T_1]} \sum_{j=0}^1 \big\|\partial_t^j u^{(1)}\big\|_{2-2j}^2(t)\Big).$$

The application of (A-III) yields

$$\big\|u_t^{(1)}\big\|^2(t) \le Ct^\nu\Big(1 + \sup_{t\in[0,T_1]} \sum_{j=0}^1 \big\|\partial_t^j u^{(1)}\big\|_{2-2j}^2(t)\Big), \tag{1.14}$$

where $\nu = \min\{1-q, 1-r\}$. Hence for sufficiently small T_1 we obtain

$$\big\|\partial_t^j u^{(1)}\big\|^2(t) \le Ct^{\nu+1-j}, \quad j=0,1. \tag{1.15}$$

In the same way as (1.13) we have

$$\big\|\partial_t^j u^{(1)}\big\|_m^2(t) \le C_m t^{\nu+1-j}, \quad j=0,1. \tag{1.16}$$

Therefore there exists a positive constant T_1 (eventually we have to choose a smaller one than in the previous step) such that the inequalities (1.16) hold for $t \in [0,T_1]$ and $(U^{(1)}, U_t^{(1)}, \nabla U^{(1)}) \subset K$.

Finally, we sketch how to handle (Q2). In (1.3) we apply the mean value theorem to

$$\rho\big(t,x;U^{(2)},U_t^{(2)},\nabla U^{(1)}\big) - \rho\big(t,x;U^{(1)},U_t^{(1)},\nabla u^{(0)}\big),$$

$$\text{and} \quad \sum_{j,k=1}^n \Big(a_{jk}(t,x;\Lambda U^{(1)})U_{x_jx_k}^{(1)} - a_{jk}(t,x;\Lambda u^{(0)})u_{x_jx_k}^{(0)}\Big).$$

Taking account of (1.16), in the same way as (1.14) we obtain

$$\begin{aligned}\big\|u_t^{(2)}\big\|^2(t) \le C\Big(&\int_0^t \|\rho\|_{(1)}(\tau)\,d\tau\Big(t^{\nu+1} + \sup_{\tau\in[0,t]} \sum_{j=0}^1 \|\partial_t^j u^{(2)}\|^2(\tau)\Big)\\ &+ \sum_{j,k=1}^n \int_0^t \|a_{jk}\|_{(1)}(\tau)\,d\tau\Big(t^\nu + \sup_{\tau\in[0,t]} \sum_{j=0}^1 \|\partial_t^j u^{(2)}\|^2(\tau)\Big)\Big),\end{aligned} \tag{1.17}$$

where C depends on $\|U^{(1)}\|_2$ and $\|U_t^{(1)}\|$. A sufficiently small t gives

$$\big\|u_t^{(2)}\big\|^2(t) \le Ct^{2\nu}.$$

In the same manner as (1.16) we have

$$\big\|\partial_t^j u^{(2)}\big\|_m^2(t) \le C_m t^{2\nu+1-j}, \quad j=0,1. \tag{1.18}$$

Thus taking T_1 small enough, we obtain (1.18) and $(U^{(2)}, U_t^{(2)}, \nabla U^{(2)}) \subset K$ for $t \in [0,T_1]$.

Repeating the above procedure, we can find a positive constant T_1 such that the following properties hold for $t \in [0, T_1]$, $m \geq 0$, and $2 \leq s \leq l$:

$$\left\|\partial_t^j u^{(s)}\right\|_m^2 \leq C_m t^{s\nu+1-j}, \quad j = 0, 1,$$

and $(U^{(s)}, U_t^{(s)}, \nabla U^{(s)}) \subset K$. This completes the proof. ∎

REMARK 1.1. In each step of the previous proof we have shown that $\Lambda U^{(s)} \subset K$, $s = 0, 1, \ldots, l$, for $(t, x) \in [0, T_1] \times \mathbb{R}^n$. Therefore we can take κ and T_1 so small that for every $g \in \Pi_\kappa$,

$$2\Lambda\big(U^{(l)} + g\big) \subset K.$$

REMARK 1.2. By using the mean value theorem the right-hand side f_l of the auxiliary Cauchy problem (CP3) can be represented with $|\alpha| = 1$ and with constants $\theta_\alpha, \theta_\beta \in (0, 1)$ in the following way:

$$\begin{aligned} f_l(t, x; \Lambda v) = &\sum_{|\alpha|=1} -\rho^{(0,0,\alpha)}\Big(t, x; U^{(l)}, U_t^{(l)}, \nabla U^{(l)} + \theta_\alpha(\nabla(u^{(l)}))^\alpha\Big)(\nabla u^{(l)})^\alpha \\ &+ \sum_{|\beta|=1} \sum_{j,k=1}^n a_{jk}^{(\beta)}\Big(t, x; \Lambda(U^{(l)} + v) + \theta_\beta(\Lambda(u^{(l)} + v))^\beta\Big)(\Lambda u^{(l)})^\beta U_{x_j x_k}^{(l)} \\ &+ \sum_{j,k=1}^n a_{jk}\big(t, x; \Lambda(U^{(l-1)})\big) u_{x_j x_k}^{(l)}. \end{aligned} \tag{1.19}$$

Taking account of Proposition 1.1 and conditions (A-III) we deduce that sufficiently small T_1 and κ imply

$$\int_0^{T_1} \tau^{-l\nu/2} \left\|f_l(\tau, x; \Lambda v)\right\|_m(\tau)\, d\tau \leq C_m \sup_{t \in [0, T_1]} t^{-l\nu/2} \sum_{j=0}^1 \left\|\partial_t^j u^{(l)}(t)\right\|_{m+2-2j}$$

for all $v \in C^1\big([0, T]; H^\infty(\mathbb{R}^n)\big) \cap \Pi_\kappa$. Thus we have explained the asymptotic behaviour of f_l near $t = 0$ and use in the following a fixed κ.

From now on, we put (without loss of generality, since the general case $\max\{q, r\} < 1$ can be studied in the same way) $q = r = \frac{1}{2}$, therefore $\nu = \frac{1}{2}$, for our convenience.

2. Energy estimates. In this section we derive energy estimates for the solution v of the auxiliary problem (CP3).

2.1. *Basic energy estimate*

PROPOSITION 2.1. *Under the assumptions* (B-I) *to* (B-III) *there exist positive constants l_0 and C_{l,L_0} such that for $v, w \in C^2\big([0, T]; H^\infty(R^n)\big)$ satisfying* (1.6) *with $E_{L_0-1}[w](t) \leq D_{L_0}$ and $w \in \Pi_\kappa$ (for the definition of Π_κ see Introduction), that is,*

$$\begin{aligned} v_{tt} - \sum_{j,k=1}^n b_{jk}(t, x; \Lambda w) v_{x_j x_k} + \sum_{j=1}^n b_j(t, x; \Lambda w) v_{x_j}& \\ + b_0(t, x; \Lambda w) v_t + b(t, x; \Lambda w) v &= f_l(t, x; \Lambda w), \\ v(0, x) = v_t(0, x) &\equiv 0, \end{aligned}$$

the following basic energy estimate holds for $l \geq l_0$:

$$E_{L_0-1}[v](t) \leq C_{l,L_0} t^{(l+1)/2}. \tag{2.1}$$

PROOF. Taking account of our partial differential equation we have

$$\partial_t \|v_t\|^2 = 2(v_{tt}, v_t) = 2\Big(\sum_{j,k=1}^{n} b_{jk} v_{x_j x_k},\, v_t\Big) - 2\Big(\sum_{j=1}^{n} b_j v_{x_j} + b_0 v_t + bv - f_l,\, v_t\Big)$$

$$= -2\Big(\sum_{j,k=1}^{n} (\partial_{x_k} b_{jk}) v_{x_j},\, v_t\Big) - 2\Big(\sum_{j,k=1}^{n} b_{jk} v_{x_j},\, v_{x_k t}\Big) - 2\Big(\sum_{j=1}^{n} b_j v_{x_j} + b_0 v_t + bv - f_l,\, v_t\Big);$$

the last equality can be derived by integration by parts. Taking account of

$$\begin{aligned} -2\Big(\sum_{j,k=1}^{n} b_{jk} v_{x_j},\, v_{x_k t}\Big) &= -2\partial_t \Big(\sum_{j,k=1}^{n} b_{jk} v_{x_j},\, v_{x_k}\Big) \\ &\quad + 2\Big(\sum_{j,k=1}^{n} (\partial_t b_{jk}) v_{x_j},\, v_{x_k}\Big) + 2\Big(\sum_{j,k=1}^{n} b_{jk} v_{t x_j},\, v_{x_k}\Big), \end{aligned}$$

we see that

$$-2\Big(\sum_{j,k=1}^{n} b_{jk} v_{x_j},\, v_{x_k t}\Big) = -\partial_t \Big(\sum_{j,k=1}^{n} b_{jk} v_{x_j},\, v_{x_k}\Big) + \Big(\sum_{j,k=1}^{n} (\partial_t b_{jk}) v_{x_j},\, v_{x_k}\Big).$$

Therefore it follows from the above equality that

$$\begin{aligned} \partial_t E[v](t) &= -2\Big(\sum_{j,k=1}^{n} (\partial_{x_k} b_{jk}) v_{x_j},\, v_t\Big) \\ &\quad + \Big(\sum_{j,k=1}^{n} (\partial_t b_{jk}) v_{x_j},\, v_{x_k}\Big) - 2\Big(\sum_{j=1}^{n} b_j v_{x_j} + b_0 v_t + bv - f_l,\, v_t\Big). \end{aligned}$$

On the other hand,

$$\partial_t b_{jk}\big(t, x; \Lambda v\big) = \sum_{|\gamma|=1} b_{jk}^{(\gamma)}\big(t, x; \Lambda v\big) \partial_t (\Lambda v)^\gamma + \big(\partial_t b_{jk}\big)(t, x; \Lambda v).$$

Since $w \in C^2\big([0,T]; H^\infty(R^n)\big)$, using (B-III) we then deduce that

$$|\partial_t b_{jk}|, |\partial_{x_k} b_{jk}|, |b_j|, |b_0| \text{ and } |b| \leq C\tfrac{1}{t}.$$

Hence

$$\partial_t E[v](t) \leq \frac{C_1}{t} E[v](t) + 2|(f_l, v_t)|. \tag{2.2}$$

Note that according to (1.19) the function f_l can be written in the form

$$f_l(t, x; \Lambda v) = \sum_{|\nu|=1} F_{\nu,l}(t, x; \Lambda v)(\Lambda u^{(l)})^\nu + \sum_{|\nu'|=2} F_{\nu',l}(t, x) D_x^{\nu'} u^{(l)}, \tag{2.3}$$

where $F_{\nu,l}$ and $F_{\nu',l}$ are appropriate functions satisfying (B-III)ii) for $h = F_{\nu,l}$. Then we have with a positive parameter ε

$$2|(f_l, v_t)| \le C \sum_{|\nu|=1,|\nu'|=2} t^{l/2}\Big(\|F_{\nu,l}\|_{(0)}(t) + \|F_{\nu',l}\|_{(0)}(t)\Big) \times \Big(\frac{1}{\varepsilon}\, t^{-l/2}\sum_{j=0}^{1}\big\|\partial_t^j u^{(l)}\big\|^2_{2-2j} + \varepsilon t^{-l/2}\big\|v_t\big\|^2\Big), \tag{2.4}$$

where C depends on $\sum_{j=0}^{1}\|\partial_t^j u^{(l)}\|_{L_0-j} < \kappa$. Differentiating m times with respect to x both sides of (1.6), $0 \le m \le [\frac{n}{2}]+1$, we get for $|\alpha| = m$ the identity

$$(D_x^\alpha v)_{tt} - \sum_{j,k=1}^{n} b_{jk}(D_x^\alpha v)_{x_jx_k} + \sum_{j=1}^{n} b_j(D_x^\alpha v)_{x_j} + b_0(D_x^\alpha v)_t + b(D_x^\alpha v) = \sum_{\eta<\alpha}\binom{\alpha}{\eta}$$
$$\times\Big(\sum_{j,k=1}^{n}(D_x^\eta b_{jk})D_x^{\alpha-\eta}v_{x_jx_k} + \sum_{j=1}^{n}(D_x^\eta b_j)D_x^{\alpha-\eta}v_{x_j} + (D_x^\eta b_0)D_x^{\alpha-\eta}v_t + (D_x^\eta b)D_x^{\alpha-\eta}v\Big).$$

Therefore applying the same way as for deriving (2.2) to the left-hand side, we have using Lemma 1.3 with a constant $C_{1,n} > C_1$ the energy estimate

$$\partial_t E_{L_0-1}[v](t) \le \frac{C_{1,n}}{t}\, E_{L_0-1}[v](t) + C\sum_{|\nu|=1,|\nu'|=2} t^{l/2}\Big(\|F_{\nu,l}\|_{(L_0-1)}(t) + \|F_{\nu',l}\|_{(L_0-1)}(t)\Big) \times\Big(\frac{1}{\varepsilon}\, t^{-l/2}\sum_{j=0}^{1}\big\|\partial_t^j u^{(l)}\big\|^2_{L_0+1-2j} + \varepsilon t^{-l/2}\big\|v_t\big\|^2_{L_0-1}\Big).$$

Take an integer l_0 so that $l_0/2 > C_{1,n}$. The application of Lemma 1.4 yields for $l \ge l_0$ the energy estimate

$$E_{L_0-1}[v](t) \le C\int_0^t \exp\Big(\int_s^t \frac{C_{1,n}}{\tau}\, d\tau\Big)s^{l/2}\sum_{|\nu|=1,|\nu'|=2}\Big(\big\|F_{\nu,l}\big\|_{(L_0-1)} + \big\|F_{\nu',l}\big\|_{(L_0-1)}\Big)(s)\, ds \times \sup_{\tau\in[0,t]}\Big(\frac{1}{\varepsilon}\,\tau^{-l/2-1}\sum_{j=0}^{1}\big\|\partial_t^j u^{(l)}\big\|^2_{L_0+1-2j} + \varepsilon\tau^{-l/2}\big\|v_t\big\|^2_{L_0-1}\Big).$$

Since $l/2 \ge C_{1,n}$, taking ε sufficiently small we get

$$E_{L_0-1}[v](t) \le Ct^{l/2}\int_0^t \sum_{|\nu|=1,|\nu'|=2}\Big(\big\|F_{\nu,l}\big\|_{L_0-1} + \big\|F_{\nu',l}\big\|_{L_0-1}\Big)(s)\, ds \times \sup_{0\le\tau\le t}\tau^{-l/2}\sum_{j=0}^{1}\big\|\partial_t^j u^{(l)}(\tau)\big\|^2_{L_0+1-2j}.$$

Recalling Proposition 1.1 with $\nu = \frac{1}{2}$ and (A-III) we conclude that

$$E_{L_0-1}[v](t) \le C_{l,L_0} t^{(l+1)/2}. \tag{2.5}$$

Therefore we arrive at (2.1). ■

2.2. *Energy estimates of higher order*

PROPOSITION 2.2. *The statements of the previous proposition hold if we replace L_0-1 by m, $m \geq L_0-1$.*

PROOF. We will prove (2.1) by induction after replacing L_0-1 by m. Suppose that we have with constants $C_{l,p}$, $0 \leq p \leq m-1$, the estimates

$$E_p[v](t) \leq C_{l,p} t^{(l+1)/2}. \tag{2.6}$$

In the same way as in the proof of Proposition 2.1 we derive for $|\alpha| = m$ and with a positive parameter ε the estimate

$$\begin{aligned}
&\partial_t E[D_x^\alpha v](t) \leq \frac{C_1+\varepsilon}{t} E[D_x^\alpha v](t) \\
\text{(I)}\qquad &+ 2\big(D_x^\alpha f_l(t,x;\Lambda v), D_x^\alpha v_t\big) \\
\text{(II)}\qquad &+ \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \sum_{j,k=1}^{n} 2\big(D_x^{\alpha-\gamma} b_{jk} D_x^\gamma v_{x_j x_k}, D_x^\alpha v_t\big) \\
\text{(III)}\qquad &+ \frac{t}{\varepsilon} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \Big(\sum_{j=1}^{n} \big\|D_x^{\alpha-\gamma} b_j D_x^\gamma v_{x_j}\big\|^2 + \big\|D_x^{\alpha-\gamma} b_0 D_x^\gamma v_t\big\|^2 + \big\|D_x^{\alpha-\gamma} b D_x^\gamma v\big\|^2 \Big).
\end{aligned}$$

Analogously to the derivation of (2.4), by using Lemma 1.3 we have

$$\begin{aligned}
2\big|(D_x^\alpha f_l, D_x^\alpha v_t)\big| \leq C \sum_{|\nu|=1,|\nu'|=2} &\Big(\big\|F_{\nu,l}\big\|_{(m)} + \big\|F_{\nu',l}\big\|_{(m)}\Big) \\
&\times \Big(\frac{1}{\varepsilon} \sum_{j=0}^{1} \big\|\partial_t^j u^{(l)}\big\|_{m+2-2j}^2 + \varepsilon \big\|v_t\big\|_m^2 \Big).
\end{aligned} \tag{2.7}$$

Moreover,

$$\text{(II)} \leq \sum_{j,k=1}^{n} \Big(C(m) \|b_{jk}\|_{(1)} \big(\|v_{x_j x_k}\|_{m-1}^2 + \|v_t\|_m^2 \big) \Big) + \frac{C}{t} E_{m-1}[v](t). \tag{2.8}$$

Finally, it follows from (B-III)ii) that

$$\text{(III)} \leq \frac{C}{t} E_{m-1}[v](t). \tag{2.9}$$

Put $A_m(t) = \sup_{j,k} C(m)\|b_{jk}\|_{(1)}(t)$. Summarizing (2.7) to (2.9) and taking ε so small that $l_0/2$ $(> C_{1,n})$ $> C_1+\varepsilon$ we obtain for $l \geq l_0$ the inequality

$$\begin{aligned}
\partial_t E_m[v](t) \leq{}& \frac{l}{2t} E_m[v](t) + A_m(t) E_m[v](t) \\
&+ C \sum_{|\nu|=1,|\nu'|=2} \Big(\|F_{\nu,l}\|_{(m)} + \|F_{\nu',l}\|_{(m)} \Big) \Big(\frac{1}{\varepsilon} \sum_{j=0}^{1} \big\|\partial_t^j u^{(l)}\big\|_{m+2-2j}^2 + \varepsilon \big\|v_t\big\|_m^2 \Big) \\
&+ \frac{C}{t} E_{m-1}[v].
\end{aligned} \tag{2.10}$$

By applying Lemma 1.4 we conclude that

$$\begin{aligned}
E_m[v](t) \le C \int_0^t \exp\Big(\int_r^t \Big(\frac{l/2}{\tau} + A_m(\tau)\Big)\, d\tau\Big) \\
\times \Bigg(r^{l/2} \sum_{|\nu|=1,|\nu'|=2} \big(\|F_{\nu,l}\|_{(m)} + \|F_{\nu',l}\|_{(m)}\big)(r) \\
\times \sup_{0\le\tau\le t} \Big(\frac{\tau^{-l/2}}{\varepsilon} \sum_{j=0}^{1} \big\|\partial_t^j u^{(l)}(\tau)\big\|^2_{m+2-2j} + \varepsilon\tau^{-l/2}\big\|v_\tau\big\|^2_m\Big) + \frac{C}{r}\, E_{m-1}[v](r)\Bigg)\, dr.
\end{aligned} \tag{2.11}$$

From (B-III)i) it follows immediately that $\exp\big(\int_0^t A_m(\tau)\, d\tau\big)$ is bounded for $t \in [0,T]$. Taking ε sufficiently small, in the same way as in the derivation of (2.5) we conclude that

$$E_m[v](t) \le c_{l,m} t^{(l+1)/2} + C E_{m-1}[v](t) \tag{2.12}$$

and with (2.6)

$$E_m[v](t) \le C_{l,m} t^{(l+1)/2}. \tag{2.13}$$

Thus the proof is complete. ∎

Remark 2.1. From both propositions we conclude immediately that $E_m[v](t) \le D_m$ and $v \in \Pi_\kappa$ for sufficiently small $T = T(m)$.

3. Linear problem. In this section we consider the following linear Cauchy problem (LP) corresponding to (CP3) in $(0,T)\times\mathbb{R}^n$:

$$\text{(LP)} \qquad \begin{cases} L[v] = \partial_t^2 v - \displaystyle\sum_{j,k=1}^n b_{jk}(t,x)\partial^2_{x_j x_k} v + \sum_{j=1}^n b_j(t,x)\partial_{x_j} v + b_0(t,x)\partial_t v \\ \qquad\qquad\qquad\qquad\qquad\qquad + b(t,x)v = f_l(t,x), \\ v(0,x) = v_t(0,x) = 0, \end{cases} \tag{3.1}$$

where b_{jk} satisfy (B-I), (B-II)i) and (B-III)i), b_j, b_0 and b satisfy (B-II)ii) and (B-III)ii) and $f_l(t,x) \in C\big([0,T]; H^\infty(\mathbb{R}^n)\big)$ satisfies $\|f_l\|_s^2(t) \le C_s t^l$ for every integer $s \ge 0$.

Proposition 3.1. *There exists a natural number l_1 such that the Cauchy problem* (LP) *with $l \ge l_1$ has a uniquely determined solution $v \in C^1\big([0,T]; H^\infty(\mathbb{R}^n)\big)$ satisfying for every integer $s \ge 0$ the energy estimate*

$$E_s[v](t) \le C_s t^l \int_0^t \tau^{-l} \big\|f_l\big\|_s^2(\tau)\, d\tau. \tag{3.2}$$

Proof. For a positive parameter ε we consider the following ε-shifted problem $(\mathrm{LP})_\varepsilon$ of (LP) in $(0,T-\varepsilon)\times\mathbb{R}^n$:

$$(\mathrm{LP})_\varepsilon \qquad \begin{cases} L_\varepsilon[v] = \partial_t^2 v_\varepsilon - \displaystyle\sum_{j,k=1}^n b^\varepsilon_{jk}(t,x)\partial^2_{x_j x_k} v_\varepsilon + \sum_{j=1}^n b^\varepsilon_j(t,x)\partial_{x_j} v_\varepsilon + b^\varepsilon_0(t,x)\partial_t v_\varepsilon \\ \qquad\qquad\qquad\qquad\qquad\qquad + b^\varepsilon(t,x)v_\varepsilon = f_l(t,x), \\ v_\varepsilon(0,x) = \partial_t v_\varepsilon(0,x) = 0, \end{cases}$$

where for any function $h = h(t,x)$ defined in $(0,T)\times\mathbb{R}^n$ we write $h^\varepsilon = h^\varepsilon(t,x) := h(t+\varepsilon,x)$. It is well-known that there exists a smooth solution v_ε of $(LP)_\varepsilon$ belonging to

$C^2\big([0,T-\varepsilon];H^\infty(\mathbb{R}^n)\big)$. On the same way as for (2.5) we can show that there exists an integer l_1 such that

$$E_s[v_\varepsilon](t) \le C_s t^l \int_0^t \tau^{-l}\|f_l\|_s^2(\tau)\,d\tau \tag{3.3}$$

for all $l \ge l_1$ and $\varepsilon \in (0,\varepsilon_0]$. Therefore $E_s[v_\varepsilon](t)$ are uniformly bounded in ε. We consider a sequence $\{v_{\varepsilon_i}\}_{i=0}^\infty$ of solutions of $(\mathrm{LP})_{\varepsilon_i}$ with a sequence $\{\varepsilon_i\}$, $\varepsilon_i > \varepsilon_{i+1}$ and $\varepsilon_i \to 0$ for $i \to \infty$. The difference $w_{\varepsilon_i} = v_{\varepsilon_i} - v_{\varepsilon_{i-1}}$, $i \ge 1$, satisfies in $(0,T-\varepsilon_0)\times\mathbb{R}^n$ the Cauchy problem

$$\begin{cases} L_{\varepsilon_i}[w_{\varepsilon_i}] = -(L_{\varepsilon_i} - L_{\varepsilon_{i-1}})[v_{\varepsilon_{i-1}}],\\ w_{\varepsilon_i}(0,x) = (\partial_t w_{\varepsilon_i})(0,x) = 0. \end{cases}$$

For a fixed $s \ge \left[\frac{n}{2}\right]+1$ we obtain, using the uniform boundedness of $E_{s+1}[v_\varepsilon](t)$ with respect to ε and following the approach to derive (2.5), the energy estimate

$$\begin{aligned} E_s[w_{\varepsilon_i}](t) &\le C_i t^{l_1} \sup_{0\le\tau\le t} \tau^{-l_1} \sum_{|\alpha|\le 2} \big\|(\partial_\tau v_{\varepsilon_{i-1}}, \partial_x^\alpha v_{\varepsilon_{i-1}})\big\|_{s_0}^2(\tau)\\ &\le C_i t^{l_1} \sup_{0\le\tau\le t} \tau^{-l_1}\big(E_s[v_{\varepsilon_{i-1}}] + E_{s+1}[v_{\varepsilon_{i-1}}]\big) \le C_i C_{s+1} t^l, \end{aligned} \tag{3.4}$$

where the constant C_i depends on

$$\int_0^t \big\|b_{jk}^{\varepsilon_i}(\tau,x) - b_{jk}^{\varepsilon_{i-1}}(\tau,x)\big\|_s(\tau)\,d\tau, \quad j,k=1,\dots,n,$$
$$\int_0^t \big\|h^{\varepsilon_i}(\tau,x) - h^{\varepsilon_{i-1}}(\tau,x)\big\|_s(\tau)\,d\tau$$

for $h = b_j$, $j=1,\dots,n$, b_0 and b.

Thus we can choose ε_i in such a way that $C_i \le 2^{-i}$ because of (B-III). Consequently, $\{v_{\varepsilon_i}\}_{i\ge 0}$ is a Cauchy sequence in $\bigcap_{j=0}^1 C^j\big([0,T];H^{s+1-j}(\mathbb{R}^n)\big)$. The limit element v is the uniquely determined solution of (LP) belonging to $\bigcap_{j=0}^1 C^j\big([0,T];H^{s_0+1-j}(\mathbb{R}^n)\big)$. Repeating this approach for all s with suitable sequences $\{\varepsilon_{i,s}\}$ gives immediately the result $v \in C^1\big([0,T];H^\infty(\mathbb{R}^n)\big)$. ■

4. H^∞-well posedness for (CP3)**.** The results from the previous sections allow us now to study the quasi-linear Cauchy problem (CP3).

Proposition 4.1. *Under the assumptions* (B-I)–(B-III) *there exists a positive constant T^* such that* (CP3) *with $w = v$ has a uniquely determined solution v belonging to $C^2\big([0,T^*];H^\infty(\mathbb{R}^n)\big)\cap\Pi_\kappa$ and satisfying*

$$E_s[v](t) \le C_s t^{(l+1)/2}$$

for each $s \ge \left[\frac{n}{2}\right]+1$ and with a fixed $l \ge l_0$.

Proof. The proof will be divided into several steps.

1. *An iteration scheme.* In order to prove the time local existence of solution to (CP3), we consider for $i = 1, 2, \ldots$ the following iteration scheme in $(0, T) \times \mathbb{R}^n$:

$$
(\mathrm{CP3})_i \quad \begin{cases} Q_{i-1}[v_i] = \partial_t^2 v_i - \displaystyle\sum_{j,k=1}^{n} b_{jk}(t, x; \Lambda v_{i-1}) \partial_{x_j x_k}^2 v_i + \sum_{j=1}^{n} b_j(t, x; \Lambda v_{i-1}) \partial_{x_j} v_i \\ \qquad\qquad + b_0(t, x; \Lambda v_{i-1}) \partial_t v_i + b(t, x; \Lambda v_{i-1}) v_i = f_l(t, x; \Lambda v_{i-1}), \\ v_i(0, x) = (\partial_t v_i)(0, x) = 0, \end{cases} \tag{4.1}
$$

where $v_0 \equiv 0$ and $T \leq T_1$ with T_1 taken from Proposition 1.1.

Lemma 4.1. *For a fixed integer $s \geq [\frac{n}{2}] + 1$, there exist positive constants C_s and T_s such that the solution v_i of $(\mathrm{CP3})_i$ satisfies for $t \in [0, T_s]$ the energy estimate*

$$
E_s[v_i] \leq C_s t^{(l+1)/2}, \tag{4.2}
$$

where C_s is independent of i.

Proof. Let us start our iteration scheme with $v_0 \equiv 0$. Then the application of Proposition 3.1 gives a solution $v_1 \in C^1\big([0, T]; H^\infty(\mathbb{R}^n)\big)$, where $T \leq T_1$. Here we used Remark 1.2. Due to (3.2) this solution satisfies the energy estimate

$$
E_s[v_1] \leq C_s t^{(l+1)/2}.
$$

Together with Remark 1.2 again we see that $v_1 \in C^2\big([0, T]; H^\infty(\mathbb{R}^n)\big)$. A sufficiently small T_s gives $E_s[v_1] \leq D_s$ for all $t \in [0, T_s]$ and $v_1 \in \Pi_\kappa$. Proposition 2.2 yields immediately (eventually with a larger C_s) the energy estimate $E_s[v_1] \leq C_s t^{(l+1)/2}$. Then Proposition 2.2 and Remark 2.1 imply (eventually with a smaller T_s) the statement of Lemma 4.1, especially (4.2), for v_2. Now we are able to apply Proposition 2.2 step by step, where the constants C_s and T_s are unchanged. This brings the statement of Lemma 4.1, especially (4.2), for v_i. Hence we have completed the proof of the lemma. ∎

2. *Cauchy sequence property.* We show the Cauchy sequence property for $\{v_i\}_{i \geq 0}$ in $C\big([0, T^*]; H^{s_0+1}(\mathbb{R}^n)\big) \cap C^1\big([0, T^*]; H^{s_0}(\mathbb{R}^n)\big) \cap \Pi_\kappa$ with a fixed $s_0 \geq \left[\frac{n}{2}\right] + 1$. The difference $w_i = v_i - v_{i-1}$ solves in $(0, T) \times \mathbb{R}^n$ the linear Cauchy problem

$$
\begin{cases} Q_{i-1}[w_i] = -(Q_{i-1} - Q_{i-2})[v_{i-1}] + f(t, x; \Lambda v_{i-1}) - f(t, x; \Lambda v_{i-2}), \\ w_i(0, x) = \partial_t w_i(0, x) = 0. \end{cases} \tag{4.3}
$$

Then we will prove the next result.

Lemma 4.2. *There exists a positive constant C_{s_0} such that for $t \in [0, T_{s_0}]$ and fixed $l \geq l_0$ the differences w_{i+1} and w_i, $i = 0, 1, 2, \ldots$, satisfy*

$$
E_{s_0}[w_{i+1}](t) \leq C_{s_0} t^{(l+1)/2} E_{s_0}[w_i](t), \tag{4.4}
$$

where C_{s_0} depends on $\sum_{j=0}^{1} \|\partial_t^j v_k\|_{s_0+2-j}^2$, $k = i, i-1$.

Proof. By the mean value theorem the equation from (4.3) can be written in the form

$$Q_i[w_{i+1}] = \sum_{|\gamma|=1} g_\gamma\big(t,x;v_{i-1},v_i\big)(\Lambda w_i)^\gamma, \tag{4.5}$$

where g_γ satisfies (B-III). Then there exists a positive constant C_{s_0} such that in the same way as in the proof of Proposition 2.2 by taking account of Corollary 1.1 we obtain for a fixed $l \ge l_0$ the energy estimate

$$E_{s_0}[w_{i+1}](t) \le C_{s_0} t^{(l+1)/2} \sup_{0\le\tau\le t} \tau^{-l/2} E_{s_0}[w_i](\tau),$$

where C_{s_0} depends on $\exp\Big(\int_0^t A_{s_0}(\tau)\,d\tau\Big)$, $\int_0^t \|h\|_{(s_0+1)}(\tau)\,d\tau$ for $h = g_\gamma$, $|\gamma| = 1$, and $\sum\limits_{j=0}^{1} \|\partial_t^j v_k\|^2_{s_0+2-j}$, $k = i, i-1$. Thus we obtain the desired result. ∎

The relation (4.2) implies that $E_{s_0}[v_i](t)$ is uniformly bounded for $t \in [0, T_{s_0}]$. Therefore the constant C_{s_0} from (4.4) can be taken independently of i. Repeatedly using (4.4) we obtain

$$E_{s_0}[w_{i+1}](t) \le \big(C_{s_0} t^{(l+1)/2}\big)^{i+1} E_{s_0}[w_0](t).$$

Consequently,

$$E_{s_0}[w_{i+1}](t) \le 2^{-i} E_{s_0}[w_0](t) \tag{4.6}$$

for all $t \in [0, T^*]$, where we take $T^*(\le T_{s_0})$ so small that $C_{s_0}(T^*)^{(l+1)/2} \le 2^{-1}$. This gives the Cauchy sequence property for $\{v_i\}_{i\ge0}$ in $\bigcap_{j=0}^1 C^j\big([0,T^*]; H^{s_0+1-j}(\mathbb{R}^n)\big)$. The limit element v represents the uniquely determined solution of (CP3) with $w = v$.

3. *A continuation argument.* Finally, we will show that the solution v of (CP3) with $w = v$ belongs to $C^2\big([0,T^*]; H^\infty(\mathbb{R}^n)\big)$ with T^* taken as above.

Following the above reasoning we show that there exists a constant $T^*_s \le T^*$ such that (CP3) with $w = v$ has the solution $v \in \bigcap_{j=0}^1 C^j\big([0,T^*_s]; H^{s+1-j}(R^n)\big)$ for any fixed $s > s_0$. By the well-known continuation theorem for solutions of Cauchy problems for quasi-linear strictly hyperbolic equations (see [10]) it is easily seen that the solution v persists in $[0,T^*]$. Here we use that the life span of solutions depends only on a lower order energy. Thus we have $v \in \bigcap_{j=0}^1 C^j\big([0,T^*]; H^{s+1-j}(R^n)\big)$. In fact, for any fixed t_0 with $0 < t_0 < T^*_s$ our problem is strictly hyperbolic on $[t_0, T^*_s]$ and $v \in \Pi_\kappa$ in $[0, T^*_s]$. Therefore $v = v(t,x)$ is the desired solution of (CP3) with $w = v$. From the equation we conclude that $v \in C^2\big([0,T^*]; H^\infty(\mathbb{R}^n)\big)$. This completes the proof of Proposition 4.1. ∎

5. Proof of the main results

5.1. *Proof of Theorem* 0.1. The existence of a solution $u \in C^1\big([0,T^*]; H^\infty(\mathbb{R}^n)\big)$ of (CP1) follows from Proposition 4.1 and Proposition 1.1 by putting $u = u(t,x) = U^{(l)}(t,x) + v(t,x)$. To finish the proof it remains to derive a uniqueness result.

PROPOSITION 5.1. *Under the assumptions* (A-I) *to* (A-III) *there exists an integer* s_0 *such that* (CP1) *has at most one solution*

$$u \in \bigcap_{j=0}^{1} C^j\big([0,T^*]; H^{s_0+1-j}(\mathbb{R}^n)\big).$$

PROOF. Let u_1, u_2 be solutions of (CP1) belonging to $\bigcap_{j=0}^1 C^j\big([0,T^*]; H^{s_0+1-j}(\mathbb{R}^n)\big)$. The difference $w = u_1 - u_2$ solves in $(0,T^*) \times \mathbb{R}^n$ the Cauchy problem

$$\begin{cases} P_{u_1}[w] = \displaystyle\sum_{j,k=1}^n \big(a_{jk}(t,x;\Lambda u_1) - a_{jk}(t,x;\Lambda u_2)\big)\partial^2_{x_j x_k} u_2 \\ \qquad\qquad\qquad\qquad - \big(\rho(t,x;\Lambda u_1) - \rho(t,x;\Lambda u_2)\big), \\ w(0,x) = w_t(0,x) = 0. \end{cases} \tag{5.1}$$

By the mean value theorem the right-hand side of (5.1) is represented in the form

$$\sum_{i=1}^n f_{i+1}(t,x;u_1,u_2) w_{x_i} + f_1(t,x;u_1,u_2) w_t + f_0(t,x;u_1,u_2) w,$$

where each f_i $(i = 0,1,\ldots,n+1)$ satisfies for every integer $s \geq 0$ the condition

$$\|f_i(t)\|_{(s)} \in L^1(0,T)$$

because of (A-III). The equation from (5.1) can be rewritten in $(0,T^*) \times \mathbb{R}^n$ in the form (see [8])

$$\begin{aligned} w_{tt} - f_1(t,x;u_1,u_2) w_t - f_0(t,x;u_1,u_2) w \\ = \sum_{j,k=1}^n a_{jk}(t,x;\Lambda u_1) w_{x_j x_k} + \sum_{j=1}^n f_{i+1}(t,x;u_1,u_2) w_{x_i}. \end{aligned} \tag{5.2}$$

Repeating the approach from the second step of the proof of Proposition 1.1 we have for any integer $s > 0$, taking eventually T^* smaller if necessary,

$$\|w_t\|_s^2(t) \leq C_s t^{1/2} \sup_t \|w\|_{s+2}^2(t), \tag{5.3}$$

where C_s depends on $\int_0^t \|h\|_{(s)}(\tau)\, d\tau$ for $h = a_{jk}$, $a_{jk}^{(\gamma)}$, $\rho^{(\gamma)}$, $|\gamma| = 1$. Then (5.3) implies

$$\|\partial_t^j w\|_s^2(t) \leq C_s t^{3/2-j}, \quad j = 0,1. \tag{5.4}$$

Applying (5.3) and (5.4) and repeating this procedure, we obtain for an integer $l \geq 0$ the estimate

$$\big\|\partial_t^j w\big\|_{s-2l}^2(t) \leq C_{s,l} t^{3l/2-j}, \quad j = 0,1.$$

Take l_0 and s_0 so that $\frac{3}{2} l_0$ $(> C_{1,n}) > C_1$ and $s_0 \geq \left[\frac{n}{2}\right] + 1 + 2l_0$. Then similarly to the proof of (2.13) with $f_{l_0} \equiv 0$ for the above l_0 we have after application of Lemma 1.4 the energy estimate

$$E_{s_0-2l_0}[w](t) \leq 0.$$

Hence we conclude that $w \equiv 0$ in $\bigcap_{j=0}^1 C^j\big([0,T^*]; H^{s_0+1-j}(\mathbb{R}^n)\big)$. ∎

5.2. *Proof of Theorem* 0.2. To complete the proof we have only to show the existence of a domain of dependence. This leads together with Theorem 0.1 to C^∞-well posedness for our starting problem.

PROPOSITION 5.2. *The solution of problem* (CP1) *possesses the domain of dependence property.*

PROOF. For solutions u_1 and u_2 satisfying (CP1) with initial data (φ_1, ψ_1) and (φ_2, ψ_2) respectively, the proof will be carried out by showing that the difference $u_1 - u_2$ vanishes in some set $K(t_{i_0}, x_0)$ if $\varphi_1 \equiv \varphi_2$ and $\psi_1 \equiv \psi_2$ on $K(t_{i_0}, x_0) \cap \{t = 0\} =: D$. The difference $z := u_1 - u_2$ satisfies (5.1) and is, consequently, in $(0, T^*) \times R^n$ a solution of the linear Cauchy problem

$$\begin{aligned}&\partial_t^2 z - \sum_{j,k=1}^n a_{jk}(t,x)\partial^2_{x_j x_k} z - \sum_{j=1}^n f_{j+1}(t,x)\partial_{x_j} z - f_1(t,x)z_t - f_0(t,x)z = 0,\\ &z(0,x) = \varphi_1(x) - \varphi_2(x),\ z_t(0,x) = \psi_1(x) - \psi_2(x).\end{aligned} \tag{5.5}$$

Thus $z = z_t = 0$ on D. Take i_0 so that $2^{-i_0} = T^*$ (eventually we have to decrease T^* a bit). Let us define with $f(t,x,\xi;\eta) := \sum_{j,k=1}^n a_{jk}(t,x,\xi)\eta_j\eta_k$ for $t \in (0, T^*]$ the function

$$\lambda_f = \lambda_f(t) = \sup_{(x,\xi,\eta)\in\mathbb{R}^n\times K\times S_{n-1}} \big\{|\tau| : \tau^2 - f(t,x,\xi;\eta) = 0,\ |\eta| = 1\big\},$$

and for $i \geq i_0$ the functions

$$\lambda_{f,i} = \lambda_{f,i}(t) = \sup_{t\in[2^{-i-1},2^{-i}]} \lambda_f(t).$$

For $i \geq i_0$ we put

$$H_{i_0}(t_{i_0}, x_0) = \Big\{(t,x) \in [2^{-i_0-1}, 2^{-i_0}] \times \mathbb{R}^n : |x - x_0| < C_f\sqrt{2}^{(i_0+1)\nu}|t_{i_0} - t|\Big\},$$

$$\begin{aligned}H_{i_0+1}(t_{i_0}, x_0) = \Big\{(t,x) \in [2^{-i_0-2}, 2^{-i_0-1}] \times \mathbb{R}^n : |x - y| < C_f\sqrt{2}^{(i_0+2)\nu}|2^{-(i_0+1)} - t|,\\ y \in H_{i_0}(t_{i_0}, x_0) \cap \{t = 2^{-(i_0+1)}\}\Big\},\end{aligned}$$

and in general for $i \geq i_0 + 1$ we set

$$\begin{aligned}H_i(t_{i_0}, x_0) = \Big\{(t,x) \in [2^{-i-1}, 2^{-i}] \times \mathbb{R}^n : |x - y| < C_f\sqrt{2}^{(i+1)\nu}|2^{-i} - t|,\\ y \in H_{i-1}(t_{i_0}, x_0) \cap \{t = 2^{-i}\}\Big\}.\end{aligned}$$

Here we used $\sqrt{f(t,x,\xi;\eta)} \leq C_f t^{-\nu/2}$ for $|\eta| = 1$ which comes from (A-III)i). Finally, let us define $K(t_{i_0}, x_0) = \bigcup_{i\geq i_0} H_i(t_{i_0}, x_0)$. Then, taking account of the diameter of $H_i(t_{i_0}, x_0) \cap \{t = 2^{-1-i}\}$, equal to

$$2C_f \sum_{l=i_0}^{i} \frac{2^{(l+1)\nu/2}}{2^{l+1}},$$

we deduce that the diameter d of D satisfies

$$d = 2C_f \sum_{l=i_0}^{\infty} \frac{2^{(l+1)\nu/2}}{2^{l+1}} < +\infty.$$

We take a cut-off function $\chi = \chi(x)$ which is identical to 1 in a neighbourhood of D and supported in a ball $D^* \supset\supset D$. Then we consider the following ε-shifted problem in $(0, T^* - \varepsilon_i) \times R^n$ corresponding to (5.5):

$$(\mathrm{DD})_{\varepsilon_i} \qquad \begin{cases} \partial_t^2 u_{\varepsilon_i} - \displaystyle\sum_{j,k=1}^{n} a_{jk}^{\varepsilon_i}(t,x)\partial_{x_j x_k}^2 u_{\varepsilon_i} = \sum_{j=1}^{n} f_{j+1}^{\varepsilon_i}(t,x)\partial_{x_j} u_{\varepsilon_i} \\ \qquad\qquad\qquad\qquad + f_1^{\varepsilon_i}(t,x)\partial_t u_{\varepsilon_i} + f_0^{\varepsilon_i}(t,x) u_{\varepsilon_i}, \\ u_{\varepsilon_i}(0,x) = (1-\chi)\varphi(x),\ (\partial_t u_{\varepsilon_i})(0,x) = (1-\chi)\psi(x), \end{cases} \tag{5.6}$$

where $\varepsilon_i = 1/2^i$ for $i \geq i_0 + 1$. Note that $C_f \sqrt{2}^{i\nu} \leq \lambda_{f^{\varepsilon_i},i}^{-1}$. The Cauchy problem $(\mathrm{DD})_{\varepsilon_i}$ satisfies a domain of dependence property in the sense that

$$u_{\varepsilon_i} \equiv 0 \quad \text{in } K(t_{i_0}, x_0).$$

In fact, for a forward cone $K(\varepsilon_i)$ on D which is defined by

$$K(\varepsilon_i) = \Big\{(t,x) \in \big[0, \lambda_{f^{\varepsilon_i},i} \cdot \tfrac{d}{2}\big] \times \mathbb{R}^n : |x - x_0| < \lambda_{f^{\varepsilon_i},i}^{-1} \big|t - \lambda_{f^{\varepsilon_i},i} \cdot \tfrac{d}{2}\big|\Big\},$$

we have $u_{\varepsilon_i} \equiv 0$ in $K(\varepsilon_i) \cup K(t_{i_0}, x_0)$.

Let us now consider (5.5), (5.6) with data $\phi(1-\chi)\varphi$ and $\phi(1-\chi)\psi$, where ϕ is some cut-off function which yields Sobolev behaviour of data. Due to our existence and uniqueness results (Propositions 4.1 and 5.1) we have unique solutions u_{ε_i}, $z \in C^1\big([0,T^*]; H^\infty(R^n)\big)$. Then the differences $w_{\varepsilon_i} := z - u_{\varepsilon_i}$ satisfy in $(0, T^* - \varepsilon_i) \times R^n$ the Cauchy problem

$$\begin{cases} \partial_t^2 w_{\varepsilon_i} - \displaystyle\sum_{j,k=1}^{n} a_{jk}^{\varepsilon_i}(t,x)\partial_{x_j x_k}^2 w_{\varepsilon_i} = \sum_{j=1}^{n} f_{j+1}^{\varepsilon_i}(t,x)\partial_{x_j} w_{\varepsilon_i} \\ \qquad\qquad + f_1^{\varepsilon_i}(t,x)\partial_t w_{\varepsilon_i} + f_0^{\varepsilon_i}(t,x) w_{\varepsilon_i} + F_{\varepsilon_i}(t,x,z), \\ w_{\varepsilon_i}(0,x) = (\partial_t w_{\varepsilon_i})(0,x) = 0. \end{cases}$$

Here F_{ε_i} depends on $a_{jk} - a_{jk}^{\varepsilon_i}$ and $f_j - f_j^{\varepsilon_i}$. Using (A-II)ii) and (A-III)i) we deduce that

$$\int_0^t \big\|a_{jk}^{\varepsilon_i}(t,x) - a_{jk}(t,x)\big\|_s(\tau)\,d\tau + \sum_{j=0}^{n+1} \int_0^t \big\|f_j^{\varepsilon_i}(t,x) - f_j(t,x)\big\|_s(\tau)\,d\tau \to 0,$$

if $i \to \infty$ for all $s \in N$. Thus we can follow the approach of Section 1 and 4 and obtain $E_{L_0}(z - u_{\varepsilon_i}) \to 0$ as $i \to \infty$. By using $u_{\varepsilon_i} \equiv 0$ in $K(t_{i_0}, x_0)$ we get $z \equiv 0$, there. Hence, $u_1 \equiv u_2$ in $K(t_{i_0}, x_0)$ if $u_1 = u_2$, $\partial_t u_1 = \partial_t u_2$ on D. This completes the proof. ■

Remark 5.1. Our assumptions to coefficients of P_u are weaker than those from [1] in the case of C^∞-well posedness in the following sense: the assumptions on the coefficients in [1] are:

$$\begin{aligned} |D_x^\alpha a_{jk}(t,x)| &\leq C_\alpha t^{-p}, & 0 \leq p < 1,\ \alpha \geq 0, \\ |D_x^\alpha \partial_t a_{jk}(t,x)| &\leq C_\alpha^1 t^{-1-\gamma|\alpha|}, & 0 \leq \gamma < 1,\ \alpha \geq 0. \end{aligned}$$

If we compare these with (A-III), then we need only an assumption for the asymptotic behaviour near $t = 0$ of $\partial_t a_{jk}$, but not of $D_x^\alpha \partial_t a_{jk}$.

6. Concluding remarks. The goal of this paper is to study second order quasi-linear model Cauchy problems in the case of coefficients non-Lipschitz in t and smooth in x. We show that a straightforward approach leads to C^∞-well posedness results. With this respect our strategy, especially the reduction scheme, is applicable. This scheme is too rough if we are interested in Sobolev solutions, because the loss of derivatives is in general not precise. One should generalize some of our ideas in several directions. One interesting point is to study linear Cauchy problems of higher order with non-Lipschitz coefficients. The authors expect that models can be studied, where time derivative of coefficients possesses the singular behaviour $O(\frac{1}{t})$. It would be interesting to include the optimal singular behaviour $O(\frac{1}{t}\log(\frac{1}{t}))$ as in [3], [7] or at least $O(\frac{1}{t}(\log(\frac{1}{t}))^\gamma)$ with $\gamma \in (0,1)$. But these problems seem to be complicate. With the singular behaviour $O(\frac{1}{t})$ one can try to study general quasi-linear models of higher order. Here one can follow ideas are developed in [6]. Finally, one can deal with the Cauchy problem for classes of fully nonlinear hyperbolic equations by using our method (cf. [4]).

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