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ON THE ANOMALOUS SINGULARITIES OF THE SOLUTIONS TO SOME CLASSES OF WEAKLY HYPERBOLIC SEMILINEAR SYSTEMS. EXAMPLES

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Abstract. This paper deals with the newly observed singularities of the solutions of some specific examples of weakly hyperbolic semilinear systems in $\mathbf{R}^2$. Two, respectively three, characteristics are supposed to be mutually tangential at the origin only and the initial data are continuous only. The exact strength of the new-born singularities is investigated too.

1. Introduction. When studying the propagation of singularities of the solutions to semilinear hyperbolic and non-strictly hyperbolic equations and systems, interesting new effects in comparison with the linear case can appear. The interaction of the singularities propagating along several characteristics crossing at some point (surface) could give rise of new singularities propagating along the outgoing characteristics starting from that point (surface). In many cases the new-born singularities are weaker than the initial ones. We shall mention only the papers of Beals [1], Bony [2], [3], Chemin [4], Hörmander [6], Melrose-Ritter [9] (cf. also Gramchev [5] for appearance of Gevrey ultradistributional singularities). The propagation of jump type discontinuities for hyperbolic and weakly hyperbolic systems in $\mathbf{R}^2$ was considered by Rauch-Reed [11], [12], John [8], Micheli [10],

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Iordanov [7] and others. We specify here the main results from [10], [7] in the case of continuous Cauchy data and we prove that the strength of the newly observed singularity propagating along a transversal outgoing characteristic is rather different from the strength of that singularity in the case of sufficiently smooth Cauchy data. We are dealing with jump discontinuities in all our investigations. Without loss of generality we assume that 2 or 3 characteristics are tangential to each other of finite order at the origin.

2. Formulation and proof of the main results

2.1. Consider the weakly hyperbolic system in the plane $\mathbf{R}^2$:

$$\begin{cases} Xu = (\partial_t + pt^{p-1}\partial_x)u = 0 \\ \partial_t v = u \\ Dz = (\partial_t + \partial_x)z = uv, \end{cases} \tag{1}$$

where $p > 0$ is an even integer and the initial data $u_0(x)$, $v_0(x) = z_0(x) = 0$ are prescribed for $t = -T < 0$.

Assume first that $u_0(x) = c_1(x - x_0)^{k+1}$, $k \geq 1$, k — integer, $x \leq x_0$; $u_0(x) = c_2(x - x_0)^{k+1}$, $x > x_0$, $c_1 c_2 > 0$, $c_1 \neq c_2$. Obviously, $u_0 \in C^k(\mathbf{R}^1)$ and $\partial_x^{k+1} u$ has a jump discontinuity at x_0. We shall say then that u_0 has a k-order finite jump. The characteristics of the vector fields X and $\dfrac{\partial}{\partial t}$ are tangential each to other at 0 and D is transversal with respect to them. L. Micheli has proved in [10] that a new-born singularity of the component z appears at the origin and propagates along the characteristic $x - t = 0$, $t > 0$. Moreover, $z \in C^{(2k+3)p}$ near that outgoing characteristic and it has jump type discontinuities of order $(2k+3)p$, i.e. some derivative of z of order $(2k+3)p+1$ in transversal direction to D possesses jump along the characteristic mentioned above. Certainly, $z \in C^\infty$ in a neighbourhood of $x - t = 0$, $t < 0$. We shall see that the picture is quite different if $k = 0$.

Denote by C_1, C_2, C_3 the characteristics of X, ∂_t, D passing through the origin (see Fig. 1). The characteristics of the same vector fields passing through some point $(\bar{x}, \bar{t})$ are given respectively by: $x = t^p + \bar{x} - \bar{t}^p$, $x = \bar{x}$, $x = t + \bar{x} - \bar{t}$. Put $x_0 = T^p$. Then

$$u(\bar{x}, \bar{t}) \equiv u(T^p + \bar{x} - \bar{t}^p, -T) = u_0(x_0 + \bar{x} - \bar{t}^p). \tag{2}$$

Consider now the following cases of Cauchy data:

a) $u_0(x) = \begin{cases} x - x_0, & x \leq x_0 \\ 2(x - x_0), & x > x_0, \end{cases}$ b) $u_0(x) = \begin{cases} x - x_0, & x \leq x_0 \\ 0, & x > x_0, \end{cases}$

c) $u_0(x) = \begin{cases} 0, & x \leq x_0 \\ x - x_0, & x > x_0, \end{cases}$ d) $u_0(x) = \begin{cases} 1, & x \leq x_0 \\ 1 + x - x_0, & x > x_0, \end{cases}$

e) $u_0(x) = \begin{cases} 1 + x - x_0, & x \leq x_0 \\ 1, & x > x_0. \end{cases}$

There are no difficulties to give explicit formulas for the component u of (1) in the cases a)–e) by applying (2). For example, in case a) $u = x - t^p$ for $x \leq t^p$, $u = 2(x - t^p)$, $x \geq t^p$; in case b) $u = x - t^p$ for $x \leq t^p$, $u = 0$ if $x \geq t^p$; in case c) $u = 0$ for $x \leq t^p$, $u = x - t^p$ if $x \geq t^p$ etc.

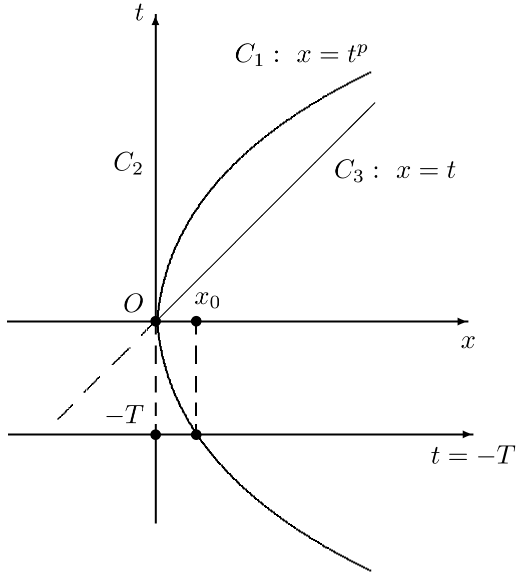

Fig. 1

Having in mind that $v(x,t) = \int_{-T}^{t} u(x,\tau)\,d\tau$ we introduce the functions:

$$A(x,t) = \int_{-T}^{t} (x - \tau^p)\,d\tau,$$

$$B(x,t) = \int_{-x^{1/p}}^{t} u(x,\tau)\,d\tau,\ x > 0, \tag{3}$$

$$C(x) = \int_{-x^{1/p}}^{x^{1/p}} (x - \tau^p)\,d\tau = \frac{2p}{p+1}\,x^{1+1/p},\ x > 0.$$

It is rather easy to compute the values of v in the following three subdomains of $\mathbf{R}^2$: $\{x < 0\} \cup \{0 < x < t^p, t < 0\}$, $\{t^p < x\}$ and the curvilinear angle between the characteristics C_1^+, C_2^+, $C_1^+ = C_1 \cap \{x > 0\}$, $C_2^+ = C_2 \cap \{x > 0\}$ (see Fig. 1). We shall denote these values by v_1, v_2, v_3. Then in case a) $v_1 = A$, $v_2 = A + B$, $v_3 = A + C$; in case b) $v_1 = A$, $v_2 = A - B$, $v_3 = A - C(x)$, in case c) $v_1 = 0$, $v_2 = B$, $v_3 = C(x)$ etc.

Suppose that $(x,t) \in C_3$, $x > 0$ and consider the D-characteristics $C_3^{\pm}$ passing through the points (x^+, t^+), (x^-, t^-) located near (x,t) above and, respectively, under the characteristic C_3, i.e. C_3^+: $x - t = \sigma^+ = x^+ - t^+ < 0$, C_3^-: $x - t = \sigma^- = x^- - t^- > 0$. The symbols a_1, b_1 stand for the ordinates of the crossing points of $C_3^{\pm}$ with the ordinate axes while a_2, b_2 stand for the ordinates of the crossing points of $C_3^{\pm}$ with C_1. Then $a_1 + x - t = 0$, $b_1 + x - t = 0$ and $a_2(\sigma)$, $\sigma < 0$, $b_2(\sigma)$, $\sigma > 0$ satisfy in a neighbourhood of $\sigma = 0$ the equation

$$c^p(\sigma) = c(\sigma) + \sigma, \quad c(0) = 0. \tag{4}$$

According to the implicit function theorem $a_2, b_2 \in C^\infty$, $c(\sigma) = -\sigma + \sigma^p + O(\sigma^{p+1})$, $\sigma \to 0$, $\sigma = \sigma^{\pm}$.

Our next step is to compute in a small neighbourhood of (x^+, t^+) (or (x^-, t^-)) the corresponding values z^+ (z^-) of $z(x,t)$ and to find out the limit of z^+ (z^-) and its derivatives up to some order in some direction transversal to C_3 for $\sigma \to 0^+$ ($\sigma \to 0^-$). Thus, we have in case a):

$$z^+(x,t) = \int_{-T}^{a_1} Au\,d\tau + \int_{a_1}^{a_2} (A+C)u\,d\tau + \int_{a_2}^{t} (A+B)u\,d\tau,$$
$$z^-(x,t) = \int_{-T}^{b_1} Au\,d\tau + \int_{b_1}^{b_2} Au\,d\tau + \int_{b_2}^{t} (A+B)\,d\tau,$$

and we are integrating in both cases along the straight line $(\tau + x - t, \tau)$, i.e. $(\tau + \sigma^\pm, \tau)$.

One can guess that jump discontinuities can appear from the "asymmetric" term $\int_{a_1}^{a_2} C(\tau + x - t)u(\tau + x - t, \tau)\,d\tau$ only, i.e. from the term with underintegral function participating in z^+ but not in z^-. The standard change $\tau + x - t = s^p$ in the last integral shows that

$$\Gamma = \int_{a_1}^{a_2} Cu\,d\tau = p\int_0^{a_2} \frac{2p}{p+1}\, s^{p+1}\big[s^p - (s^p - \sigma)^p\big] s^{p-1}\,ds, \tag{5}$$

as $a_2 + \sigma = a_2^p$, $a_1 + \sigma = 0$, $\sigma < 0$.

The terms having minimal powers in (s, σ) and participating in the polynomial under integral (5) are s^{3p} and $s^{2p}\sigma^p$. Thus, the leading term in (5) near $\sigma^+ = x - t \to 0^-$ is const. $\Big(\dfrac{a_2^{3p+1}}{3p+1} - \dfrac{a_2^{2p+1}}{2p+1}\,\sigma_+^p\Big)$. The same change shows that $\int_{a_2}^{t} Bu\,d\tau \in C^\infty(x > 0)$ etc. In this way we conclude that each derivative of the asymmetric term Γ of order not exceeding $3p$ tends to 0 for $\sigma^+ \to 0$. On the other hand, the leading term in Γ is given by

$$\frac{p(\sigma^+)^{3p+1}}{(2p+1)(3p+1)}\big(1 + O(\sigma^+)\big), \quad \sigma^+ \to 0.$$

Consequently, $z \in C^{3p}$ in a neighbourhood of C_3 contained inside the parabola C_1, while some transversal with respect to C_3 derivative of z of order $3p+1$ has a jump discontinuity at C_3. The same result holds in case b).

In case c) $z^+ = \int_{a_2}^{t} Bu\,d\tau$, $z^- = \int_{b_2}^{t} Bu\,d\tau$ and therefore $z \in C^\infty$ in a neighbourhood of C_3 located inside C_1.

In case d) jump discontinuities can appear by the following term only (leading term):

$$\int_{a_1}^{a_2} \frac{2p}{p+1}(\tau + x - t)^{(p+1)/p}\,d\tau = \frac{2p^2}{(p+1)(2p+1)}\,a_2^{2p+1}.$$

Thus, $z \in C^{2p}$ near C_3 and the transversal derivatives of z: $\partial_x^{2p+1} z$, ∂_t^{2p+1} have finite jumps along C_3 inside C_1.

In case e) the component z has finite jump discontinuity of order $2p$. So we come to the following proposition.

Proposition 1. *Consider the weakly hyperbolic semilinear system* (1) *with Cauchy data u_0 having finite jump of order $k = 0$ at some point x_0 and $v_0 \equiv z_0 \equiv 0$. Then in general the component z does not have jump of exact order* $(2k+3)p$ *along the characteristic C_3 located inside the parabola C_1.*

The examples a), b) correspond to the central results in Micheli [10], Iordanov [7], claiming that the newly created singularity of the component $z(x,t)$ has a finite jump discontinuity of order $(2k+3)p$, $k = 0$ along the part of C_3 located inside C_1. On the other hand, examples c), d), e) contradict the main results in the same papers as either $z \in C^\infty$ near C_3 or z possesses a finite jump discontinuity along C_3 and inside C_1 of order $(k+2)p < (2k+3)p$, $k = 0$. Certainly, this is a new effect.

2.2. Consider now the weakly hyperbolic semilinear system

$$\begin{cases} Xu = (\partial_t + pt^{p-1}\partial_x)u = 0 \\ \partial_t v = u \\ Yw = (\partial_t - pt^{p-1}\partial_x)w = u + v \\ Dz = (\partial_t + \partial_x)z = \alpha uv + \beta uw + \gamma vw, \\ \qquad \alpha, \beta, \gamma = \text{const.},\ p - \text{even integer} \geq 2 \end{cases} \tag{6}$$

with Cauchy data $u|_{t=-T} = u_0(x)$, $T > 0$, $v_0 = w_0 = z_0 = 0$. The constants α, β, γ are assumed to be different from 0.

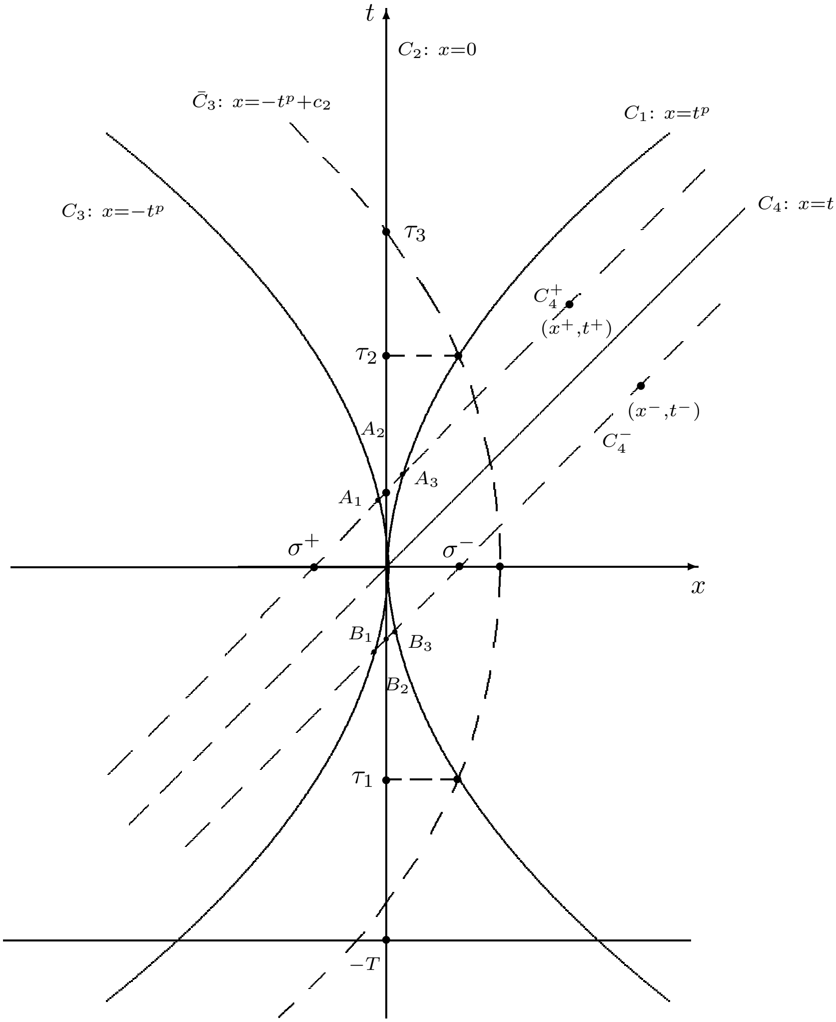

Fig. 2

The characteristics of the vector fields X, ∂_t, Y, D passing through the point $(\bar{x},\bar{t})$ are respectively: $x=t^p+c_1$, $c_1=\bar{x}-\bar{t}^p$, $x=\bar{x}$, $x=-t^p+c_2$, $c_2=\bar{x}+\bar{t}^p$, $x-t=\bar{x}-\bar{t}$. Put $C_1 : x=t^p$, $C_2 : x=0$, $C_3 : x=-t^p$, $\bar{C}_3 : x=-t^p+c_2$, $C_4 : x=t$. Obviously, C_1, C_2, C_3 are tangential each to other characteristics at 0 and C_4 is transversal to them at 0. The initial function u_0 is defined by d) from (1) (see Fig. 2). Then $u=u_1=1$ outside the parabola C_1 and $u=u_2=1+x-t^p$ inside it. The values of v in the three subdomains introduced above are:

$$v_1=t+T,\quad v_2=v_1+xt-\frac{t^{p+1}}{p+1}+\frac{p}{p+1}\,x^{1+1/p},\quad x>0,$$
$$v_3=v_1+\frac{2p}{p+1}\,x^{1+1/p},\quad x>0.$$

Technical remark. When estimating the strength of the new-born singularities along $C_3\cap\{x>0\}$ we need some additional calculations. Thus consider the function $\theta(s)=s/(s^p-\sigma)$, $s\geq 0$, where $\sigma<0$, $|\sigma|\ll 1$, $a_3=-\sigma+\sigma^p+O(\sigma^{p+1})$, $a_3^p=a_3+\sigma$. Then $\theta(0)=0$ and θ is monotonically increasing on the interval $[0,s_0)$ and is monotonically decreasing for $s\geq s_0$, where $s_0=\big(\sigma/(1-p)\big)^{1/p}>0$. On the other hand, $0\leq a_3\leq s_0(\sigma)$. Therefore, $0\leq s\leq a_3\Rightarrow 0\leq\theta(s)\leq\theta(a_3)=1$. There are no difficulties to see that $\int_0^{a_3}(s^p-\sigma)^{p+1}s^{p-1}\,ds=-\sigma^{p+1}a_3^p/p+O(\sigma^{3p})$, $\sigma\to 0$, $\sigma<0$.

Let $c_2=x+t^p$. Then $w(x,t)=\int_{-T}^t(u+v)(c_2-\tau^p,\tau)\,d\tau$. The crossing points of $\bar{C}_3$ with C_1 and C_2^+ have the following ordinates: $\tau_1=-(c_2/2)^{1/p}$, $\tau_2=-\tau_1$, $\tau_3=c_2^{1/p}$. Define now the characteristics $C_4^\pm : x-t=\sigma^\pm$ (see Fig. 2). The crossing points of C_4^+ with C_3, C_2, C_1 are denoted by A_1, A_2, A_3 and their ordinates are a_1, a_2, a_3, respectively. The crossing points B_1, B_2, B_3 and their ordinates b_1, b_2, b_3 are introduced in a similar way as above (see Fig. 2). The smooth functions $a_3(\sigma)>0$, $b_3(\sigma)<0$ satisfy the equation $\alpha+\sigma=\alpha^p$, $\sigma=x-t$, $\alpha(0)=0$, the functions $a_1(\sigma)>0$, $b_1(\sigma)<0$ satisfy the equation $\beta(\sigma)+\sigma=-\beta^p(\sigma)$, $\beta(0)=0$ and a_2, b_2 are the solutions of $\gamma(\sigma)+\sigma=0$. The values of w under the parabolas C_1, C_3 are denoted by $w_1(x,t)$, the values of w inside C_1 are denoted by $w_2(x,t)$, the values of w in the curvilinear angle between C_2^+, C_1^+ are denoted by $w_3(x,t)$, $C_1^+=C_1\cap\{x>0\}$. The definition of w_4 is obvious. One can easily see that

$$w_1=\int_{-T}^t(1+v_1)\,d\tau,\quad t<\tau_1,$$
$$w_2=w_1+\int_{\tau_1}^t(c_2-2\tau^p)\,d\tau+\int_{\tau_1}^t(v_2-v_1)\,d\tau,\quad \tau_1\leq t<\tau_2$$

and we are integrating along the curve $(c_2-\tau^p,\tau)$,

$$w_3=\int_{-T}^t(1+v_1)\,d\tau+\int_{\tau_1}^{\tau_2}(c_2-2\tau^p)\,d\tau+\int_{\tau_1}^{\tau_2}(v_2-v_1)\,d\tau+\int_{\tau_2}^t\frac{2p}{p+1}(c_2-\tau^p)^{1+1/p}\,d\tau,$$

$\tau_2\leq t<\tau_3$. A similar expression can be found for w_4. The previous integrals can be investigated via the standard change $\tau=c_2^{1/p}s$. Obviously,

$$z^+=\int_{-T}^{a_1}(\,\cdot\,)\,d\tau+\int_{a_1}^{a_2}(\,\cdot\,)\,d\tau+\int_{a_2}^{a_3}(\,\cdot\,)\,d\tau+\int_{a_3}^{t}(\,\cdot\,)\,d\tau,$$

where the underintegral function is $\alpha uv + \beta uw + \gamma vw$ and we are integrating along the straight line $(\tau+\sigma, \tau)$, $\sigma = x - t$. Similar decomposition holds for z^-. As in (1), case d), the term αuv leads to a jump discontinuity of sharp order $2p$ along C_4 and inside C_1. The singularities could appear because of the presence of "asymmetric terms" in the expressions for $z^{\pm}$. These terms are of the type

$$\int_{a_1}^{a_2} c_2^{1+1/p}\,d\tau,\ \int_{a_1}^{a_2} c_2^{1+2/p}\,d\tau,\ \int_{a_2}^{a_3} c_2^{1+1/p}\,d\tau,\ \ldots,\ c_2 = (\tau+\sigma+\tau^p)^{1+1/p}. \tag{7}$$

In order to find out the leading singularity, say in the third integral in (7), we make the standard change $\tau + \sigma = s^p$, having in mind that $\sigma < 0$. Applying the binomial power series we come to the conclusion that the integral under consideration possesses the following leading term: const. $\sigma^{p+1} a_3^p = o(\sigma^{2p+1})$, $\sigma \to 0^-$. This way we come to the following proposition.

PROPOSITION 2. *Consider the semilinear weakly hyperbolic system* (6) *with Cauchy data u_0 having finite jump of order $k = 0$ at some point x_0 and $v_0 = w_0 = z_0 = 0$. Then in general the component z does not have finite jump of exact order* $(2k+3)p$ *along the characteristic C_4 located inside the parabola C_1.*

In our case the jump discontinuity is of order $(k+2)p < (2k+3)p$. This is a new effect, of course.

2.3. We shall deal now with the following weakly hyperbolic system

$$\begin{cases} Xu = (\partial_t + qt^{q-1}\partial_x)u = 0 \\ \partial_t v = u \\ Yw = (\partial_t - pt^{p-1}\partial_x)w = u + v \\ Dz = (\partial_t + \partial_x)z = \alpha uv + \beta uw + \gamma vw, \\ \qquad \alpha, \beta, \gamma = \text{const.},\ p, q - \text{even},\ q > p \ge 2, \end{cases} \tag{8}$$

with Cauchy data $u|_{t=-T} = u_0(x)$, $T > 0$, $v_0 = w_0 = z_0 \equiv 0$, where u_0 is defined as in case d) for system (1).

The characteristics of the vector fields X, ∂_t, Y, D passing through the point $(\bar{x}, \bar{t})$ are given respectively by the equations $x = t^q + c_1$, $c_1 = \bar{x} - \bar{t}^q$, $x = \bar{x}$, $x = -t^p + c_2$, $c_2 = \bar{x} + \bar{t}^p$, $x - t = \bar{x} - \bar{t}$. Put $C_1 : x = t^q$, $C_2 : x = 0$, $C_3 : x = -t^p$, $\bar{C}_3 : x = -t^p + c_2$, $C_4 : x = t$. Evidently, C_1, C_2, C_3 are tangential each to other characteristics at the origin and C_4 is transversal to them at O. The main problem we are interested in is whether the new-born singularity at O of the component z will have a finite jump along C_4 and inside C_1 depending on both integers p and q. As our considerations are similar to that realized for the system (6) we shall omit some details. One can easily see that $u = u_1 = 1$ outside the parabola C_1 and $u = u_2 = 1 + x - t^q$ inside it. By v_i, $i = 1, 2, 3$, we denote the values of v in the subdomains $\{x < 0\} \cup \{0 < x < t^q,\ t < 0\}$, $\{t^q < x\}$ and the curvilinear angle between C_1^+, C_2^+. Then

$$v_1 = t + T, \quad v_2 = v_1 + xt - \frac{t^{q+1}}{q+1} + \frac{q}{q+1}\,x^{1+1/q}, \quad v_3 = v_1 + \frac{2q}{q+1}\,x^{1+1/q}.$$

Put $c_2 = x + t^p$. Obviously, $w(x,t) = \int_{-T}^{t} (u+v)(c_2 - \tau^p, \tau)\,d\tau$.

The crossing points of $\bar{C}_3$ with C_1 have ordinates $\tau_2 = -\tau_1 > 0$ which satisfy the equation $\tau^p + \tau^q = c_2 > 0$. Let $\xi = c_2^{1/p}$ and $\rho(\xi) = \tau(\xi)$, $\rho(\xi) = \xi(1+\pi)$. Therefore, $(1+\pi)^p + \xi^{q-p}(1+\pi) - 1 = 0$. Applying the implicit function theorem we conclude that in a neighbourhood of the point $\xi = 0$, $\pi = 0$ there exists a unique function $\pi(\xi) \in C^\infty$, $\pi(0) = 0$, satisfying $\rho^p(\xi) + \rho^q(\xi) = \xi$. More detailed analysis shows that $\pi(\xi) = -\frac{1}{p}\,\xi^{q-p}(1 + O(\xi))$, $\xi \to 0$. So $\tau_2 = c_2^{1/p}(1 + \pi(c_2^{1/p}))$, $\pi(0) = 0$. The ordinate of the crossing point of C_2^+ and $\bar{C}_3$ is $\tau_3 = c_2^{1/p}$. The crossing points of C_4^+ with C_3, C_2, C_1 are denoted by A_1, A_2, A_3 and their ordinates are a_1, a_2, a_3, respectively. The crossing points B_1, B_2, B_3 and their ordinates b_1, b_2, b_3 are defined similarly. Put $\sigma = x - t$. Then (a_1, b_1), (a_2, b_2), (a_3, b_3) satisfy the equations $\tau + \tau^p + \sigma = 0$, $\tau + \sigma = 0$, $\tau^q = \tau + \sigma$, respectively. An application of the inverse function theorem to $\tau^q - \tau = \sigma$ near $\tau = 0$ shows that one can find a unique C^∞ smooth inverse function $\tau = \tau(\sigma)$, $|\sigma| \ll 1$, $\tau(0) = 0$ and $\tau(\sigma) = -\sigma + \sigma^q + O(\sigma^{q+1})$, $\sigma \to 0$. So $a_3, b_3 = -\sigma_\pm + \sigma^q + O(\sigma_\pm^{q+1})$. The values w_i, $1 \le i \le 4$ of w are found in a similar way as in the case of system (6) in the intervals $t < \tau_1$, $\tau_1 \le t < \tau_2$, $\tau_2 \le t < \tau_3$, $t \ge \tau_3$. We have $w_1 \in C^\infty$. Then for $t \in (\tau_1, \tau_2]$,

$$w_2 = w_1 + \int_{\tau_1}^{t} (c_2 - \tau^p - \tau^q)\, d\tau + \int_{\tau_1}^{t} \Big[(c_2 - \tau^p)\tau - \frac{\tau^{q+1}}{q+1} + \frac{q}{q+1}(c_2 - \tau^p)^{1+1/q}\Big]\, d\tau.$$

Making the change $\tau = c_2^{1/p} s$ we get

$$\int_{\tau_1}^{t} (c_2 - \tau^p)^{1+1/q}\, d\tau = \int_{-(1+\pi(c_2^{1/p}))}^{t/c_2^{1/p}} (1 - s^p)^{1+1/q}\, ds\, c_2^{1+1/p+1/q}.$$

As we are inside the parabola C_1 we have $x > 0$ and therefore the previous integral is C^∞ smooth there. Having in mind the formulas giving the values of τ_i, $1 \le i \le 3$, we obtain

$$w_2 = w_1 + \frac{p}{p+1}\, c_2^{1+1/p}\big(1 + O(c_2^{1/q}) + O(c_2^{(q-p)/p})\big), \quad c_2 \to 0,$$
$$w_1 \in C^\infty, \quad w_1 = \int_{-T}^{t} (1 + v_1)\, d\tau.$$

Similar considerations show that

$$w_3 = w_1 + \frac{p}{p+1}\, c_2^{1+1/p}\big(1 + O(c_2^{1/q}) + O(c_2^{(q-p)/p})\big), \quad c_2 \to 0,$$
$$w_4 = w_1 + \frac{2p}{p+1}\, c_2^{1+1/p}\big(1 + O(c_2^{1/q}) + O(c_2^{(q-p)/p})\big), \quad c_2 \to 0.$$

The expressions for z^+ and z^- are the same as in the previous case (6) and we are integrating along the straight lines $(\tau + \sigma_+, \tau)$, $(\tau + \sigma_-, \tau)$, $\sigma_\pm = x - t$. The term αuv, $\alpha \ne 0$, leads to a jump discontinuity of sharp order $2q$ along the part of the characteristic C_4 located inside C_1. As was mentioned above, the jump discontinuities could appear because of the presence of several "asymmetric terms" in the expressions for $z^\pm$. Let us concentrate now on the right-hand side βuv, $\beta \ne 0$. The leading "asymmetric terms" are: $\int_{a_1}^{a_2} w_4\, d\tau$, $\int_{a_2}^{a_3} w_3\, d\tau$, $\int_{b_1}^{b_2} w_1\, d\tau$, $\int_{b_2}^{b_3} w_1\, d\tau$. To fix the ideas we shall deal with the first two integrals only. One can easily guess that the leading singularities in that situation (i.e. the terms containing the lowest powers of σ) are

$$\mathrm{I} = \int_{a_1}^{a_2} c_2^{1+1/p}\, d\tau, \quad \mathrm{II} = \int_{a_2}^{a_3} c_2^{1+1/p}\, d\tau$$

and the integration in the previous integrals is along the line $(\tau+\sigma,\tau)$.

There are no difficulties to verify the existence of a smooth function $a(\sigma)$, $\sigma \ll 1$, $0 < a(\sigma) < -\sigma$, $\sigma > 0$ with the properties $\tau \in [a(\sigma), +\infty) \Rightarrow \tau + \sigma + \tau^p = h(\tau) \geq 0$, $h(a) = 0$, $h'(\tau) > 0$. Then h^{-1} exists on the interval $[0,\infty)$ and is strictly monotone there. Following this way we conclude that the smooth function $s(\tau) = (\tau+\sigma+\tau^p)^{1/p}$ is well defined and invertible on $[a(\sigma),\infty)$. Moreover, the inverse function

$$\tau(s) = a(\sigma) + \frac{s^p}{1+pa^{p-1}(\sigma)} + O(s^{2p}), \quad s \to 0,\ s \geq 0.$$

Now we are able to make the following change in I: $s^p = \tau + \sigma + \tau^p$, $\tau \geq a(\sigma) \geq 0$. Then $\tau = a_1 \Rightarrow s_1 = 0$, $\tau = a_2 \Rightarrow s_2 = a_2 > 0$. Thus

$$\mathrm{I} = p\int_0^{a_2} \frac{s^{2p}}{1+p\tau^{p-1}(s)}\, ds = \text{const.}\, a_2^{2p+1}(1+O(a_2^{2p+2})), \quad a_2 \to 0; \quad \text{const.} \neq 0.$$

Consider now the function $s^p + s^q = h_1(s)$, $s \geq 0$. Certainly, it is strictly monotonically increasing and $s = h^{-1}(w) = w^{1/p} + O(w^{2/p})$, $w \to +0$. Let us make now in the second integral II the change $s^p + s^q = \tau + \sigma + \tau^p$, $\tau \geq a(\sigma)$ (i.e. $s = h_1^{-1}(\tau+\sigma+\tau^p)$, $\tau \geq a(\sigma)$, σ — parameter). Then $\tau = a_2 \Rightarrow s_1 = h_1^{-1}(a_2^p) = a_2(1+O(a_2))$.

Thus

$$\mathrm{II} = \int_{h_1^{-1}(a_2^p)}^{a_3} s^{2p}(1+s^{q-p})^{1+1/p}\frac{(p+qs^{q-p})}{1+p\tau^{p-1}(s)}\, ds = O(a_2^{2p+2}), \quad a_2 \to 0.$$

Hence, $\mathrm{I} = \text{const.}\, \sigma^{2p+1}(1+O(\sigma^{2p+2}))$, $\mathrm{II} = O(\sigma^{2p+2})$; const. $\neq 0$.

Proposition 3. *Consider the semilinear hyperbolic system* (8) *with Cauchy data* u_0 *having a finite jump of order* $k=0$ *at some point* x_0 *and* $v_0 = w_0 = z_0 = 0$. *Then in general the component* z *does not have a finite jump of exact order* $(2k+3)p$ *along the characteristic* C_4 *located inside the parabola* C_1. *In our case the jump discontinuity is of order* $(k+2)p$, $k=0$ *and therefore it does not depend on* q.

In other words we cannot expect interaction between the orders of tangency p, respectively q, of the characteristics C_1, C_2 and C_2, C_3 at the origin in determining the strength of the new-born jump discontinuity of the component z along the semicharacteristic $C_4 \cap \{x > 0\}$.

Short summary of results in the communication. Two, respectively three, characteristics of the semilinear systems under consideration are assumed to be mutually tangential of order $p \geq 2$ at 0 and the Cauchy data are continuous with a jump discontinuity of the first derivative at a given point x_0. Then a newly created singularity appears at 0 and propagates along a transversal outgoing characteristic. The sharp order of that jump discontinuity is $2p$ instead of the expected order $3p$ according to [7], [10].

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