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SOLVABILITY FOR SEMILINEAR PDE WITH MULTIPLE CHARACTERISTICS

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Abstract. We prove local solvability in Gevrey spaces for a class of semilinear partial differential equations. The linear part admits characteristics of multiplicity $k \geq 2$ and data are fixed in G^σ, $1 < \sigma < \frac{k}{k-1}$. The nonlinearity, containing derivatives of lower order, is assumed of class G^σ with respect to all variables.

1. Introduction. Local solvability has been widely studied for linear partial differential equations $P(x,D)u = f$. The problem consists in finding a local solution u in a neighborhood Ω of a point x_0, for any f in a given class of data. The regularity of the data is usually prescribed according to the regularity of the coefficients of $P(x,D)$. For an operator P with C^∞ coefficients one takes $f \in C_0^\infty(\Omega)$, whereas for analytic or Gevrey coefficients it is natural to fix data in the Gevrey class $G_0^\sigma(\Omega) = G^\sigma(\Omega) \cap C_0^\infty(\Omega)$, $1 < \sigma < \infty$. We recall that f belongs to $G^\sigma(\Omega)$ if the following local estimates are satisfied:

$$|D^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^\sigma.$$

Let us refer, for example, to Mascarello-Rodino [9] for a survey concerning local solvability for linear equations in the C^∞ or Gevrey frame.

During the last 10 years the attention of the scholars has been addressed to local solvability for nonlinear equations. The functional frame is given in this case by Sobolev spaces, or similar spaces with Hilbert or Banach structure; here one can apply Functional Analysis results, as Inverse Function Theorem, Fixed Point Theorem etc., and then reduce to the study of linearization. A representative example of this proceeding is local solvability for fully nonlinear equations; the result is well-known, however we have not for it a precise reference. We shall give in the next Section 2 a short proof.

As for the non-elliptic case, some recent results concern semilinear equations

$$P(x,D)u + F(x,Du) = f(x),$$

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where the linear part $P(x,D)$ is assumed to be locally solvable and the nonlinear term $F(x,Du)$ contains derivatives of lower order with respect to $P(x,D)$; see for example Hounie-Santiago [7], Gramchev-Rodino [6], Gramchev-Popivanov [5], Garello-Rodino [3], [4], Messina-Rodino [10], Marcolongo-Oliaro [8], De Donno-Oliaro [2].

In Section 3 we shall consider as linear part $P(x,D)$ an operator with characteristics of multiplicity $k \geq 2$, cf. Gramchev-Rodino [6], and fix data $f \in G_0^\sigma(\Omega)$, $1 < \sigma < \frac{k}{k-1}$. Following the lines of Bourdaud-Reissig-Sickel [1], we shall be able to treat here the case when the nonlinearity belongs to the class G^σ with respect to all the variables, improving the result of [6] about analytic nonlinearity.

2. Nonlinear elliptic equations. First of all let us fix some notation: we indicate by $x = (x_1, \dots, x_n)$ a point of $\mathbf{R}^n$; moreover for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ we write $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, where $D_{x_j} := (-i)\partial_{x_j}$. The length of α is by definition $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Now we recall the following well-known theorem:

THEOREM 1 (Inverse Function Theorem in Banach spaces). *Let B_1 and B_2 be Banach spaces; we fix $u_0 \in B_1$ and an open neighborhood $U \subset B_1$ of u_0. Let us consider a map $F : U \to B_2$, continuously differentiable. Assume moreover that $F'[u_0]$ is invertible, i.e. there exists $A \in \mathcal{L}(B_2, B_1)$ such that $F'[u_0]A = I_{B_2}$. Then we can find a sufficiently small neighborhood V of $v_0 = F[u_0]$ such that $Fu = v$ admits at least one solution $u \in U$, for any $v \in V$.*

DEFINITION 1. Let us fix $s \in \mathbf{R}$. As standard, we say that $f \in H^s$ if and only if $\hat{f}(\xi)(1+|\xi|^2)^{s/2} \in L^2(\mathbf{R}^n)$, $\hat{f}(\xi)$ being the Fourier transform of f. We can also define the following spaces:

— $H^s_{\mathrm{comp}}(\Omega) = \{f \in H^s \text{ with compact support in } \Omega\}$;
— $H^s_{\mathrm{loc}}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \varphi f \in H^s_{\mathrm{comp}}(\Omega) \text{ for every } \varphi \in C_0^\infty(\Omega)\}$.

Now fix $x_0 \in \mathbf{R}^n$ and let Ω be an open neighborhood of x_0; consider the map

$$F[u] := F(x, D^\alpha u)_{|\alpha| \leq m},\ x \in \Omega \subset \mathbf{R}^n.$$

The following result is a consequence of Schauder's Lemma.

PROPOSITION 2. *Let $F(x, u_\alpha)$ be a C^∞ function with respect to $x \in \Omega$, entire in $u_\alpha \in \mathbf{C}^M$, $M = \sum_{\alpha \in \mathbf{Z}_+^n, |\alpha| \leq m} 1$. Then if $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$, $s > \frac{n}{2}$, the functional composition $F(x, D^\alpha u)_{|\alpha| \leq m}$ belongs to $H^s_{\mathrm{loc}}(\Omega)$.*

Let us now consider the equation

$$F(x, D^\alpha u)_{|\alpha| \leq m} = \epsilon f(x),\ \epsilon > 0. \tag{1}$$

DEFINITION 2. We say that F is *locally solvable* at x_0 if for every $f \in C^\infty(\Omega)$ (or $f \in H^s_{\mathrm{loc}}(\Omega)$, s sufficiently large), equation (1) admits a classical solution u in a neighborhood $\Omega' \subset \Omega$ of x_0, for ϵ sufficiently small.

THEOREM 3. *As before, let $F(x, u_\alpha)$ be C^∞ in x and entire in u_α, $F(x, 0) = 0$. We suppose that the nonlinear operator $F[u] := F(x, D^\alpha u)_{|\alpha| \leq m}$ is elliptic. Then $F[u]$ is locally solvable.*

PROOF. Let us linearize the operator F at 0; we obtain

$$F'[0] = \sum_{|\alpha|\leq m} \frac{\partial F}{\partial u_\alpha}(x,0)D^\alpha,\ x\in\Omega. \tag{2}$$

We have

$$F'[0] \in \mathcal{L}(H^{s+m}_{\mathrm{loc}}(\Omega), H^s_{\mathrm{loc}}(\Omega)).$$

To obtain the local solvability of equation (1), we intend to apply Theorem 1. Note however that $H^s_{\mathrm{loc}}(\Omega)$ has not the structure of a Banach space. We shall then reconsider (1) in the frame of the Hilbert spaces H^s, by using suitable cut-off functions.

Let us first choose P linear with constant coefficients: $P = \sum_{|\alpha|\leq m} c_\alpha D^\alpha$, $c_\alpha \in \mathbf{C}$. In this case P is elliptic if and only if $\sum_{|\alpha|=m} c_\alpha \xi^\alpha \neq 0$ for $\xi \neq 0$. The well-known result of Malgrange-Ehrenpreis-Hörmander states the existence of a fundamental solution, i.e. a distribution $E \in \mathcal{D}'(\mathbf{R}^n)$ such that $PE = \delta$. A solution of the equation $Pu = f \in \mathcal{E}'(\mathbf{R}^n)$ is then given by $u = E * f \in \mathcal{D}'(\mathbf{R}^n)$. Observe that in the elliptic case $E* : H^s_{\mathrm{comp}}(\mathbf{R}^n) \to H^{s+m}_{\mathrm{loc}}(\mathbf{R}^n)$.

Now let P be a linear operator with C^∞ coefficients in Ω, i.e. $P = \sum_{|\alpha|\leq m} c_\alpha(x) D^\alpha$, satisfying the ellipticity condition $\sum_{|\alpha|=m} c_\alpha(x)\xi^\alpha \neq 0$ for all $\xi \neq 0$ and $x \in \Omega$. In this case we can proceed as follows: we fix $x_0 \in \Omega$ and write $P_0(D) := \sum_{|\alpha|\leq m} c_\alpha(x_0) D^\alpha$. Then we see that $P(x,D) = P_0(D) + Q(x,D)$, where $Q(x,D) = P(x,D) - P_0(D)$; observe that $Q(x,D) = \sum_{|\alpha|\leq m} \tilde{c}_\alpha(x) D^\alpha$ with $\tilde{c}_\alpha(x_0) = 0$. We want now to solve

$$P_0(D)u + Q(x,D)u = f. \tag{3}$$

Since $P_0(D)$ is elliptic with constant coefficients, there exists a fundamental solution E_0 of P_0. Let us fix a cut-off function $\psi \in C_0^\infty(\mathbf{R}^n)$, $\psi(x) = 1$ in a neighborhood of 0. For $\delta > 0$ we define $\psi_\delta(x) := \psi(\frac{x-x_0}{\delta})$. We choose another cut-off function $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi(x) = 1$ on $\operatorname{supp}\psi_\delta$. Let us now consider the equation

$$v + \psi_\delta Q(x,D)E_0 * \varphi v = f \in H^s. \tag{4}$$

Writing $Kv := \psi_\delta Q(x,D)E_0 * \varphi v$ we deduce that $K : H^s \to H^s_{\mathrm{comp}}$ is bounded; moreover $\|K\|_{\mathcal{L}(H^s,H^s)} \to 0$ as $\delta \to 0$. Shrinking the neighborhood we then obtain the existence of the inverse $(I+K)^{-1} : H^s \to H^s$, i.e. we have the existence of a solution $v \in H^s$ of equation (4). It follows that $u = E_0 * \varphi v$ is a local solution of (3).

Let us now pass to the fully nonlinear case, considering $F[u] := F(x, D^\alpha u)_{|\alpha|\leq m} = \epsilon f(x)$, $\epsilon > 0$. The linear operator $F'[0] = P(x,D)$, cf. (2), is elliptic by definition. As in the precedent case we write $P_0(D) := P(x_0, D)$. We have

$$F[u] = P_0(D)u + Q(x,D)u + G[u],$$

where $Q(x,D) = P(x,D) - P_0(D)$ and $G[u] = F(x, D^\alpha u)_{|\alpha|\leq m} - P(x,D)$; observe that $G'[0] = 0$. Let us now consider the equation

$$R[v] := v + Kv + \psi_\delta G[E_0 * \varphi v] = \epsilon f, \tag{5}$$

where $Kv := \psi_\delta Q(x,D)(E_0 * \varphi v)$ and E_0 is a fundamental solution of $P_0(D)$. Since $R : H^s \to H^s$ and $R'[0] = I + K$, shrinking δ we have as in the preceding case $(R'[0])^{-1} = (I+K)^{-1} : H^s \to H^s$. So we obtain a solution v of (5) by means of Theorem 1. A local solution of (1) is then given by $u = E_0 * \varphi v$. ∎

3. Semilinear equations with multiple characteristics. In this section we shall discuss the local solvability in Gevrey classes of equations of the form

$$P(x,D)v + F(x, \partial^\alpha v)_{|\alpha|\le m-1} = f(x) \tag{6}$$

with a linear part having analytic coefficients:

$$P(x,D) = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha.$$

As usual we write $p_m(x,\xi) = \sum_{|\alpha|=m} c_\alpha(x)\xi^\alpha$ for the principal symbol of $P(x,D)$ and $\Sigma = \{(x,\xi) : p_m(x,\xi) = 0,\ \xi \neq 0\}$ for the characteristic manifold. We shall now assume that the characteristics are multiple, i.e. P is not elliptic and for some $(x,\xi) \in \Sigma$ we may have $\mathrm{d}_{x,\xi} p_m(x,\xi) = 0$, see below.

Before giving the main result we need some preliminary definitions. We write $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$ and $\xi = (\xi', \xi_n) = (\xi_1, \dots, \xi_{n-1}, \xi_n)$ for the dual variables and with a fixed $\delta > 0$ we argue for $x_n \in (-\delta, \delta)$.

Definition 3. A nonnegative function $\psi(x_n, \xi') \in C^\infty((-\delta,\delta)\times \mathbf{R}^{n-1})$ is said to be a *weight function of order* $\rho \in (0,1)$ if for every $j \in \mathbf{Z}_+$ and $\beta' \in \mathbf{Z}_+^{n-1}$ there exist $C_{j\beta'}$ and M such that

$$|D^j_{x_n} D^{\beta'}_{\xi'} \psi(x_n, \xi')| \le C_{j\beta'}(1+|\xi'|)^{\rho - |\beta'|}$$

for $x_n \in (-\delta, \delta)$ and $|\xi'| > M$.

Definition 4. Let us fix $s > 0$, $\tau > 0$, and a weight function $\psi(x_n, \xi')$ of order ρ. We write $\sigma := \frac{1}{\rho}$ and define $H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1}\times(-\delta,\delta)) := \{f \in L^2(\mathbf{R}^{n-1}\times(-\delta,\delta)) : \|f\|_{H^{s,\psi}_{\tau,\sigma}} = \|e^{\tau\psi(x_n,D')} f(x)\|_{H^s(\mathbf{R}^{n-1}\times(-\delta,\delta))} < \infty\}$, where we mean $e^{\tau\psi(x_n,D')}f(x) = (2\pi)^{-n+1}\int e^{ix'\xi'} e^{\tau\psi(x_n,\xi')} \tilde{f}(\xi', x_n)\, d\xi'$. Here $\tilde{f}$ denotes the Fourier transform of f with respect to x'.

It is shown in Gramchev-Rodino [6, Lemma 2.7] that the following inclusion holds:

$$G^\lambda_0 \subset H^{s,\psi}_{\tau,\sigma} \text{ for } \lambda < \sigma \text{ and every } \tau, s \text{ and } \psi. \tag{7}$$

Now we are interested in the local solvability at the origin of equation (6). We define $\Omega_\delta := \{x \in \mathbf{R}^n : |x| < \delta\}$.

Theorem 4 (Local solvability for semilinear equations). *Consider equation* (6) *and suppose that the following conditions hold*:

(HL) *regarding the linear part we require that for every $\zeta = (x_0, \xi_0) \in \Sigma$ there exists a neighborhood Γ_ζ of ζ, conical with respect to the ξ-variables, in which we can write*:

$$p_m(x,\xi) = e_{m-k}(x,\xi) a_1(x,\xi)\cdots a_k(x,\xi),\ 2 \le k \le m, \tag{8}$$

where $e_{m-k}(x,\xi)$ is elliptic, analytic in x,ξ and homogeneous in ξ of order $m-k$; $a_j(x,\xi)$, $j = 1,\dots,k$, are analytic in x,ξ, homogeneous in ξ of order 1 *and moreover one of the following conditions is satisfied*:

(a) *For all $\zeta \in \Sigma$ we can choose Γ_ζ and the decomposition* (8) *in such a way that $a_j(x,\xi)$, $j = 1,\dots,k$, is real-valued and $\partial_{\xi_n} a_j(x,\xi) > 0$ in Γ_ζ.*

(b) *Alternatively in the two-dimensional case, we may require that we have a global factorization of* $p_m(x,\xi)$ *of the type*

$$p_m(x,\xi) = e_{m-k}(x,\xi)\big(\xi_2 + \lambda_1(x)\xi_1\big)\cdots\big(\xi_2 + \lambda_k(x)\xi_1\big),$$

for $x = (x_1, x_2) \in \Omega_\delta \subset \mathbf{R}^2$, *where* $e_{m-k}(x,\xi)$ *is elliptic and* $\lambda_1(x), \dots, \lambda_k(x)$ *are analytic in* Ω *satisfying* $\Im\lambda_j(x) \geq 0$ *for* $j = 1, \dots, k$ *and* $x \in \Omega_\delta$.

Let σ *be a fixed number such that*

$$1 < \sigma < \tfrac{k}{k-1}. \tag{9}$$

If condition (a) *is satisfied we choose the weight function in the following way*:

$$\psi(x_n, \xi') = \Big(1 + \frac{x_n}{2\delta}\Big)(1 + |\xi'|)^{1/\sigma}; \tag{10}$$

if (b) *is fullfilled we fix*

$$\psi(x_n, \xi') = \Big(1 + \frac{x_2}{2\delta}\,\varphi(\xi_1)\operatorname{sign}\xi_1\Big)(1 + |\xi_1|)^{1/\sigma}, \tag{11}$$

where $\varphi(\xi_1) \in C^\infty(\mathbf{R})$, $\varphi(\xi_1) = 1$ *for* $|\xi_1| > 1$, $\varphi(\xi_1) = 0$ *for* $|\xi_1| < \frac{1}{2}$.

(HN) *Let us regard* $F(x, \partial^\alpha v)_{|\alpha|\leq m-1}$ *in* (6) *as* $\mathbf{F}(x; \Re(\partial^\alpha v), \Im(\partial^\alpha v))_{|\alpha|\leq m-1}$ *and suppose that* $\mathbf{F}(x; z)$, $z \in \mathbf{R}^N$, $N = 2\sum_{\alpha\in\mathbf{Z}^n_+,\ |\alpha|\leq m-1} 1$, *satisfies the following conditions for* σ *fixed as above*:

— $\mathbf{F}(x; z_0) \in G^\lambda_0(\mathbf{R}^{n-1} \times (-\delta, \delta))$, *for some* λ, $1 < \lambda < \sigma$, *for every* $z_0 \in \mathbf{R}^N$;
— $\mathbf{F}(x_0; z) \in \bigcup_{\lambda<\sigma} G^\lambda(\mathbf{R}^N)$ *for every* $x_0 \in \mathbf{R}^{n-1} \times (-\delta, \delta)$;
— $\mathbf{F}(x; 0) = 0$.

Then equation (6) *admits a classical solution in* Ω_δ *for* $f \in H^{s,\psi}_{\tau,\sigma}$, $s \gg 0$, σ *and* ψ *fixed as in* (9), (10) *and* (11), *if* δ *and* $\|f\|_{H^{s,\psi}_{\tau,\sigma}}$ *are sufficiently small, cf.* (24) *below.*

REMARK 1. Taking into account the inclusion (7) we see that Theorem 4 implies the G^λ-local solvability of the equation (6) for $\lambda < \frac{k}{k-1}$.

The previous theorem is a generalization of a result contained in Gramchev-Rodino [6], in which the local solvability is proved under the same hypotheses of Theorem 4 on the linear part, but requiring that the nonlinearity $F(x, \partial^\alpha v)_{|\alpha|\leq m-1}$ is analytic in $\partial^\alpha v$; the case in which F is Gevrey also in $\partial^\alpha v$ is treated in [6] with stronger hypotheses on the linear part. Here, following the lines of Bourdaud-Reissig-Sickel [1] we allow F to be Gevrey in all variables without any additional condition on $P(x, D)$.

Let us first analyze the linear equation

$$P(x, D)v = f(x). \tag{12}$$

It is proved in Gramchev-Rodino [6, Theorem 3.5] that under the hypotheses (HL) there exists a linear map $E : H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta)) \to H^{s+m-k(1-1/\sigma),\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$ such that $P(x, D)Eu = \chi(x)u + Ru$, where $\chi \in G^\lambda_0(\Omega)$, $1 < \lambda < \sigma$, $\chi(x) = 1$ in a neighborhood of the origin; R is a linear regularizing map, in the sense that $R : H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta)) \to H^{t,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$ for every $t \geq 0$. As it is shown in [6] we can find a positive nondecreasing continuous function $\mathcal{C}(\delta) : [0, \delta_0] \to [0, +\infty)$ satisfying $\mathcal{C}(0) = 0$ such that

$$a_s(\delta) := \sup_{0 \neq w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta)} \frac{\|Rw\|_{H^{s,\psi}_{\tau,\sigma}}}{\|w\|_{H^{s,\psi}_{\tau,\sigma}}} \leq \mathcal{C}(\delta), \tag{13}$$

$$b_s(\delta) := \sup_{0 \neq w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta)} \frac{\|Ew\|_{H^{s+m-1,\psi}_{\tau,\sigma}}}{\|w\|_{H^{s,\psi}_{\tau,\sigma}}} \leq \mathcal{C}(\delta), \tag{14}$$

for every $\delta \in (0, \delta_0]$.

Now we want to analyze the nonlinearity $\mathbf{F}(x; \Re(\partial^\alpha v), \Im(\partial^\alpha v))_{|\alpha| \leq m-1}$. In Bourdaud-Reissig-Sickel [1] the composition $\mathbf{F}(x; u(x))$, $u(x) = (u_1(x), \ldots, u_N(x))$, is studied for $u_j(x)$ belonging to spaces that are similar to $H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$. The results proved there can be adapted to our situation. First of all we see that $H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$ is an algebra for $s > 1$ if $\psi(x_n, \xi')$ satisfies the condition

$$\psi(x_n, \xi') - \psi(x_n, \xi' - \eta') - \psi(x_n, \eta') \leq -b \min\{(1 + |\xi' - \eta'|), (1 + |\eta'|)\}^\rho, \tag{15}$$

$b > 0$ independent of $\xi', \eta' \in \mathbf{R}^{n-1}$ and $x_n \in (-\delta, \delta)$; for $u, v \in H^{s,\psi}_{\tau,\sigma}$ we can estimate

$$\|uv\|_{H^{s,\psi}_{\tau,\sigma}} \leq C(s) \|u\|_{H^{s,\psi}_{\tau,\sigma}} \|v\|_{H^{s,\psi}_{\tau,\sigma}}. \tag{16}$$

Observe that this result is lightly different from the corresponding one in [6]: the condition (15), stronger than the one requested there, is fundamental in order to prove the following lemma.

Lemma 5. *Let us suppose that $s > 1$ and fix the weight function ψ of order $\rho = \frac{1}{\sigma}$ satisfying* (15)*. Then there exist two constants c and a such that*

$$\|e^{iu(x)} - 1\| \leq \begin{cases} c\, e^{a\|u\|^\rho \log \|u\|} & \text{if } \|u\| > 1 \\ c\, \|u\| & \text{if } \|u\| \leq 1 \end{cases} \tag{17}$$

for every real-valued function $u \in H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$, where the norms in (17) *are taken in $H^{s,\psi}_{\tau,\sigma}$, and where the constants c and a depend only on n and s.*

The proof of this lemma is omitted. A similar result is proved in [1, Sections 2.9 and 2.10]. Using the estimate (17) we can prove the next proposition.

Proposition 6. *Let us consider a function $\mathbf{F}(x; z)$ satisfying the hypotheses* (HN) *of Theorem* 4*. Moreover we consider $X \subset H^{s+m-1,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$ bounded with respect to $\|\cdot\|_{H^{s+m-1,\psi}_{\tau,\sigma}}$ and we take $u(x) \in X$, where the weight function $\psi(x_n, \xi')$ satisfies* (15) *and $s > 1$. If $J(u) := \mathbf{F}(x; \Re(\partial^\alpha u), \Im(\partial^\alpha u))_{|\alpha| \leq m-1}$, then $J(u) \in H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1} \times (-\delta, \delta))$ and there exists a continuous nondecreasing function $\Phi : [0, +\infty) \to [0, +\infty)$ satisfying $\Phi(0) = 0$ such that*

$$\|J(u)\|_{H^{s,\psi}_{\tau,\sigma}} \leq \Phi(\|u\|_{H^{s+m-1,\psi}_{\tau,\sigma}}). \tag{18}$$

Moreover, we can find a constant C_X such that for all $u, v \in X$

$$\|J(u) - J(v)\|_{H^{s,\psi}_{\tau,\sigma}} \leq C_X \|u - v\|_{H^{s+m-1,\psi}_{\tau,\sigma}}, \tag{19}$$

where the constant C_X depends on the bounded set X.

PROOF. Observe first that if $s>1$ then $H^{s,\psi}_{\tau,\sigma}(\mathbf{R}^{n-1}\times(-\delta,\delta))\hookrightarrow L^\infty(\mathbf{R}^{n-1}\times(-\delta,\delta))$. Indeed, this follows from

$$\|u\|_{L^\infty}\le \sup_{x_n\in(-\delta,\delta)}\|\tilde{u}(x_n,\xi')\|_{L^1(\mathbf{R}^{n-1}_{\xi'})}\le C\sup_{x_n\in(-\delta,\delta)}\|e^{\tau\psi(x_n,D')}u\|_{L^2(\mathbf{R}^{n-1}_{x'})}\le C'\|u\|_{H^{s,\psi}_{\tau,\sigma}},$$

where $\tilde{u}(x_n,\xi')$ is the Fourier transform of u with respect to x'. So we deduce that X is bounded in $L^\infty(\mathbf{R}^{n-1}\times(-\delta,\delta))$. We can suppose without loss of generality that, for every $x_0\in\mathbf{R}^{n-1}\times(-\delta,\delta)$, $\mathbf{F}(x_0;z)\in\bigcup_{r<\sigma}G^r_0(\mathbf{R}^N)$. Since $\mathbf{F}(x;0)=0$ we have

$$\mathbf{F}(x;z)=(2\pi)^{-N-n+1}\int e^{ix'\xi'}(e^{iz\eta}-1)\widehat{F}(x_n,\xi';\eta)\,d\xi'\,d\eta,$$

where $\widehat{F}(x_n,\xi';\eta):=\int e^{-ix'\xi'}e^{-iz\eta}F(x',x_n;z)\,dx'\,dz$. So we can estimate the norm of $J(u)$ in the following way:

$$\|J(u)\|_{H^{s,\psi}_{\tau,\sigma}}\le C\sum_{j=0}^{s}\int\big\|e^{ix'\xi'}(e^{i\langle\eta;u(x)\rangle}-1)\big\|_{H^{s,\psi}_{\tau,\sigma}}\sup_{x_n\in(-\delta,\delta)}\big|D^j_{x_n}\widehat{F}(x_n,\xi';\eta')\big|\,d\xi'\,d\eta,$$

where $\langle\eta;u(x)\rangle:=\sum_{|\alpha|\le m-1}\big(\eta^1_{(\alpha)}\Re(\partial^\alpha u)+\eta^2_{(\alpha)}\Im(\partial^\alpha u)\big)$. Here we write $\eta^1_{(\alpha)}$ to indicate the covariable of the variable of place $\Re(\partial^\alpha u)$ in $\mathbf{F}(x;\Re(\partial^\alpha u),\Im(\partial^\alpha u))$, $\eta^2_{(\alpha)}$ the one of the variable of place $\Im(\partial^\alpha u)$. By standard properties of the Fourier transform we deduce that for every $j=0,\dots,s$

$$|D^j_{x_n}\widehat{F}(x_n,\xi';\eta)|\le Ce^{-\epsilon(|\xi'|^{1/\lambda}+|\eta|^{1/r})},\tag{20}$$

$\lambda<\sigma$, $r<\sigma$, where C and ϵ do not depend on j. Moreover, by the definition of $\|\cdot\|_{H^{s,\psi}_{\tau,\sigma}}$ we can deduce that

$$\|e^{ix'\xi'}f(x)\|_{H^{s,\psi}_{\tau,\sigma}}\le Ce^{|\xi'|^{1/\sigma}}\|f\|_{H^{s,\psi}_{\tau,\sigma}}.\tag{21}$$

Since $\lambda<\sigma$, by (20) and (21) we have

$$\begin{aligned}\|J(u)\|_{H^{s,\psi}_{\tau,\sigma}}&\le C\int e^{|\xi'|^{1/\sigma}}e^{-\epsilon|\xi'|^{1/\lambda}}d\xi'\cdot\int\|e^{i\langle\eta;u(x)\rangle}-1\|_{H^{s,\psi}_{\tau,\sigma}}e^{-\epsilon|\eta|^{1/r}}d\eta\\&=C'\int\|e^{i\langle\eta;u(x)\rangle}-1\|_{H^{s,\psi}_{\tau,\sigma}}e^{-\epsilon|\eta|^{1/r}}d\eta.\end{aligned}\tag{22}$$

It is not difficult to prove, by induction on N, that for every N-tuple of complex numbers $(a_1,\dots,a_N)$ the following identity holds:

$$a_1\cdots a_N-1=\sum_{l=1}^{N}\sum_{0<j_1<\ldots<j_l\le N}(a_{j_1}-1)\cdots(a_{j_l}-1).\tag{23}$$

Applying the decomposition (23) to $(e^{i\langle\eta;u(x)\rangle}-1)$ and using the algebra property of the space $H^{s,\psi}_{\tau,\sigma}$, cf. (16), we can estimate $\|e^{i\langle\eta;u(x)\rangle}-1\|_{H^{s,\psi}_{\tau,\sigma}}$ with factors of the type $\|e^{i\eta^1_{(\alpha)}\Re(\partial^\alpha u)}-1\|_{H^{s,\psi}_{\tau,\sigma}}$ and $\|e^{i\eta^2_{(\alpha)}\Im(\partial^\alpha u)}-1\|_{H^{s,\psi}_{\tau,\sigma}}$. These factors can be treated by using Lemma 5. Since $\|\Re(\partial^\alpha u)\|_{H^{s,\psi}_{\tau,\sigma}}\le\|u\|_{H^{s+m-1,\psi}_{\tau,\sigma}}$ and $\|\Im(\partial^\alpha u)\|_{H^{s,\psi}_{\tau,\sigma}}\le\|u\|_{H^{s+m-1,\psi}_{\tau,\sigma}}$ for $|\alpha|\le m-1$, recalling that $r<\sigma$ we can deduce by (22) that

$$\|J(u)\|_{H^{s,\psi}_{\tau,\sigma}}\le\Phi(\|u\|_{H^{s+m-1,\psi}_{\tau,\sigma}}).$$

Observe that if $\|u\|_{H^{s,\psi}_{\tau,\sigma}}$ is sufficiently small we have by (23), (17) and (22)

$$\|J(u)\|_{H^{s,\psi}_{\tau,\sigma}} \leq C \sum_{l=1}^{N} \|u\|^{l}_{H^{s+m-1,\psi}_{\tau,\sigma}}$$

which implies that $\Phi(\|u\|_{H^{s+m-1,\psi}_{\tau,\sigma}}) \to 0$ for $\|u\|_{H^{s+m-1,\psi}_{\tau,\sigma}} \to 0$. This proves (18).

In order to verify (19) we write

$$\mathbf{F}(x;z) - \mathbf{F}(x;y) = \sum_{j=1}^{N} (z_j - y_j) \int_0^1 \big((\partial_{z_j}\mathbf{F})(x; z + ty) - (\partial_{z_j}\mathbf{F})(x;0)\big)\, dt + \sum_{j=1}^{N} (z_j - y_j)(\partial_{z_j}\mathbf{F})(x;0).$$

So

$$\|J(u) - J(v)\|_{H^{s,\psi}_{\tau,\sigma}} \leq C\|u - v\|_{H^{s+m-1,\psi}_{\tau,\sigma}} \times \sum_{j=1}^{N} \bigg[\int_0^1 \big\|(\partial_{z_j}\mathbf{F})\big(x; \Re(\partial^\alpha u + t\partial^\alpha v), \Im(\partial^\alpha u + t\partial^\alpha v)\big) - (\partial_{z_j}\mathbf{F})(x;0)\big\|_{H^{s,\psi}_{\tau,\sigma}}\, dt + \|(\partial_{z_j}\mathbf{F})(x;0)\|_{H^{s,\psi}_{\tau,\sigma}} \bigg].$$

Now, since $\mathbf{G}(x;z) := (\partial_{z_j}\mathbf{F})(x;z) - (\partial_{z_j}\mathbf{F})(x;0)$ satisfies all the hypotheses of the theorem, by (18) we get

$$\|J(u) - J(v)\|_{H^{s,\psi}_{\tau,\sigma}} \leq C\|u - v\|_{H^{s+m-1,\psi}_{\tau,\sigma}} \sum_{j=1}^{N} \bigg[\int_0^1 \Phi(\|u + tv\|_{H^{s+m-1,\psi}_{\tau,\sigma}})\, dt + \|(\partial_{z_j}\mathbf{F})(x;0)\|_{H^{s,\psi}_{\tau,\sigma}} \bigg];$$

recall that $u(x)$ and $v(x)$ are in X. The function Φ is continuous, and so (19) is true on X. ∎

Remark 2. Since $\sigma > 1$, cf. (9), the weight functions (10) and (11) satisfy the condition (15). For this reason we can use Proposition 6 in order to prove Theorem 4.

Proof of Theorem 4. We shall obtain the local solvability of equation (6) applying the Fixed Point Theorem in the space $X := \{w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta) : \|w - f\|_{H^{s,\psi}_{\tau,\sigma}} \leq 1\}$. Let us suppose that δ and f satisfy the following conditions:

$$\begin{cases} a_s(\delta)(1 + \|f\|_{H^{s,\psi}_{\tau,\sigma}}) + \Phi\big(b_s(\delta)(1 + \|f\|_{H^{s,\psi}_{\tau,\sigma}})\big) \leq 1, \\ a_s(\delta) + C_X b_s(\delta) < 1, \end{cases} \tag{24}$$

where $a_s(\delta)$ and $b_s(\delta)$ are given by (13) and (14) respectively. The function Φ and the constant C_X are the ones of Proposition 6.

We look for a solution $v(x)$ of (6) of the form $v(x) = Ew(x)$. Hence we can rewrite the equation (6) in the following way:

$$w(x) = \mathcal{K}w(x) + f(x),$$

where

$$\mathcal{K}w(x) := -Rw - F(x, \partial^\alpha(Ew))_{|\alpha|\le m-1}.$$

It is then sufficient to prove that the operator $\Theta w := \mathcal{K}w + f$ is a contraction in X.

(i) $\Theta : X \to X$: indeed using (13), (14), (18) and the first inequality in (24) we have for $w \in X$:

$$\begin{aligned}\|\Theta w - f\|_{H^{s,\psi}_{\tau,\sigma}} &\le \|Rw\|_{H^{s,\psi}_{\tau,\sigma}} + \|F(x, \partial^\alpha(Ew))\|_{H^{s,\psi}_{\tau,\sigma}} \\ &\le a_s(\delta)(1 + \|f\|_{H^{s,\psi}_{\tau,\sigma}}) + \Phi\big(b_s(\delta)(1 + \|f\|_{H^{s,\psi}_{\tau,\sigma}})\big) \le 1.\end{aligned}$$

(ii) Θ is a contraction in X: by (13), (14), (19) and the second inequality in (24) we obtain

$$\|\Theta w_1 - \Theta w_2\|_{H^{s,\psi}_{\tau,\sigma}} \le \|w_1 - w_2\|_{H^{s,\psi}_{\tau,\sigma}}(a_s(\delta) + C_X b_s(\delta)) = L\|w_1 - w_2\|_{H^{s,\psi}_{\tau,\sigma}}$$

with $L < 1$, for all $w_1, w_2 \in X$.

By the Fixed Point Theorem we then obtain a solution $v = Ew \in H^{s+m-k(1-1/\sigma),\psi}_{\tau,\sigma}$ of the equation (6). If we take s sufficiently large by the Sobolev inclusions the solution v is classical. ■

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