

SELF-SIMILAR SOLUTIONS IN REACTION-DIFFUSION SYSTEMS

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Abstract. In this paper we examine self-similar solutions to the system

$$\begin{aligned}u_{it} - d_i \Delta u_i &= \prod_{k=1}^m u_k^{p_k^i}, \quad i = 1, \dots, m, \quad x \in \mathbb{R}^N, \quad t > 0, \\u_i(0, x) &= u_{0i}(x), \quad i = 1, \dots, m, \quad x \in \mathbb{R}^N,\end{aligned}$$

where $m > 1$ and $p_k^i > 0$, to describe asymptotics near the blow up point.

1. Introduction. In this paper we present the results that describe solutions to the system

$$(1.1) \quad \begin{aligned}u_{it} - d_i \Delta u_i &= \prod_{k=1}^m u_k^{p_k^i}, \quad i = 1, \dots, m, \quad x \in \mathbb{R}^N, \quad t > 0, \\u_i(0, x) &= u_{0i}(x), \quad i = 1, \dots, m, \quad x \in \mathbb{R}^N,\end{aligned}$$

near the blow up point in terms of self-similar solutions to the system. Namely, we are going to show that blowing up solutions to the system are asymptotically close to the respective self-similar solutions. To formulate our results we have to introduce some convenient notation according to the system under consideration, i.e. (1.1).

We observe that without loss of generality we can assume henceforth that

$$(1.2) \quad \min_i \sum_{k=1}^m p_k^i = \sum_{k=1}^m p_k^1.$$

Let

$$(1.3) \quad A_m = [p_k^i], \quad i, k = 1, \dots, m,$$

with row i and column k .

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Let

$$(1.4) \quad \delta = \det(A_m - I),$$

and suppose $\alpha = (\alpha_1, \dots, \alpha_m)$ satisfies

$$(1.5) \quad (A_m - I)\alpha^t = (1, 1, \dots, 1)^t.$$

We notice that if $\delta \neq 0$ then the unique solution α of (1.5) is given by

$$(1.6) \quad \alpha_k = \delta^{-1} \det(D_k(A_m - I)), \quad k = 1, \dots, m,$$

where $D_k(A_m - I)$ denotes the matrix $A_m - I$ with column k replaced by the vector $(1, 1, \dots, 1)^t$.

THEOREM 1.1 ([R3]). *Let u be a solution of (1.1) defined in $\mathbb{R}^N \times (0, T)$. Assume that $\min_i \alpha_i > 0$ and $\max_i \alpha_i = \alpha_1$, where the α_i are given by (1.3)–(1.6). Suppose that one of the conditions below holds:*

$$(1.7) \quad N = 1, 2 \quad \text{or} \quad N \geq 3, \quad \alpha_1 \geq \frac{N - 2}{4},$$

$$(1.8) \quad d_1 \Delta u_{0_1} + A_1 u_{0_1}^{1+1/\alpha_1} > 0.$$

Then for some constant $C > 0$,

$$(1.9) \quad u_i(x, t) \leq C(T - t)^{-\alpha_i}, \quad i = 1, \dots, m.$$

This result makes it possible to proceed further in order to derive a finer description of a solution near a blow up. Our approach starts with introducing similarity variables in the following way:

$$(1.10) \quad \begin{aligned} u_i(x, t) &= (T - t)^{-\alpha_i} U_i(y, s), \\ y &= x(T - t)^{-1/2}, \quad s = -\log(T - t). \end{aligned}$$

Then the new functions U_i solve

$$(1.11) \quad U_{is} = d_i \Delta U_i - \frac{1}{2} y \nabla U_i - \alpha_i U_i + \prod_{k=1}^m U_k^{p_k^i}, \quad i = 1, \dots, m.$$

Our task is to characterize all global bounded nonnegative solutions of the rescaled system (1.11). Since $t \rightarrow T$ corresponds to $s \rightarrow \infty$, the analysis of nonglobal solutions of (1.1) is now equivalent to examining the large-time asymptotics of solutions of (1.11). However, the last problem seems to be easier to solve. Consequently, our task is to characterize all global bounded nonnegative solutions of the rescaled system (1.11). It turns out that under some restriction on p_k^i such solutions are constant. We can divide the argument into two steps. We start with the assuring ourselves that as $s \rightarrow \infty$, $U_i(y, s)$ are independent of s , i.e. should approach a stationary solution of the system. Next, we characterize such solutions as the constants. In this way we will show that the solution u of (1.1) is “asymptotically self-similar”. Our main result is

THEOREM 1.2. *Let $U = (U_1, \dots, U_m)$ be any nonnegative bounded solution of (1.11) defined for $(y, s) \in \mathbb{R}^N \times \mathbb{R}$. Assume that $U \not\equiv \mathbf{0}$ where $\mathbf{0} = (0, \dots, 0)$ and $\max_i \alpha_i = \alpha_1$. If*

$$(1.12) \quad \min_i \alpha_i > 0$$

and

$$N = 1, 2$$

or

$$(1.13) \quad N \geq 3, \quad 1 < \frac{\sum_{k=i}^m p_k^i \alpha_k}{\alpha_i} \leq \frac{N+2}{N-2}, \quad i = 1, \dots, m,$$

then

$$(1.14) \quad \lim_{s \rightarrow \infty} (U_1, \dots, U_m) = (C_{\alpha_1}, \dots, C_{\alpha_m})$$

where $(C_{\alpha_1}, \dots, C_{\alpha_m})$ are constant solutions of the system (1.11).

We emphasize that the assumptions of Theorem 1.2 imply the growth rate estimates (1.9) since condition (1.7) is satisfied. Therefore U_i can be defined by the system (1.1) and formulas (1.10).

2. Preliminaries. First, we recall some auxiliary facts established in [R3] and [R2]. We put

$$(2.1) \quad b_k = \frac{\alpha_k}{\alpha_1}, \quad k = 1, \dots, m,$$

assuming that $\alpha_1 \neq 0$ and $b_k > 0$ (the last condition is guaranteed if $\min_k \alpha_k > 0$). We also define

$$(2.2) \quad r = \sum_{k=1}^m b_k p_k^1 = \frac{1 + \alpha_1}{\alpha_1}.$$

Let us consider the following kinetic system corresponding to (1.1):

$$(2.3) \quad \begin{cases} u_i' = \prod_{k=1}^m u_k^{p_k^i} \\ u_i(0) = u_{0i}, \end{cases} \quad i = 1, \dots, m.$$

LEMMA 2.1. *The set*

$$\partial M = \{(u_1, \dots, u_m) : u_i \geq 0, F(u_1, \dots, u_m) = (F_1 \dots, F_{m-1})(u_1, \dots, u_m) = \mathbf{0}; \\ F_j(u_1, \dots, u_m) = u_{j+1} - a_{j+1} u_j^{b_{j+1}/b_j}, \quad j = 1, \dots, m-1\},$$

where a_j are constants given by the conditions

$$(2.4) \quad a_1 = 1, \quad b_i \prod_{j=1}^i a_j^{b_j/b_i} = \prod_{k=1}^m \left(\prod_{j=1}^k a_j^{b_k/b_j} \right)^{p_k^i}, \quad i = 2, \dots, m.$$

is an invariant manifold for (2.3).

DEFINITION 2.1 (Invariant region). Let $\frac{du}{dt} + Au = f(t, u)$, $u = (u_1, \dots, u_m)$. A region D is called a *regular invariant region* for this system, if the conditions $u_0 \in C^2$, $u_0 \in \partial D$ imply that $(t, u(x, t)) \in D$ (where u denotes the solution of this system).

LEMMA 2.2. *Let*

$$u = (u_1, \dots, u_m), \quad F(u) = (F_1(u), \dots, F_{m-1}(u)),$$

$$F_i(u) = u_{i+1} - a_{i+1}u_i^{b_{i+1}/b_i}, \quad i = 1, \dots, m - 1$$

and

$$M = \{u : F(u) \leq \mathbf{0}, u_i \geq 0, i = 1, \dots, m$$

$$\text{and } (d_{i+1} - d_i)\Delta u_{0i} \leq 0 \text{ for } u_{0i} = u_i(0) \in C^2\}.$$

Then M is a regular invariant region for (1.1) if $\frac{b_{i+1}}{b_i} \leq 1$ for all $i = 1, \dots, m - 1$ (i.e., $u_0 \in C^2, u_0 \in \partial M$ imply that $u(x, t) \in M$ for all $t < T$).

3. Asymptotical self-similarity of solutions. In this section we consider the natural question about a finer description of u near blow up. Assuming estimates on the growth rate established in [R3], i.e.

$$u_i(x, t) \leq C(T - t)^{-\alpha_i}, \quad i = 1, \dots, m.$$

we define the following change of both independent and dependent variables

$$(3.1) \quad \begin{aligned} U_i(y, s) &= (T - t)^{\alpha_i} u_i(x, t) \\ y &= x(T - t)^{-1/2}, \quad s = -\log(T - t). \end{aligned}$$

We introduce the system corresponding to new functions and variables called the similarity variables

$$(3.2) \quad U_{is} = d_i \Delta U_i - \frac{1}{2} y \nabla U_i - \alpha_i U_i + \prod_{k=1}^m U_k^{p_k^i}, \quad i = 1, \dots, m.$$

We argue by means of the invariant region technique. The utility of the method consists in the possibility of rewriting the system (1.1) to a system of inequalities. Nevertheless, all are the scalar ones. Moreover, it turns out that adapting a technique used to discuss the scalar problem is then available.

Applying Lemma 2.2 we can rewrite the system (1.1) (so respectively (3.2)) as follows, provided that initial values belong to ∂M :

$$(3.3) \quad u_{it} - d_i u_i \leq M_i u_1^{\sum_{k=1}^{i-1} b_k p_k^i} u_i^{\sum_{k=i}^m b_k p_k^i / b_i}, \quad i = 1, \dots, m$$

and

$$(3.4) \quad \begin{aligned} U_{is} - d_i \Delta U_i + \frac{1}{2} y \nabla U_i + \alpha_i U_i &\leq M_i \prod_{k=1}^{i-1} U_1^{b_k p_k^i} \prod_{k=i}^m U_i^{b_k p_k^i / b_i} \\ &= M_i U_1^{\sum_{k=1}^{i-1} b_k p_k^i} U_i^{\sum_{k=i}^m b_k p_k^i / b_i}, \end{aligned}$$

where

$$(3.5) \quad \begin{aligned} M_1 &= A_1 = \prod_{k=1}^m \left(\prod_{j=1}^k a_j^{b_k/b_j} \right)^{p_k^i}, \\ M_i &= \prod_{k=1}^{i-1} \left(\prod_{j=1}^k a_j^{b_k/b_j} \right)^{p_k^i} \prod_{k=i}^m \left(\prod_{j=i}^k a_j^{b_k/b_j} \right)^{p_k^i}, \quad i = 2, \dots, m. \end{aligned}$$

We now focus on the first inequality of (3.4) which in fact is scalar.

$$(3.6) \quad U_{1s} - d_1 \Delta U_1 + \frac{1}{2} y \nabla U_1 + \alpha_1 U_1 - M_1 U_1^{(1+\alpha_1)/\alpha_1} \leq 0.$$

From here we pursue the existing trail for the scalar equation. We refer also to Section 3 in [R4], where the case $m = 2, d_i = 1$ is discussed. According to this problem we start with the assuring ourselves that as $s \rightarrow \infty, U_1$ is necessarily self-similar, i.e. independent of s . Therefore, U_1 should approach a stationary solution of (3.6). From this point of view we shall next analyze the global nonnegative and bounded solutions of the equation

$$d_1 \Delta U_1 - \frac{1}{2} y \nabla U_1 - \alpha_1 U_1 + M_1 U_1^{(1+\alpha_1)/\alpha_1} = 0.$$

It turns out that they are constants. We set

$$(3.7) \quad \begin{aligned} U_1^{+\infty}(y, s) &= \lim_{s_j \rightarrow \infty} U_1(y, s + s_j) \\ U_1^{-\infty}(y, s) &= \lim_{s'_j \rightarrow -\infty} U_1(y, s + s'_j). \end{aligned}$$

This corresponds to limits $\lim_{t \rightarrow T} u_1(x, t)$ and $\lim_{t \rightarrow -\infty} u_1(x, t)$, respectively.

The first step of our analysis can be expressed as follows:

PROPOSITION 3.1. *If U_1 is a bounded global solution of (3.6), then both limits U_1^∞ and $U_1^{-\infty}$ exist and are independent of s . Moreover, $U_1(y, s + s_j) \rightarrow U_1^{\pm\infty}$ as $s_j \rightarrow \pm\infty$ uniformly on compact subsets of \mathbb{R}^N with $\lim_{s \rightarrow \pm\infty} \nabla U_1(y, s) = 0$ for almost every y .*

PROOF. Since the approach is completely parallel with the case of the single equation, we shall just recall main ideas, pointing out differences emerging for the inequality (3.6).

We quote a technical result of Giga and Kohn ([GK1], Proposition 1).

LEMMA 3.2. *If u_1 is a function satisfying (3.3) and if (1.9) holds, then*

$$(3.8) \quad |\nabla^k u_1(x, t)| \leq C(T - t)^{-\beta - k/2}.$$

This implies

LEMMA 3.3. *If U_1 is a bounded function satisfying (3.6) defined on \mathbb{R}^{N+1} then*

$$(3.9) \quad |\nabla U_1| + |\nabla^2 U_1| \leq C', \quad |U_{1s}| \leq C'(1 + |y|).$$

Our problem can be represented as

$$(3.10) \quad \rho U_{1s} - d_1 \nabla \cdot (\rho \nabla U_1) + \frac{1}{r-1} \rho U_1 - M_1 \rho U_1^r \leq 0,$$

where $r = 1 + \frac{1}{\alpha_1}, \rho(y) = \exp(-\frac{1}{4}|y|^2)$. We claim

LEMMA 3.4. *If U_1 solves (3.6) and is bounded then*

$$(3.11) \quad \int_a^b \int_{\mathbb{R}^N} |U_{1s}|^2 \rho \, dy \, ds \leq E[U_1](a) - E[U_1](b),$$

where $a < b$ are real and

$$(3.12) \quad E[U_1](s) = \frac{d_1}{2} \int_{\mathbb{R}^N} |\nabla U_1|^2 \rho \, dy + \frac{1}{2(r-1)} \int_{\mathbb{R}^N} |U_1|^2 \rho \, dy - \frac{M_1}{r+1} \int_{\mathbb{R}^N} |U_1|^{r+1} \rho \, dy.$$

PROOF. We derive the above assertion analogously to Proposition 3 in [GK1]. Multiplying (3.6) by U_{1s} and integrating over $B_R = \{y : |y| < R\}$ one gets

$$(3.13) \quad \begin{aligned} & \int_{B_R} |U_{1s}|^2 \rho \, dy \\ & \leq \int_{B_R} \left(d_1 U_{1s} \nabla \cdot (\rho \nabla U_1) - \frac{1}{2(r-1)} \frac{d}{ds} |U_1|^2 \rho + \frac{M_1}{r+1} \frac{d}{ds} |U_1|^{r+1} \rho \right) dy \\ & = -\frac{d_1}{2} \frac{d}{ds} \int_{B_R} |\nabla U_1|^2 \rho \, dy + \int_{\partial B_R} U_{1s} \frac{\partial U_1}{\partial r} \rho \, d\sigma \\ & \quad - \frac{d_1}{2(r-1)} \frac{d}{ds} \int_{B_R} |U_1|^2 \rho \, dy + \frac{M_1}{r+1} \frac{d}{ds} \int_{B_R} |U_1|^{r+1} \rho \, dy. \end{aligned}$$

By Lemma 3.3 we observe that the surface term tends to zero as $R \rightarrow \infty$. Therefore, the integration from a to b and the passage with R to the limit gives

$$(3.14) \quad \int_a^b \int_{\mathbb{R}^N} |U_{1s}|^2 \rho \, dy \, ds \leq E[U_1](a) - E[U_1](b).$$

This concludes the proof. ■

The last inequality is basic to the further elaboration.

LEMMA 3.5. *Assume that U_1 is a bounded solution of (3.6) on \mathbb{R}^{N+1} and a sequence $s_j \rightarrow +\infty$ (respectively, $s_j \rightarrow -\infty$) is monotonically increasing (decreasing) with $s_{j+1} - s_j \rightarrow \pm\infty$ as $j \rightarrow \infty$. Let $U_1^j(y, s) = U_1(y, s + s_j)$ converge to a limit $U_1^{\pm\infty}$ uniformly on compact sets of \mathbb{R}^{N+1} with $\nabla U_1^j(y, m) \rightarrow \nabla U_1^{\pm\infty}(y, m)$ almost everywhere in \mathbb{R}^N for all integer m . Then $U_1^{\pm\infty}$ does not depend on s and $E[U_1^{\infty}]$, $E[U_1^{-\infty}]$ are independent of the choice of $\{s_j\}$.*

PROOF. This lemma is an analogue of Proposition 4 in [GK1]. We consider $s_j \rightarrow \infty$. By Lemma 3.4, putting $U_1 = U_1^j$, $a = m$, $b = m + s_{j+1} - s_j$ in (3.11) we get

$$(3.15) \quad \begin{aligned} \int_m^{m+s_{j+1}-s_j} \int_{\mathbb{R}^N} |U_{1s}^j|^2 \rho \, dy \, ds & \leq E[U_1^j](m) - E[U_1^j](m + s_{j+1} - s_j) \\ & = E[U_1^j](m) - E[U_1^{j+1}](m). \end{aligned}$$

Lemma 3.3 yields that $|\nabla U_1^j| < C$, $|U_1^j| < c$ for all i . Thus, we can pass with $j \rightarrow \infty$ in (3.15) and since $E[U_1^j](m) \rightarrow E[U_1^{\infty}](m)$, $s_{j+1} - s_j \rightarrow \infty$, we see that

$$(3.16) \quad \lim_{j \rightarrow \infty} \int_m^K \int_{\mathbb{R}^N} |U_{1s}^j|^2 \rho \, dy \, ds \leq 0, \quad m < K.$$

Therefore, we must have

$$(3.17) \quad \lim_{j \rightarrow \infty} \int_m^K \int_{\mathbb{R}^N} |U_{1s}^j|^2 \rho \, dy \, ds = 0.$$

It follows by $|U_{1s}^j| \leq C(1 + |y|)$ (i.e. (3.9)) and a lower-semicontinuity of the integral in (3.17) that also

$$(3.18) \quad \int_m^K \int_{\mathbb{R}^N} |U_{1s}^\infty|^2 \rho \, dy \, ds = 0$$

for even m and K , so U_1^∞ is necessarily independent of s .

Now, we discuss properties of $E[U_1^\infty]$. We argue by contradiction. Let \check{s}_j be such a sequence that \check{s}_j satisfies assumptions of Lemma 3.5, $\check{U}_1^j(y, s) = U_1(y, s + \check{s}_j)$, $\check{U}_1^\infty = \lim_{\check{s}_j \rightarrow \infty} \check{U}_1^j$ and $E[U_1^\infty] \neq E[\check{U}_1^\infty]$. Let $E[U_1^\infty] \leq E[\check{U}_1^\infty]$, $s_j < \check{s}_j$ (we can obtain it passing to a subsequence). We use Lemma 3.4 setting in (3.11) $a = s_j$, $b = \check{s}_j$. Then

$$(3.19) \quad \int_{s_j}^{\check{s}_j} \int_{\mathbb{R}^N} |U_{1s}|^2 \rho \, dy \, ds \leq E[U_1](s_j) - E[U_1](\check{s}_j) = \\ = E[U_1^j](0) - E[\check{U}_1^j](0) \rightarrow E[U_1^\infty] - E[\check{U}_1^\infty] < 0.$$

Thus, for j large enough, there is a contradiction with a nonnegativity of the integral in (3.19). We conclude that $E[U_1^\infty] = E[\check{U}_1^\infty]$, i.e. $E[U_1^\infty]$ is independent of \check{s}_j . We complete the proof considering the case $s_j \rightarrow -\infty$ similarly.

Finally, by (3.9) we have

$$|\nabla U_1| \leq C', \quad |U_{1s}| \leq C'(1 + |y|),$$

then for some subsequence $\{s_j\}$, $s_j \rightarrow \infty$, $U_1(y, s + s_j) \rightarrow U_1^\infty(y, s)$ uniformly on compact sets. The bound $|\nabla^2 U_1| \leq C$ assures that $\nabla U_1(y, m + s_j) \rightarrow \nabla U_1^\infty(y, m)$ almost everywhere for every integer m and some subsequence s_j . Clearly, we can take such a subsequence to $s_{j+1} - s_j \rightarrow \infty$, also. Then we obtain a desired conclusion. ■

Summarizing Lemmas 3.4 and 3.5, we infer Proposition 3.1. ■

To complete the classification of bounded solutions of (3.2) we prove that such solutions of corresponding stationary problem are constant.

PROPOSITION 3.6. *If U_1 is a stationary, global and bounded solution of (3.6), so satisfies*

$$(3.20) \quad d_1 \Delta U_1 - \frac{1}{2} y \nabla U_1 - \frac{U_1}{r-1} + M_1 U_1^r \geq 0,$$

then U_1 is a constant such that $U_1 \leq [M_1(r-1)]^{-1/(r-1)}$ provided $N = 1, 2$ or $N \geq 3$ and $1 < r < \frac{N+2}{N-2}$ (i.e. $\alpha_1 \geq \frac{N-2}{4}$).

PROOF. As we have mentioned, U_1 is a subsolution of a corresponding scalar equation, namely $U_1(y) \leq \bar{U}_1(y)$, where

$$(3.21) \quad d_1 \Delta \bar{U}_1 - \frac{y}{2} \nabla \bar{U}_1 - \frac{\bar{U}_1}{r-1} - M_1 \bar{U}_1^r = 0.$$

However, the above problem has been studied by several authors, so we can employ the known result (cf. Theorem 1 of [GK1], also [GS]) to infer that (3.21) has only constant global solutions: $\bar{U}_1 \equiv 0$ or $\bar{U}_1 \equiv K$, where

$$(3.22) \quad K = \left(\frac{\alpha_1}{M_1}\right)^{\alpha_1} = [M_1(r-1)]^{-1/(r-1)}$$

provided $N = 1, 2$ or $N \geq 3, 1 < r < \frac{N+2}{N-2}$.

We slightly transform (3.20) into

$$(3.23) \quad \nabla \cdot (\rho \nabla U_1) - \frac{\rho}{r-1} U_1 + M_1 \rho |U_1|^r \geq 0$$

with $\rho(y) = \exp(-\frac{1}{4}|y|^2)$. Multiplying (3.23) by $-U_1$, integrating over \mathbb{R}^N and integrating by parts in the first term, we get

$$(3.24) \quad \int_{\mathbb{R}^N} |\nabla U_1|^2 \rho \, dy + \frac{1}{r-1} \int_{\mathbb{R}^N} |U_1|^2 \rho \, dy - M_1 \int_{\mathbb{R}^N} |U_1|^{r+1} \rho \, dy \leq 0.$$

This yields

$$(3.25) \quad \int_{\mathbb{R}^N} |\nabla U_1|^2 \rho \, dy \leq \int_{\mathbb{R}^N} |U_1|^2 \rho \left(M_1 |U_1|^{r-1} - \frac{1}{r-1} \right) \, dy.$$

Consequently, by $|U_1| \leq \left[\frac{1}{M_1(r-1)}\right]^{1/(r-1)}$ or $|U_1| = 0$ we obtain

$$(3.26) \quad \int_{\mathbb{R}^N} |\nabla U_1|^2 \rho \, dy \leq 0 \quad \text{or} \quad U_1 \equiv 0.$$

Thus U_1 is some constant. Moreover, $U_1 \equiv 0$ or $0 < U_1 \leq K = [M_1(r-1)]^{-1/(r-1)}$ if we assume that r is subcritical. ■

Now we return to the system (3.4). If one considers the asymptotic behavior of U_i as $s \rightarrow \infty$, then the assertion proved for U_1 yields

$$(3.27) \quad U_{is} - d_i \Delta U_i + \frac{y}{2} \nabla U_i + \alpha_i U_i \leq M_i K \sum_{k=1}^{i-1} b_k p_k^i U_i \sum_{k=i}^m b_k p_k^i / b_i, \quad i = 2, \dots, m,$$

where K is given by (3.22).

Therefore, for the problem (3.27) for each i we can repeat the analysis analogous to that shown for (3.6) in the above Propositions and Lemmas, to conclude that $U_i(y, s)$ ($i = 2, \dots, m$) should approach, as $s \rightarrow \infty$, a stationary solution of (3.27) and then, if $1 < \sum_{k=i}^m \frac{b_k}{b_i} p_k^i \leq \frac{N+2}{N-2}$, $N \geq 3$ or $N = 1, 2$, the only global bounded nonnegative solution of (3.27) independent of s is a constant. Moreover, since y is the blow up point, the argument parallel to that presented for the equation in [GK3] excludes the trivial solution. On the other hand, we know that U_i are the solutions of system (3.2). Then $U_i \rightarrow C_{\alpha_i}$ as $s \rightarrow \infty$, where C_{α_i} are constants satisfying (3.2).

It remains to give some thought to the case when initial data do not belong to ∂M . Then one can find \bar{u}_{0i} and \underline{u}_{0i} such that

$$(3.28) \quad u_{0i} \leq \bar{u}_{0i} = \prod_{j=1}^i a_j^{b_i/b_j} (\bar{u}_{01})^{b_i}$$

and

$$(3.29) \quad \underline{u}_{0i} = \prod_{j=1}^i a_j^{b_i/b_j} (\underline{u}_{01})^{b_i} \leq u_{0i}.$$

Clearly, as \bar{u}_i and $\underline{u}_i \in \partial M$, we conclude the same asymptotics properties as above for corresponding \bar{U}_i and \underline{U}_i . Moreover by comparison

$$(3.30) \quad \underline{U}_\infty \leq U_\infty \leq \bar{U}_\infty.$$

In this way we have showed that the solution u of (1.1) is “asymptotically self-similar”. Thus, the proof is complete.

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