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ANALYSIS OF SINGULARITIES AND OF INTEGRABILITY OF ODE'S BY ALGORITHMS OF POWER GEOMETRY

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Abstract. Here we present basic ideas and algorithms of Power Geometry and give a survey of some of its applications. In Section 2, we consider one generic ordinary differential equation and demonstrate how to find asymptotic forms and asymptotic expansions of its solutions. In Section 3, we demonstrate how to find expansions of solutions to Painlevé equations by this method, and we analyze singularities of plane oscillations of a satellite on an elliptic orbit. In Section 4, we consider the problem of local integrability of a planar ODE system. In Section 5, we expound the spacial generalizations of planar constructions. Power Geometry gives alternatives to some methods of Algebraic Geometry, Differential Algebra, Nonstandard Analysis, Microlocal Analysis, Group Analysis and to other algebraic methods in Dynamical Systems.

1. Introduction. Traditional differential calculus is effective for linear and quasilinear problems. It is less effective for essentially nonlinear problems. A linear problem is the first approximation to a quasilinear problem. The linear problem is usually solved by methods of functional analysis, then the solution to the quasilinear problem is found as a perturbation of the solution to the linear problem. For an essentially nonlinear problem, we need to isolate its first approximations, to find their solutions, and to construct perturbations of these solutions. This is what Power Geometry (PG) is aimed at. For equations and systems of equations (algebraic, ordinary differential, and partial differential), PG allows to compute asymptotic forms of solutions as well as asymptotic and local expansions of solutions at infinity and at any singularity of the equation (including boundary layers and singular perturbations) [1, 2].

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Indeed PG is the third level of Differential Calculus. Elements of plane PG were proposed by Newton for algebraic equations (1670); and by Briot and Bouquet for ordinary differential equations (1856). Space PG for a nonlinear autonomous system of ODEs was proposed by the author (1962), and for a linear PDE, by Mikhailov (1963).

In this talk we intend to give basic notions of PG, present some of its algorithms, results, and applications. It is clear that this calculus cannot be mastered during this presentation. The Calculus is subject for one–year course of lectures "Nonlinear Analysis" in Lomonosov Moscow State University.

2. Plane Power Geometry. Theory

2.1. Statement of the problem. First, consider one differential equation and power-logarithmic expansions of its solutions (although there are possible more complex expansions).

Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. A *differential monomial* $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l$, $l \in \mathbb{N}$.

A sum of differential monomials

$$f(x, y) = \sum a_i(x, y) \tag{2.1}$$

is called a *differential sum*.

Let a differential equation be given

$$f(x, y) = 0, \tag{2.2}$$

where $f(x, y)$ is a differential sum. As $x \to 0$, or as $x \to \infty$, for solutions $y = \varphi(x)$ to the equation (2.2), find all expansions of the form

$$y = c_r x^r + \sum c_s x^s, \quad c_r = \text{const} \in \mathbb{C}, \quad c_r \neq 0, \tag{2.3}$$

where c_s are polynomials in $\log x$, and power exponents $r, s \in \mathbb{R}$,

$$\omega r > \omega s, \tag{2.4}$$

and

$$\omega = \begin{cases} -1, & \text{if } x \to 0, \\ 1, & \text{if } x \to \infty. \end{cases} \tag{2.5}$$

The procedure to compute expansions (2.3) consists of two steps: computation of the first approximations

$$y = c_r x^r, \quad c_r \neq 0 \tag{2.6}$$

and computation of further expansion terms in (2.3) [3].

2.2. Computation of truncated equations. To each differential monomial $a(x, y)$, we assign its (vector) *power exponent* $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ by the following rules:

$$Q(cx^{r_1}y^{r_2}) = (r_1, r_2); \quad Q(d^l y/dx^l) = (-l, 1);$$

when differential monomials are multiplied, their power exponents must be added as vectors $Q(a_1 a_2) = Q(a_1) + Q(a_2)$.

The set $\mathbf{S}(f)$ of power exponents $Q(a_i)$ of all differential monomials $a_i(x, y)$ present in the differential sum (2.1) is called the *support* of $f(x, y)$. Obviously, $\mathbf{S}(f) \in \mathbb{R}^2$. The convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polygon* of $f(x, y)$. Its boundary $\partial\Gamma(f)$ consists of the vertices $\Gamma_j^{(0)}$ and the edges $\Gamma_j^{(1)}$. They are called (generalized) *faces* $\Gamma_j^{(d)}$, where the upper index indicates the dimension of the face, and the lower one is its number. Each face $\Gamma_j^{(d)}$ corresponds to the *truncated sum*

$$\hat{f}_j^{(d)}(x, y) = \sum a_i(x, y) \text{ over } Q(a_i) \in \Gamma_j^{(d)} \cap \mathbf{S}(f). \tag{2.7}$$

Example. Consider the third Painlevé equation

$$f(x, y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' + ay^3 + by + cxy^4 + dx = 0, \tag{2.8}$$

assuming the complex parameters $a, b, c, d \neq 0$. Here the first three differential monomials have the same power exponent $Q_1 = (-1, 2)$, then $Q_2 = (0, 3)$, $Q_3 = (0, 1)$, $Q_4 = (1, 4)$, $Q_5 = (1, 0)$. They are shown in Fig. 1 in coordinates q_1, q_2. Their convex hull $\Gamma(f)$ is the

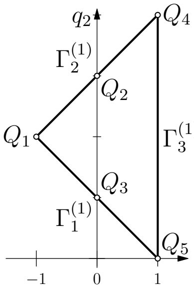

Fig. 1. Support $\mathbf{S}(f)$, polygon $\Gamma(f)$ and its edges $\Gamma_j^{(1)}$ for the third Painlevé equation (2.8)

triangle with three vertices $\Gamma_1^{(0)} = Q_1$, $\Gamma_2^{(0)} = Q_4$, $\Gamma_3^{(0)} = Q_5$, and with three edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$, $\Gamma_3^{(1)}$. The vertex $\Gamma_1^{(0)} = Q_1$ corresponds to the truncation

$$\hat{f}_1^{(0)}(x, y) = -xyy'' + xy'^2 - yy',$$

and the edge $\Gamma_1^{(1)}$ corresponds to the truncation

$$\hat{f}_1^{(1)}(x, y) = \hat{f}_1^{(0)}(x, y) + by + dx.$$

Let the plane $\mathbb{R}_*^2$ be dual to the plane $\mathbb{R}^2$ such that for $P = (p_1, p_2) \in \mathbb{R}_*^2$ and $Q = (q_1, q_2) \in \mathbb{R}^2$, the scalar product

$$\langle P, Q \rangle \stackrel{\text{def}}{=} p_1 q_1 + p_2 q_2$$

is defined. Each face $\Gamma_j^{(d)}$ in $\mathbb{R}_*^2$ corresponds to its own *normal cone* $\mathbf{U}_j^{(d)}$ formed by the outward normal vectors P to the face $\Gamma_j^{(d)}$. For the edge $\Gamma_j^{(1)}$, the normal cone $\mathbf{U}_j^{(1)}$ is the ray orthogonal to the edge $\Gamma_j^{(1)}$ and directed outward the polygon $\Gamma(f)$. For the vertex $\Gamma_j^{(0)}$, the normal cone $\mathbf{U}_j^{(0)}$ is the open sector (angle) in the plane $\mathbb{R}_*^2$ with the vertex at the origin $P = 0$ and limited by the rays which are the normal cones of the edges adjacent to the vertex $\Gamma_j^{(0)}$.

Example. For the the equation (2.8), the normal cones $\mathbf{U}_j^{(d)}$ of the faces $\Gamma_j^{(d)}$ are shown in Fig. 2.

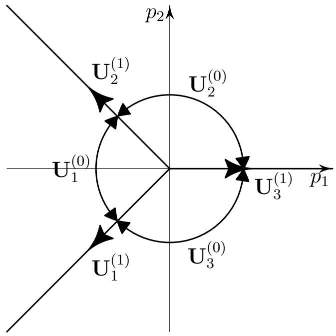

Fig. 2. Normal cones $\mathbf{U}_j^{(d)}$ to the vertices and to the edges $\Gamma_j^{(1)}$ of the polygon in Fig. 1

Thus, each face $\Gamma_j^{(d)}$ corresponds to the normal cone $\mathbf{U}_j^{(d)}$ in the plane $\mathbb{R}_*^2$ and to the truncated equation

$$\hat{f}_j^{(d)}(x, y) = 0. \tag{2.9}$$

Theorem 2.1. *If the expansion* (2.3) *satisfies the equation* (2.2)*, and* $\omega(1, r) \in \mathbf{U}_j^{(d)}$*, then the truncation* $y = c_r x^r$ *of the solution* (2.3) *is the solution to the truncated equation* $\hat{f}_j^{(d)}(x, y) = 0$.

Hence, to find all truncated solutions $y = c_r x^r$ to the equation (2.2), we need to compute: the support $\mathbf{S}(f)$, the polygon $\Gamma(f)$, all its faces $\Gamma_j^{(d)}$, and their normal cones $\mathbf{U}_j^{(d)}$. Then for each truncated equation $\hat{f}_j^{(d)}(x, y) = 0$, we need to find all its solutions $y = c_r x^r$ which have one of the vectors $\pm(1, r)$ lying in the normal cone $\mathbf{U}_j^{(d)}$. The vertex $\Gamma_j^{(0)} = \{Q\}$ corresponds to the truncated equation $\hat{f}_j^{(0)}(x, y) = 0$ the support of which consists of one point $Q = (q_1, q_2)$. Take $g(x, y) = x^{-q_1} y^{-q_2} \hat{f}_j^{(0)}(x, y)$, then $g(x, cx^r)$ does not depend on x and c, and it is a polynomial in r. Consequently, for the solution $y = c_r x^r$ to the equation $\hat{f}_j^{(0)}(x, y) = 0$, the power exponent r is the root of the *characteristic equation*

$$\chi(r) \stackrel{\text{def}}{=} g(x,\, x^r) = 0, \tag{2.10}$$

with an arbitrary coefficient c_r. We need only those roots r of the equation (2.10) for which the vector $\omega(1, r)$ lies in the normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\Gamma_j^{(0)}$.

Example. For the equation (2.8), the vertex $\Gamma_1^{(0)} = Q_1 = (-1, 2)$ corresponds to the truncated equation

$$\hat{f}_1^{(0)}(x, y) \stackrel{\text{def}}{=} -xyy'' + xy'^2 - yy' = 0, \tag{2.11}$$

and $\hat{f}_1^{(0)}(x, x^r) = x^{2r-1}[-r(r-1) + r^2 - r] \equiv 0$, i. e. any expression $y = cx^r$ is a solution to the equation (2.11). Here $\omega = -1$, and we are interested only in those solutions which have the vector $-(1, r) \in \mathbf{U}_1^{(0)}$. According to Fig. 2, this means that $r \in (-1, 1)$. Thus, the vertex $\Gamma_1^{(0)}$ corresponds to the two-parameter family of power asymptotic forms of solutions

$$y = cx^r, \text{ arbitrary } c \neq 0, \quad r \in (-1, 1). \tag{2.12}$$

The edge $\Gamma_j^{(1)}$ corresponds to the truncated equation $\hat{f}_j^{(1)}(x,y)=0$, the normal cone $\mathbf{U}_j^{(1)}$ of the edge is the ray $\{P=\lambda\omega'(1,r'),\lambda>0\}$. The inclusion $\omega(1,r)\in\mathbf{U}_j^{(1)}$ means the equalities $\omega=\omega'$ and $r=r'$. This determines uniquely the power exponent r of the truncated solution $y=c_rx^r$ and the value ω. To determine the coefficient c_r, we need to substitute the expression $y=c_rx^r$ into the truncated equation $\hat{f}_j^{(1)}(x,y)=0$. After cancelation of some power of x, we obtain an algebraic equation for the coefficient c_r

$$\tilde{\hat{f}}(c_r)\stackrel{\text{def}}{=}x^{-s}\hat{f}_j^{(1)}(x,c_rx^r)=0. \tag{2.13}$$

Each root $c_r\neq 0$ of this equation corresponds to its own asymptotic form $y=c_rx^r$.

Example. For the equation (2.8), the edge $\Gamma_1^{(1)}$ corresponds to the truncated equation

$$\hat{f}_1(x,y)\stackrel{\text{def}}{=}-xyy''+xy'^2-yy'+by+dx=0. \tag{2.14}$$

Since $\mathbf{U}_1^{(1)}=\{P=-\lambda(1,1),\lambda>0\}$, then $\omega=-1$ and $r=1$. Substituting $y=c_1x$ into the truncated equation (2.14) and canceling x, we obtain the equation $bc_1+d=0$ for c_1, whence $c_1=-d/b$. Thus, the edge $\Gamma_1^{(1)}$ corresponds to a unique power asymptotic form of solutions

$$y=-(d/b)x. \tag{2.15}$$

2.3. Critical numbers of a truncated solution. If a truncated solution $y=c_rx^r$ is found, then the substitution $y=c_rx^r+z$ reduces the equation $f(x,y)=0$ to the form

$$f(x,cx^r+z)\stackrel{\text{def}}{=}\tilde{f}(x,z)\stackrel{\text{def}}{=}\mathcal{L}(x)z+h(x,z)=0, \tag{2.16}$$

where $\mathcal{L}(x)$ is a linear differential operator, and the support $\mathbf{S}(\mathcal{L}z)$ consists of only one point $(v,1)$ that is the vertex $\tilde{\Gamma}_1^{(0)}$ of the polygon $\Gamma(\tilde{f})$; the point $(v,1)$ is not in the support $\mathbf{S}(h)$. The operator $\mathcal{L}(x)$ is computed as the first variation $\delta\hat{f}_j^{(d)}/\delta y$ on the curve $y=c_rx^r$. Let $\nu(k)$ be the characteristic polynomial of the differential sum $\mathcal{L}(x)z$, i. e.

$$\nu(k)=x^{-v-k}\mathcal{L}(x)x^k. \tag{2.17}$$

The real roots $k_1,\dots,k_\varkappa$ of the polynomial $\nu(k)$ that satisfy the inequality $\omega r>\omega k_i$ are called the *critical numbers of the truncated solution* $y=c_rx^r$.

Example. For the truncated equation (2.11), the first variation is

$$\frac{\delta\hat{f}_1^{(0)}}{\delta y}=-xy''-xy\frac{d^2}{dx^2}+2xy'\frac{d}{dx}-y'-y\frac{d}{dx}.$$

On the curve $y=c_rx^r$, this variation gives the operator

$$\mathcal{L}(x)=c_rx^{r-1}\left[-r(r-1)-x^2\frac{d^2}{dx^2}+2rx\frac{d}{dx}-r-x\frac{d}{dx}\right].$$

The characteristic polynomial of the sum $\mathcal{L}(x)z$, i. e. $\mathcal{L}(x)x^k$, is

$$\nu(k)=c_r[-r(r-1)-k(k-1)+2rk-r-k]=-c_r(k-r)^2.$$

It has one double root $k_1=r$, which is not a critical number, since it does not satisfy the inequality $\omega r>\omega k_1$. Consequently, truncated solutions (2.12) have no critical numbers.

For the truncated equation (2.14), the first variation is

$$\frac{\delta \hat{f}^{(1)}}{\delta y} = \frac{\delta \hat{f}_1^{(0)}}{\delta y} + b.$$

On the curve (2.15), i. e. $y = c_1 x$, $c_1 = -d/b$, this variation gives the operator

$$\mathcal{L}(x) = c_1 \left[-x^2 \frac{d^2}{dx^2} + 2x\frac{d}{dx} - 1 - x\frac{d}{dx} - \frac{b^2}{d} \right]$$

and the characteristic polynomial

$$\nu(k) = -c_1 \left[k^2 - 2k + 1 + b^2/d \right].$$

Its roots are $k_{1,2} = 1 \pm b/\sqrt{-d}$. If $\operatorname{Im}(b/\sqrt{-d}) \neq 0$, then real critical numbers are absent. If $\operatorname{Im}(b/\sqrt{-d}) = 0$, then the inequality $\omega r > \omega k_i$ is satisfied by only one root $k_1 = 1+ \mid b/\sqrt{-d} \mid$ which is a unique critical number of the power asymptotic form (2.15).

2.4. Computation of asymptotic expansion (2.3)**.** Using support $\mathbf{S}(\tilde{f})$ of the equation (2.16) and numbers $k_1, \dots, k_\varkappa$ with $\omega r > \omega k_i$, we can find the set of numbers $\mathbf{K}(k_1, \dots, k_\varkappa) \subset \mathbb{R}$. Its elements s satisfy the inequality $\omega r > \omega s$.

Theorem 2.2. *The equation* (2.16) *has an expansion of solutions of the form*

$$z = \sum c_s(\log x) x^s \text{ over } \mathbf{K}(k_1, \dots, k_\varkappa), \tag{2.18}$$

where $k_1, \dots, k_\varkappa$ are critical numbers of the truncated solution $y = c_r x^r$; c_s are polynomials in $\log x$, which are uniquely defined for $s \neq k_i$. If all critical numbers $k_1, \dots, k_\varkappa$ are simple roots, and each k_i does not lie in the set $\mathbf{K}(k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_\varkappa)$, then all coefficients c_s are constant; for $s \neq k_i$, they are uniquely determined; and for $s = k_i$, they are arbitrary.

Example. For the truncated solution (2.12)

$$\mathbf{K} = \{s = r + l(1-r) + m(1+r), \text{ int. } l, m \geqslant 0, l + m > 0\}. \tag{2.19}$$

Since there are no critical numbers, then all c_s are constant and uniquely determined in the expansion (2.18).

For the truncated solution (2.15) $\mathbf{K} = \{s = 1 + 2l, \text{ int. } l > 0\}$. If $\operatorname{Im}(b/\sqrt{-d}) \neq 0$, then there are no critical numbers, and all power exponents s are odd integers greater than 1 in the expansion (2.18), and coefficients c_s are constant and uniquely determined. If $\operatorname{Im}(b/\sqrt{-d}) = 0$, then there is a unique critical number $k_1 = 1+ \mid b/\sqrt{-d} \mid$, and

$$\mathbf{K}(k_1) = \{s = 1 + 2l + m(k_1 - 1), \text{ int. } l, m \geqslant 0, \ l + m > 0.\} \tag{2.20}$$

Consequently, if the number k_1 is not odd, then all c_s are constant and uniquely determined in the expansion (2.18) for $s \neq k_1$, and c_{k_1} is arbitrary. Finally, if k_1 is odd, then $\mathbf{K}(k_1) = \mathbf{K}$, and c_s is a uniquely determined constant in the expansion (2.18) if $s < k_1$; c_{k_1} is a linear function of $\log x$ with an arbitrary constant term; c_s is a uniquely determined polynomial in $\log x$ if $s > k_1$.

2.5. Non-power asymptotic forms. The truncated equation $\hat{f}_j^{(d)}(x, y) = 0$ may have non-power solutions $y = \varphi(x)$ which are the asymptotic forms for solutions to the original equation $f(x, y) = 0$. These non-power solutions $y = \varphi(x)$ may be found using power and logarithmic transformations. A *power transformation* is linear in logarithms

$$\begin{aligned}\log x &= \alpha_{11}\log u + \alpha_{12}\log v,\\ \log y &= \alpha_{21}\log u + \alpha_{22}\log v,\end{aligned}\qquad \alpha = \begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix},\quad \alpha_{ij}\in\mathbb{R},\quad \det\alpha\neq 0.$$

It induces linear dual transformations in spaces $\mathbb{R}^2$ and $\mathbb{R}^2_*$. The *logarithmic transformation* has the form

$$\xi = \log u \text{ or } \eta = \log v.$$

EXAMPLE. For the truncated equation (2.14) corresponding to the edge $\Gamma_1^{(1)}$ with the normal vector $-(1,1)$, we make power transformation

$$\begin{aligned}\log x &= \log u\\ \log y &= \log u + \log v,\end{aligned}\qquad \alpha = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$

i. e. $x = u$, $y = uv$. Since $y' = xv' + v$, $y'' = xv'' + 2v'$, canceling x and collecting similar terms, the equation (2.14) takes the form

$$-x^2vv'' + x^2v'^2 - xvv' + bv + d = 0. \tag{2.21}$$

Its support consists of three points $\widetilde{Q}_1 = (0,2)$, $\widetilde{Q}_2 = (0,1)$, $\widetilde{Q}_3 = 0$ placed on the axis $\tilde{q}_1 = 0$. Now we make the logarithmic transformation $\xi = \log x$. Since $v' = \dot{v}/x$, $v'' = (\ddot{v} - \dot{v})/x^2$, where $\dot{} = d/d\xi$, collecting similar terms, the equation (2.21) takes the form

$$-v\ddot{v} + \dot{v}^2 + bv + d = 0.$$

Applying the technique described below to this equation, we obtain the expansion of its solutions

$$v = -(b/2)\xi^2 + \tilde{c}\xi + \sum_{k=0}^{\infty} c_k\xi^{-k},$$

where $\tilde{c}$ is an arbitrary constant, and the constants c_k are uniquely determined. In original variables, we obtain the family of non-power asymptotic forms of solutions to the original equation (2.8)

$$y \sim x\left[-(b/2)(\log x)^2 + \tilde{c}\log x + \sum_{k=0}^{\infty} c_k(\log x)^{-k}\right],\quad x\to 0.$$

They give the complicated expansions (2.3), where c_r and c_s are power series in $\log x$ (so called Psi series).

2.6. Complex power exponents. Expansions of solutions (2.3) with complex power exponents r and s, where $\omega\,\mathrm{Re}\,r > \omega\,\mathrm{Re}\,s$, are found in a similar way.

EXAMPLE. In the equation (2.8), for the truncated solution (2.12) with complex r, $\mathrm{Re}\,r \in (-1,1)$, the expansions (2.18) are also found by the set (2.19). And for the truncated solution (2.15) with $\mathrm{Im}(b/\sqrt{-d}) \neq 0$ and $\mathrm{Re}\,k_1 > 1$, we obtain the expansion (2.18) by the set (2.20).

Thus, in classical analysis, we encounter expansions in fractional powers and with constant coefficients, but here we obtain expansions in rather arbitrary complex powers of the independent variable with coefficients that are polynomials in logarithms of this variable. However, there are possible even more complicated expansions of solutions.

2.7. Types of expansions. As $x \to 0$, consider asymptotic expansions of solutions to the equation (2.2) of the form

$$y = c_r x^r + \sum_s c_s x^s, \tag{2.22}$$

where power exponents r and s are complex numbers without points of accumulation, $\operatorname{Re} s \geqslant \operatorname{Re} r$, $\operatorname{Re} s$ increase.

We define four types of expansions (2.22):

Type 1. c_r and c_s are constant (*power expansions*) [3];
Type 2. c_r is constant, c_s are polynomials in $\log x$ (*power-logarithmic expansions*) [3];
Type 3. c_r and c_s are power series in decreasing powers of $\log x$ (*complicated expansions*) [4];
Type 4. r and s are real ($r, s \in \mathbb{R}$), c_r and c_s are power series in x^i (*exotic expansions*) [5];
Type 5.

$$y = \sum_{k=0}^{\infty} b_k(x) C^k \exp(k\varphi(x)),$$

where $b_k(x)$ and $\varphi(x)$ are power series, and C is an arbitrary constant.

Similar technique is used for equations having small or big parameters. The power exponents of these parameters are accounted for in the same way as power exponents of variables tending to zero or infinity. Such parameter ε can be considered as a dependent variable, satisfying the equation $\varepsilon' = 0$.

2.8. Algorithms of Power Geometry. Thus, now we have the following algorithms of Power Geometry.

1. Computation of truncated equations and accompanying objects.
2. Solution of truncated equations.
3. Power transformations.
4. Logarithmic transformations.
5. Introducing independent variable x^i instead of x.
6. Computation of the first variation of a sum.
7. Computation of expansions of solutions to the initial equation, beginning by solutions to a truncated equation.

All these algorithms, except 4 and 5, can be applied to resolve algebraic equations.

3. Plain Power Geometry. Applications

3.1. The sixth Painlevé equation. It has the form

$$\begin{aligned} y'' = {} & \frac{(y')^2}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) - y'\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2}\right], \end{aligned} \tag{3.1}$$

where a, b, c, d are complex parameters, x and y are complex variables, $y' = dy/dx$. The equation (3.1) has three singular points $x = 0$, $x = 1$, and $x = \infty$. After multiplying by common denominator, we obtain the equation as a differential sum. Its support and its polygon, in the case $a \neq 0$, $b \neq 0$, are shown in Fig. 3. We found all asymptotic

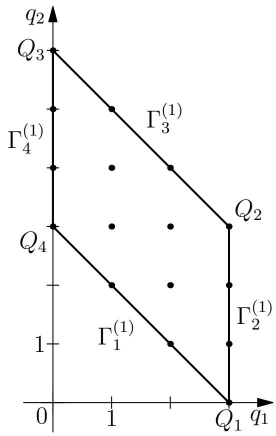

Fig. 3. The support and the polygon of the sixth Painlevé equation multiplying on its common denominator $x^2(x-1)^2y(y-1)(y-x)$ when $a \cdot b \neq 0$

expansions of solutions to the equation (3.1) near its three singular points. They comprise 117 families [6]. Among them, there are expansions of all first four types. In particular, for $a = 1$ and $c = 0$, there is an expansion of the fourth type of the form

$$y = -\frac{1}{\cos[\log(C_1 x)]} + \sum_{\operatorname{Re} s \geqslant 1} c_s x^s, \tag{3.2}$$

where C_1 is an arbitrary constant, the coefficients c_s are constants and uniquely determined. Here

$$\frac{1}{\cos[\log(x)]} = \frac{1}{x^i + x^{-i}} = x^i \sum_{k=0}^{\infty} \left(-x^{-2i}\right)^k = x^{-i} \sum_{k=0}^{\infty} \left(-x^{-2i}\right)^k.$$

For $C_1 = 1$ and real $x > 0$, the solution (2.2) has infinitely many poles accumulating at the point $x = 0$.

We found also all expansions of solutions to the equation (3.1) near its nonsingular points. They comprise 17 families [7].

3.2. The Beletsky equation. The Beletsky equation (1956)

$$(1 + e\cos\nu)\delta'' - 2e\sin\nu\delta' + \mu\sin\delta = 4e\sin\nu \tag{3.3}$$

describes plane motions of a satellite around its mass center which is moving along an elliptic orbit with an eccentricity $e = \text{const} \in [0, 1]$. In the equation, ν is the independent and δ is dependent variables, inertial parameter $\mu = \text{const} \in [-3, 3]$. The equation (3.3) is singular at $e = 1$, $\nu = \pi$, since the coefficient at the higher derivative vanishes at this point. We introduce local coordinates $x = \nu - \pi$ and $\varepsilon = 1 - e$ at the singularity. Then the equation (3.3) takes the form

$$\left[\varepsilon + \frac{1}{2}x^2 + o(x^2, \varepsilon)\right] \frac{d^2\delta}{dx^2} + 2\,[x + o(x, \varepsilon)]\,\frac{d\delta}{dx} + \mu \sin\delta = -4[x + o(x, \varepsilon)]. \tag{3.4}$$

The support and the polygon of this equation for small coordinates x, ε is shown in Fig. 4. The boundary of the polygon Γ consists of three edges and two vertices. The unit vector along the edge $\Gamma_1^{(1)}$ is $(1, 0)$, which corresponds to the variable x. The unit vector along the edge $\Gamma_2(1)$ is $(1, -1/2)$, which corresponds to the variable $x/\sqrt{\varepsilon}$. Using a variable with this type of behavior, we can regularize the equation (3.3) at the singularity and compute its solutions as relaxation oscillations. In [8], we studied the limit equations

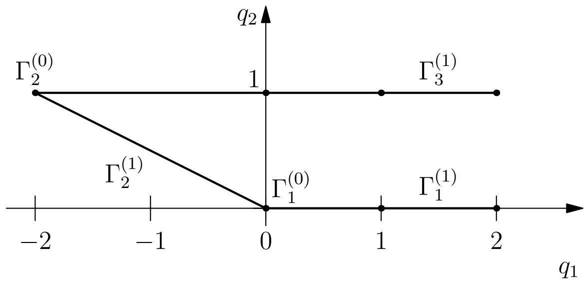

Fig. 4. The support and the polygon of the Beletsky equation near the singularity (3.4)

corresponding to the vertex $\Gamma_1^{(0)}$ and to the edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$. Using their solutions, the limits of solutions to the equation (3.3) are matched as $e \to 1$. We found that for $e = 1$, the limit families of 2π-periodic solutions form a complicated structure: the family of symmetric solutions is twisted into the spiral with infinite number of revolutions around the solution $\mathbf{C} = \{\delta = -\nu, \mu = -2\}$ and each convolution of the spiral corresponds to its own family of asymmetric 2π-periodic solutions having 4 spirals [9]. Apparently, the solution $\mathbf{C}$ is an accumulating point of infinitely many families of 2π-periodic solutions and of infinitely many their spirals [10].

4. Integrability of a planar ODE system near a degenerate stationary point

4.1. Introduction. We consider an autonomous system of ordinary differential equations

$$dx_i/dt \stackrel{\text{def}}{=} \dot{x}_i = \varphi_i(X), \quad i = 1, 2, \tag{4.1}$$

where $X = (x_1, x_2) \in \mathbb{C}^2$ and $\varphi_i(X)$ are polynomials.

In a neighborhood of the stationary point $X = X^0$, the system (4.1) is *locally integrable* if it has there first integral of the form

$$a(X)/b(X),$$

where functions $a(X)$ and $b(X)$ are analytic in a neighborhood of the point $X = X^0$. Otherwise we call the system (4.1) *locally nonintegrable* in this neighborhood. In the neighborhood of the stationary point $X = 0$ the system (4.1) can be written in the form

$$\dot{X} = A\,X + \widetilde{\Phi}(X), \tag{4.2}$$

where $\widetilde{\Phi}(X)$ has no terms linear in X.

Let λ_1, λ_2 be eigenvalues of the matrix A. If at least one of them $\lambda_i \neq 0$, then the stationary point $X = 0$ is called an *elementary* stationary point. In this case the system (4.1) has a normal form which is equivalent to a system of lower order [1].

THEOREM 4.1. *Rationality of the ratio λ_1/λ_2 and the condition* **A** *(see below) are necessary and sufficient conditions for local integrability of a system near an elementary stationary point.*

If all eigenvalues vanish, then the stationary point $X = 0$ is called a *nonelementary* stationary point. In this case there is no normal form for the system (4.1). But by using power transformations, a nonelementary stationary point $X = 0$ can be blown up to a set of elementary stationary points. After that, it is possible to compute the normal form and verify that the condition A is satisfied [1] in each elementary stationary point.

For local integrability of original system (4.1) near a degenerate (nonelementary) stationary point, it is necessary and sufficient to have local integrability near each of elementary stationary points, which are produced by the blowing up process described above [1, Ch. II, Sec. 3]. A space generalization see in [24].

4.2. Normal form and condition A. Let the linear transformation

$$X = BY \tag{4.3}$$

bring the matrix A to the Jordan form $J = B^{-1}AB$ and (4.2) to

$$\dot{Y} = JY + \widetilde{\widetilde{\Phi}}(Y). \tag{4.4}$$

Let the formal change of coordinates

$$Y = Z + \Xi(Z), \tag{4.5}$$

where $\Xi = (\xi_1, \ldots, \xi_n)$ and $\xi_j(Z)$ are formal power series, transform (4.4) into the system

$$\dot{Z} = JZ + \Psi(Z). \tag{4.6}$$

We write it in the form

$$\dot{z}_j = z_j g_i(Z) = z_j \sum g_{jQ} Z^Q \text{ over } Q \in \mathbb{N}_j,\ j = 1, 2, \tag{4.7}$$

where $Q = (q_1, q_2)$, $Z^Q = z_1^{q_1} z_2^{q_2}$, $\mathbb{N}_j = \{Q : Q \in \mathbb{Z}^2, Q + E_j \geqslant 0\}$, $j = 1, 2$, E_j means the unit vector. The diagonal $\Lambda = (\lambda_1, \lambda_2)$ of J consists of eigenvalues of the matrix A. System (4.6), (4.7) is called the *resonant normal form* if:

a) J is the Jordan matrix,
b) in (4.7), there are only the *resonant terms*, for which the scalar product

$$\langle Q, \Lambda \rangle \stackrel{\text{def}}{=} q_1\lambda_1 + q_2\lambda_2 = 0. \tag{4.8}$$

THEOREM 4.2 (Bruno [1]). *There exists a formal change* (4.5) *reducing* (4.4) *to its normal form* (4.6), (4.7).

In [1] there are conditions on the normal form (4.7), which guarantee the convergence of the normalizing transformation (4.5).

CONDITION **A**. *In the normal form* (4.7)

$$g_j = \lambda_j \alpha(Z) + \bar{\lambda}_j \beta(Z), \quad j = 1, 2,$$

where $\alpha(Z)$ *and* $\beta(Z)$ *are some power series.*

4.3. The simplest nontrivial example

EXAMPLE. We consider the system

$$\begin{aligned} dx/dt &= -y^3 - bx^3y + a_0x^5 + a_1x^2y^2, \\ dy/dt &= (1/b)x^2y^2 + x^5 + b_0x^4y + b_1xy^3, \end{aligned} \tag{4.9}$$

with arbitrary complex parameters a_i, b_i and $b \neq 0$.

After the power transformation

$$x = uv^2, \quad y = uv^3 \tag{4.10}$$

and time rescaling

$$dt = u^2v^7 d\tau,$$

we obtain the system (4.9) in the form

$$\begin{aligned} du/d\tau &= -3u - [3b + 2/b]\, u^2 - 2u^3 + (3a_1 - 2b_1)u^2v + (3a_0 - 2b_0)u^3v, \\ dv/d\tau &= v + [b + 1/b]\, uv + u^2v + (b_1 - a_1)uv^2 + (b_0 - a_0)u^2v^2. \end{aligned} \tag{4.11}$$

Under the power transformation (4.10), the point $x = y = 0$ blows up into two straight lines $u = 0$ and $v = 0$. Along the line $u = 0$ the system (4.11) has a single stationary point $u = v = 0$. Along the second line $v = 0$ this system has three elementary stationary points

$$u = 0, \quad u = -1/b, \quad u = -3b/2. \tag{4.12}$$

In these points we computed normal forms of the system (4.11). If $b^2 \neq 2/3$, the condition **A** is fulfilled for all normal forms in the four cases.

$$\begin{array}{lllll} 1) & a_0 = 0, & a_1 = -b_0\, b, & b_1 = 0, & b^2 \neq 2/3, \\ 2) & b_1 = -2\, a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3, \\ 3) & b_1 = (3/2)\, a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3, \\ 4) & b_1 = (8/3)\, a_1, & a_0 = a_1 b, & b_0 = b_1 b, & b^2 \neq 2/3 \end{array}$$

In each of the cases analytical first integrals of system (4.11) were found. Thus if $b^2 \neq 2/3$ the system (4.9) is locally integrable only in these four cases [11].

5. Space Power Geometry

5.1. Theory. Let $X \in \mathbb{C}^m$ be independent and $Y \in \mathbb{C}^n$ be dependent variables. Suppose $Z = (X, Y) \in \mathbb{C}^{m+n}$. A *differential monomial* $a(Z)$ is the product of an ordinary monomial $cZ^R = cz_1^{r_1} \cdots z_{m+n}^{r_{m+n}}$, where $c = \text{const} \in \mathbb{C}$, $R = (r_1, \dots, r_{m+n}) \in \mathbb{R}^{m+n}$, and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1} \cdots \partial x_m^{l_m}} \stackrel{\text{def}}{=} \frac{\partial^l y_j}{\partial X^L}, \quad l_j \geqslant 0, \quad \sum_{j=1}^{m} l_j = l, \quad L = (l_1, \dots, l_m).$$

A differential monomial $a(X)$ corresponds to its vector *power exponent* $Q(a) \in \mathbb{R}^{m+n}$ formed by the following rules

$$Q(cZ^R) = R, \quad Q(\partial^l y_j / \partial X^L) = (-L, E_j),$$

where E_j is unit vector. A product of monomials $a \cdot b$ corresponds to the sum of their vector power exponents:

$$Q(ab) = Q(a) + Q(b).$$

A *differential sum* is a sum of differential monomials

$$f(Z) = \sum a_k(Z).$$

A set $\mathbf{S}(f)$ of vector power exponents $Q(a_k)$ is called the *support* of the sum $f(Z)$. The closure of the convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the *polyhedron of the sum* $f(Z)$. Consider a system of equations

$$f_i(X, Y) = 0, \quad i = 1, \dots, n, \tag{5.1}$$

where f_i are differential sums. Each equation $f_i = 0$ corresponds to: its support $\mathbf{S}(f_i)$, its polyhedron $\Gamma(f_i)$ with the set of faces $\Gamma_{ij}^{(d_i)}$ in the main space $\mathbb{R}^{m+n}$, the set of their normal cones $\mathbf{U}_{ij}^{(d_i)}$ in the dual space $\mathbb{R}_*^{m+n}$, and the set of truncated equations $\hat{f}_{ij}^{(d_i)}(X, Y) = 0$. The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X, Y) = 0, \quad i = 1, \dots, n \tag{5.2}$$

is the *truncated system* if the intersection

$$\mathbf{U}_{1j_1}^{(d_1)} \cap \cdots \cap \mathbf{U}_{nj_n}^{(d_n)} \tag{5.3}$$

is not empty. A solution

$$y_i = \varphi_i(X), \quad i = 1, \dots, n$$

to the system (5.1) is associated to its normal cone $\mathbf{u} \subset \mathbb{R}^{m+n}$. If the normal cone $\mathbf{u}$ intersects with the cone (5.3), then the asymptotic form $y_i = \hat{\varphi}_i(X)$, $i = 1, \dots, n$ of this solution satisfies the truncated system (5.2), which is quasihomogeneous [2, 12].

5.2. The Euler-Poisson equations

$$\begin{aligned} Ap' + (C - B)qr &= Mg(y_0\gamma_3 - z_0\gamma_2), \quad {\gamma'}_1 = r\gamma_2 - q\gamma_3, \\ Bq' + (A - C)pr &= Mg(z_0\gamma_1 - x_0\gamma_3), \quad {\gamma'}_2 = p\gamma_3 - r\gamma_1, \\ Cr' + (B - A)pq &= Mg(x_0\gamma_2 - y_0\gamma_1), \quad {\gamma'}_3 = q\gamma_1 - p\gamma_2, \end{aligned} \tag{5.4}$$

where $\,' = d/dt$, describes the motion of a rigid body with a fixed point. In (5.4), A, B, C, x_0, y_0, z_0, and Mg are real constants. The system (5.4) has three general first integrals. In the case $B \neq C$, $x_0 \neq 0$, $y_0 = z_0 = 0$ N. Kowalewski (1908) reduced the system (5.4) to the system of two equations

$$
\begin{aligned}
f_1 &\stackrel{\mathrm{def}}{=} \ddot{\sigma}\tau + \dot{\sigma}\dot{\tau}/2 + a_1 + a_2\sigma + a_3\dot{\tau}p + a_4\tau + a_5p^2 = 0,\\
f_2 &\stackrel{\mathrm{def}}{=} \sigma\ddot{\tau} + \dot{\sigma}\dot{\tau}/2 + b_1 + b_2\dot{\sigma}p + b_3\sigma + b_4\tau + b_5p^2 = 0,
\end{aligned}
\tag{5.5}
$$

where the dot means differentiation with respect to the new independent variable p, σ and τ are new dependent variables, $a_i, b_i = \mathrm{const}$.

This system has two general first integrals. Generically, the supports $\mathbf{S}(f_i)$ and polyhedrons $\Gamma(f_i)$ of both equations (5.5) coincide; they are shown in Fig. 5.

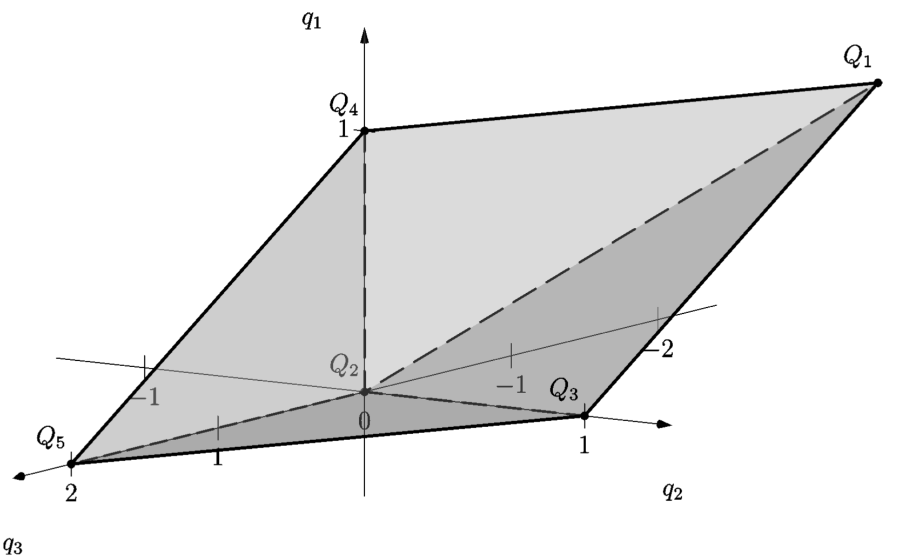

Fig. 5. Support $\mathbf{S}(f_i)$ and polyhedron $\Gamma(f_i)$ for each equation (5.5) $(i = 1, 2)$ in general case

We found all power-logarithmic expansions of solutions to the system (5.5) as $p \to 0$ and as $p \to \infty$ (they comprise 24 families) and 4 families of complicated expansions of solutions [13]. Solutions to this system do not have expansions of the 4-th type. Using power expansions we obtained all exact solutions of the form of finite sums of real powers of the variable p with complex coefficients. They comprise 12 families. Among them, 7 families were known. All new families are complex.

In the case

$$A = B, \quad Mgx_0/B = 1, \quad y_0 = z_0 = 0, \quad C/B = c$$

the system (5.4) has a unique parameter $c \in (0, 2]$. The system (5.4) has 4 two-parameter families of stationary solutions. On each of these families there are sets D_j of real stationary solutions near which the system (5.4) is locally integrable as well as the sets R_j of stationary solutions near which the system (5.4) is locally nonintegrable. In Fig. 6, there are shown the sets D_1, D_2, D_3 and the curves R_1–R_4 for one of these four families with $x = 1/c$ and $y = p^0\gamma_1^0$, where $(p, q, r, \gamma_1, \gamma_2, \gamma_3) = (p^0, 0, 0, \pm 1, 0, 0)$ is a stationary solution [13].

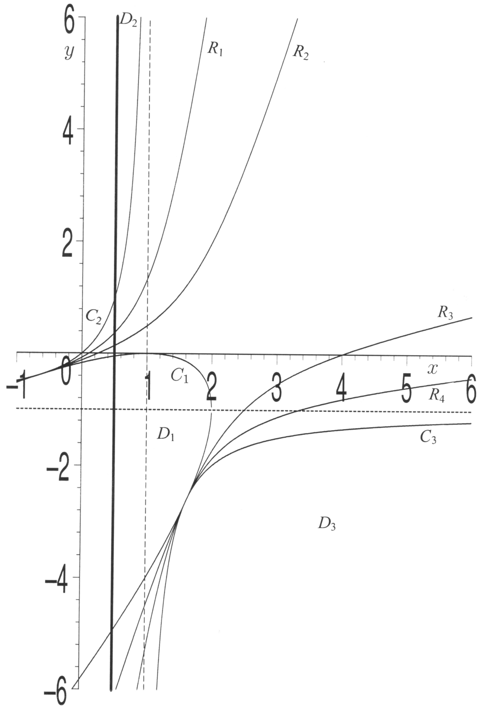

Fig. 6

5.3. Other applications of Power Geometry. Up till now Power Geometry has been applied to the following problems.

1. Asymptotic forms of solutions to Painlevé equations [14–16].
2. Periodic solutions of the restricted three-body problem [17–21].
3. Analysis of the local integrability in space problems of ODE's [22–24].
4. Boundary layer on a needle [25].
5. Evolution of the turbulent flow [12].

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