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ALGEBRAIC FOLIATIONS DEFINED BY COMPLETE VECTOR FIELDS

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Abstract. The aim of this note is to give a clearer and more direct proof of the main result of another paper of the author. Moreover, we give some complementary results related to R-complete algebraic foliations with R a rational function of type $\mathbb{C}^*$.

1. Introduction

1.1. Vector fields [7]. A vector field X on $\mathbb{C}^2$ is a section of the tangent bundle of $\mathbb{C}^2$

$$X = R\,\frac{\partial}{\partial x} + S\,\frac{\partial}{\partial y}, \qquad R, S \in \mathcal{O}_{\mathbb{C}^2}.$$

Associated to X we have the following system:

$$\begin{cases} \dot{x}(t) = R(x, y), \\ \dot{y}(t) = S(x, y). \end{cases} \tag{1}$$

According to the theorem on existence and uniqueness of local solutions of complex differential equations, for a fixed initial condition $z = (x, y) \in \mathbb{C}^2$, there exist a disk $\mathbb{D}_{r_z}$ of center zero and radius r_z and a holomorphic function $t \in \mathbb{D}_{r_z} \mapsto \varphi_z(t)$ that satisfies (1) with $\varphi_z(0) = z$. Given $t \mapsto \varphi_z(t)$, we can extend it by analytic continuation along the paths from zero to the points outside $\mathbb{D}_{r_z}$ to the maximal domain of definition Ω_z (Riemann domain spread over $\mathbb{C}$). This map $\varphi_z : \Omega_z \to \mathbb{C}^2$ is the solution of X through z and its image C_z defines the trajectory of X through z. A trajectory C_z is said to be proper if its topological closure in $\mathbb{C}^2$ defines an analytic curve of pure dimension one. The vector field X is complete if for any $z \in \mathbb{C}^2$ the solution φ_z is an entire map. In this case $(t, z) \mapsto \varphi(t, z) = \varphi_z(t)$ defines a holomorphic action of $(\mathbb{C}, +)$ on $\mathbb{C}^2$ by global

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holomorphic automorphisms. The map φ is the (global) flow of X. If for any $t \in \mathbb{C}$, $z \mapsto \varphi(t,z)$ is a polynomial automorphism of $\mathbb{C}^2$ the flow φ is said to be algebraic. If all the trajectories of X are proper the flow φ is said to be proper.

1.2. Algebraic foliations on $\mathbb{C}^2$ [1, Chapter 2]. Let X be a polynomial vector field of degree m. Let us consider the atlas $\{(U_i, \phi_i^{-1})\}_{i=0,1,2}$ of $\mathbb{CP}^2$ defined by open sets $U_i := \{[z_0 : z_1 : z_2], z_i \neq 0\}$ and homeomorphisms $\phi_0(z_1, z_2) = [1 : z_1 : z_2]$, $\phi_1(y_1, y_2) = [y_1 : 1 : y_2]$ and $\phi_2(w_1, w_2) = [w_1 : w_2 : 1]$. The vector field X defines a rational vector field on $\mathbb{CP}^2$ given by $(\phi_i^{-1} \circ \phi_0)_* X$ in each chart (U_i, ϕ_i^{-1}). The pole of X along the line at infinity L_∞ is of order $d = m-1$ or $m-2$. If we remove it we obtain on each (U_i, ϕ_i^{-1}) a polynomial vector field X_i with isolated zeroes. These vector fields $\{X_i\}_{i=0,1,2}$ define a global section $\mathcal{F}_X$ of $\mathcal{O}(d) \otimes T\mathbb{CP}^2$, for $\mathcal{O}(d)$ the line bundle of $\mathbb{CP}^2$ of degree d, which is the foliation defined by X (modulo multiplication by a non-zero complex number). The singular set $Sing\,(\mathcal{F}_X)$ of $\mathcal{F}_X$ is the set of singularities of X_i. A singular point $p \in Sing\,(\mathcal{F}_X)$ is reduced if $\mathcal{F}_X$ around p is generated by a vector field whose first jet at p has eigenvalues λ_1 and λ_2 such that either $\lambda_1 \neq 0 \neq \lambda_2$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$, or $\lambda_1 \neq 0 = \lambda_2$.

There is a foliation $\tilde{\mathcal{F}}$ defined on a rational surface M after pulling back $\mathcal{F}_X$ by a birational morphism $\pi : M \to \mathbb{CP}^2$, that is a finite composition of blowing ups, with reduced singularities only (Seidenberg's Theorem).

Associated to this resolution one has:

(a) the Zariski open set $U = \pi^{-1}(\mathbb{C}^2)$ of M. Note that X can be lifted to it as a holomorphic vector field,
(b) the exceptional divisor E of U, and
(c) the divisor at infinity

$$D = M \setminus U = \pi^{-1}(\mathbb{CP}^2 \setminus \mathbb{C}^2) = \pi^{-1}(L_\infty),$$

that is a tree of a smooth rational curves.

1.3. Results of M. Suzuki [10], [11], [12]. Let us recall some important facts about complete vector fields X on $\mathbb{C}^2$:

(I) The trajectories of X are isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$.
(II) There exists a set $E \subset \mathbb{C}^2$ invariant by X of logarithmic capacity zero such that for any $z \in \mathbb{C}^2 \setminus E$, the trajectory C_z is always of the same type. Thus X is either of type $\mathbb{C}$ or $\mathbb{C}^*$, depending on the type of its generic trajectory.
(III) A trajectory of X of type $\mathbb{C}^*$ is proper.
(IV) If X is of type $\mathbb{C}^*$ it defines a proper flow and it has a meromorphic first integral.

Suzuki's classification. M. Suzuki in [10] classified $\mathbb{C}^2$ algebraic flows and proper flows, modulo holomorphic automorphisms. The vector fields X of the two classifications together are of the form:

1) $\quad [a(x)y + b(x)]\frac{\partial}{\partial y}, \quad a(x), b(x) \in \mathbb{C}(x)$
2) $\quad \lambda x\frac{\partial}{\partial x} + \mu y\frac{\partial}{\partial y}, \quad \lambda, \mu \in \mathbb{C}$
3) $\quad \lambda x\frac{\partial}{\partial x} + (\lambda m y + x^m)\frac{\partial}{\partial y}, \quad \lambda \in \mathbb{C}^*$, $m \in \mathbb{N}$

4) $\frac{\alpha(z)}{x^\ell} \cdot \left\{ n x^{\ell+1} \frac{\partial}{\partial x} - [(m+n\ell)x^\ell y + mp(x) + nx\dot{p}(x)] \frac{\partial}{\partial y} \right\}$,

where $m, n \in \mathbb{N}^*$, p is a polynomial whose degree is not greater than $\ell - 1$ with $p(0) \neq 0$ when $\ell > 0$ or $p \equiv 0$ otherwise, and $\alpha \in \mathbb{C}(z)$ $(z = x^m(x^\ell y + p(x))^n)$ with a zero of order $\geq \ell/m$ at $z = 0$.

Proper flows are defined by vector fields of 1), 2) if $\lambda/\mu \in \mathbb{Q}$, 3) with $m = 0$, and 4). This implies that there is a rational first integral of X, modulo holomorphic automorphism, of the form x, y^p/x^q $(p/q = \lambda/\mu \in \mathbb{Q})$ or $x^m(x^\ell y + p(x))^n$.

1.4. Questions. According to Suzuki's classification a complete holomorphic vector field has a proper flow if and only if it has a rational first integral of one of the above three types, modulo holomorphic automorphism. Therefore, if X is in Suzuki's list it is of the form $f \cdot Y$, with Y a polynomial vector field and $f \in \mathcal{O}_{\mathbb{C}^2}$, and the foliation generated by X is the algebraic foliation $\mathcal{F}_Y$. It is natural to try to answer the following questions:

- Of what form are the complete vector fields on $\mathbb{C}^2$ that define an algebraic foliation? Or in other words, what can be said about the vector fields of the form $f \cdot Y$ where Y is a polynomial vector field and $f \in \mathcal{O}_{\mathbb{C}^2}$?
- Do they define other complete vector fields different from those in Suzuki's list?
- Do they define other complete vector fields until now unknown?

We can make a simplification and assume that f is transcendental by Brunella's classification of complete polynomial vector fields. The result that answers the above questions is [4, Theorem 1.1].

Theorem. *Let X be a complete vector field on $\mathbb{C}^2$ of the form $f \cdot Y$, where Y is a polynomial vector field and f is a transcendental function. Then X defines a proper flow and, up to a holomorphic automorphism, X is in Suzuki's list.*

2. Proof

2.1. Assumptions. If $X = f \cdot Y$, we will denote the foliation $\mathcal{F}_Y$ by $\mathcal{F}$. Let $\tilde{\mathcal{F}}$ be its resolution $\pi^*\mathcal{F}$ on M, and E and D its divisors.

We may assume that $\tilde{\mathcal{F}}$ has no rational first integrals and that X is of type $\mathbb{C}$ (see (IV) of §1.3). Then Y is of type $\mathbb{C}$ because $\{f = 0\}$ is $\emptyset$ or an invariant set by Y. In this situation E and D are $\tilde{\mathcal{F}}$-invariant.

On the other hand, $\tilde{\mathcal{F}}$ admits lots of tangent entire curves; most of them are Zariski dense in M (Darboux's Theorem). This implies that the *Kodaira dimension* $\mathrm{kod}(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is either 0 or 1 [8, §IV] (*see* also [1, p. 131]).

2.2. $\mathrm{kod}(\tilde{\mathcal{F}}) = 1$**.** According to [8, §IV] the absence of a first integral implies that $\tilde{\mathcal{F}}$ is *a Riccati or a turbulent foliation,* that is to say, the existence of a fibration

$$g : M \to B$$

whose generic fibre is a rational curve or an elliptic curve transverse to $\tilde{\mathcal{F}}$, respectively. *Remark that B is $\mathbb{CP}^1$ since M is a rational surface.*

Lemma 1. *$\tilde{\mathcal{F}}$ is a Riccati foliation.*

Proof. Let us suppose that $\tilde{\mathcal{F}}$ is turbulent. There is a component $D_0 \subset D$ transversal to the generic fibre $\mathcal{G}_0$ of g. Otherwise we have an elliptic curve contained in $\mathbb{C}^2$, which is impossible ($\mathbb{C}^2$ is Stein). As D_0 is $\tilde{\mathcal{F}}$-invariant, one can construct a rational first integral as pointed out in [2, Lemma1]. ■

LEMMA 2. *$g_{|U}$ is projected by π as a rational function R of type $\mathbb{C}$ or $\mathbb{C}^*$.*

Proof. Up to contraction of rational curves inside fibers of g, which can produce cyclic quotient singularities of the surface but on which the foliation is always regular, there are five possible models for the fibers of g [3, § 7], [2, p. 439]. Let L_0 be the leaf of the foliation defined by a trajectory $\tilde{C}_z$ of X transversal to g. One can conclude that the orbifold universal covering $\tilde{L}_0$ of L_0 is equal to the one of B_0, $\tilde{B}_0$, where B_0 is defined as $\mathbb{CP}^1$ minus the points over tangent fibres of g with the natural orbifold structure inherited from the orbifold structure on $\mathbb{CP}^1$ induced by (the local models of) g. Since X is complete on C_z, $\tilde{L}_0$ is biholomorphic to $\mathbb{C}$ and then L_0 is parabolic. Then $\mathrm{kod}(\tilde{\mathcal{F}}) = 1$ implies by [2, Lemma 2] that there must be at least one fibre $\mathcal{G}_0$ tangent to the foliation of one of the following classes:

(d): the fibre is rational with two saddle-nodes of the same multiplicity m, with strong separatrices inside the fibre, or
(e): the fibre is rational with two quotient singularities of order 2, and a saddle-node of multiplicity l, with strong separatrix inside the fibre.

The components of $D \cup E$ which are not contained in fibers of g define separatrices through singularities of $\tilde{\mathcal{F}}$. Then $\mathcal{G}_0$ must cut $D \cup E$ in at most one or two points. Therefore $R = g \circ \pi^{-1}$ is of type $\mathbb{C}$ or $\mathbb{C}^*$. ■

Analogously to the polynomial case one can define as in [2] that $\mathcal{F}$ is *R-complete* if there exists a finite set $\mathcal{Q} \subset \mathbb{CP}^1$ such that for all $t \notin \mathcal{Q}$: (i) $R^{-1}(t)$ is transverse to $\mathcal{F}$, and (ii) there is a neighbourhood U_t of t in $\mathbb{CP}^1$ such that $R : R^{-1}(U_t) \to U_t$ induces a holomorphic fibration on M and the restriction of $\mathcal{F}$ to $R^{-1}(U_t)$ defines a local trivialization of this fibration on M.

LEMMA 3. *$\mathcal{F}$ is R-complete.*

Proof. By the proof of Lemma 2 it is enough to note that around $\mathcal{G}_0$, removing one or two $\tilde{\mathcal{F}}$-invariant disks contained in $D \cup E$, R is a fibration trivialized by the leaves of $\mathcal{F}$. ■

2.2.1. *R of type* $\mathbb{C}$. Up to a polynomial automorphism $R = x$ (see [9]). As $\mathcal{F}$ is x-complete, Y extends to $\mathbb{CP}^1 \times \mathbb{CP}^1$ as holomorphic vector field leaving $\mathbb{CP}^1 \times \infty$ invariant. In particular

$$Y = a(x)\frac{\partial}{\partial x} + [b(x)y + c(x)]\frac{\partial}{\partial y}.$$

with a, b and $c \in \mathbb{C}[x]$. As the solutions of Y can only avoid at most one vertical line by Picard's Theorem, $a(x) = \lambda\, x^N$, $N \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Let us take $\varepsilon = 0$ if $N \geq 1$ and $\varepsilon = 1$ otherwise. Then X can be decomposed as

$$X = f \cdot x^{N-1+\varepsilon} \cdot Z = f \cdot x^{N-1+\varepsilon} \cdot 1/x^{N-1+\varepsilon} Y \tag{2}$$

As Z is of type $\mathbb{C}$ and (rational) complete, the restriction of $f \cdot x^{N-1+\varepsilon}$ to each solution φ_z of Z is constant. Then $f \cdot x^{N-1+\varepsilon}$ is a meromorphic first integral of Y, and then X defines a proper flow.

2.2.2. *R of type* $\mathbb{C}^*$. By Suzuki (see [10]) we may assume that

$$R = x^m(x^\ell y + p(x))^n,$$

where $m \in \mathbb{N}^*$, $n \in \mathbb{Z}^*$, with $(m,n) = 1$, $\ell \in \mathbb{N}$, $p \in \mathbb{C}[x]$ of degree $< \ell$ with $p(0) \neq 0$ if $\ell > 0$ or $p(x) \equiv 0$ if $\ell = 0$, up to a polynomial automorphism.

According to the relations $x = u^n$ and $x^\ell y + p(x) = v\,u^{-m}$ it is enough to take the rational map H from $u \neq 0$ to $x \neq 0$ defined by

$$(u,v) \mapsto (x,y) = (u^n, u^{-(m+n\ell)}[v - u^m p(u^n)]) \tag{3}$$

in order to get $R \circ H(u,v) = v^n$. Although R is not necessarily a polynomial ($n \in \mathbb{Z}$), it is a consequence of the proof of [5, Proposition 3.2] that $H^*\mathcal{F}$ is a Riccati foliation adapted to v^n having $u = 0$ as invariant line. Thus

$$H^*Y = u^k \cdot Z = u^k \cdot \left\{ a(v)u\frac{\partial}{\partial u} + c(v)\frac{\partial}{\partial v} \right\}, \tag{4}$$

where $k \in \mathbb{Z}$, and a, $c \in \mathbb{C}[v]$.

Applying directly the local models of [2], it is proved in [4, Lemma 2] that at least one of the irreducible components of R over 0 must be an $\mathcal{F}$-invariant line. Hence the polynomial $c(v)$ of (4) is, in fact, a monomial, and thus of the form cv^N with $c \in \mathbb{C}$ and $N \in \mathbb{N}$. Then

$$H^*X = f \circ H(u,v) \cdot u^k \cdot v^{N-1+\varepsilon} \cdot 1/v^{N-1+\varepsilon} \cdot Z \tag{5}$$

As $1/v^{N-1+\varepsilon} \cdot Z$ is complete and of type $\mathbb{C}$, the restriction of $f \circ H(u,v) \cdot u^k \cdot v^{N-1+\varepsilon}$ to each solution φ_z of $1/v^{N-1+\varepsilon} \cdot Z$ through $z \in u \neq 0$ is constant. Projecting by H,

$$f^{mn} \cdot x^{mk} \cdot \left(x^m(x^\ell y + p(x))^n\right)^{m(N-1+\varepsilon)}$$

is a meromorphic first integral of Y, and X defines a proper flow.

Remark 1. In §3, we will obtain an explicit first integral of Y which does not depend on f but that nevertheless can be multivalued. Moreover, using that integral we will give an alternative proof of the existence of an invariant line for $\mathcal{F}$ [4, Lemma 2].

2.3. $\operatorname{kod}(\tilde{\mathcal{F}}) = 0$**.** According to [8, §III and §IV], [2, p. 443] we can contract $\tilde{\mathcal{F}}$-invariant rational curves on M via a contraction s to obtain a new surface $\hat{M}$ (maybe singular with cyclic quotient singularities), a reduced foliation $\hat{\mathcal{F}}$ on this surface, and a finite covering map r from a smooth compact projective surface S to $\hat{M}$ such that: 1) r ramifies only over cyclic (quotient) singularities of $\hat{M}$ and 2) the foliation $r^*(\hat{\mathcal{F}})$ is generated by a complete holomorphic vector field Z_0 on S with isolated zeroes.

$$\begin{tikzcd}
\mathbb{CP}^2 & M \arrow[l, "\pi"'] \arrow[d, "s"] & \\
 & \hat{M} & S \arrow[l, "r"']
\end{tikzcd}$$

It follows from [2, p. 443] that the covering r can be lifted to M. That is, there are a surface T, a birational morphism $g : T \to S$ and a ramified covering $h : T \to M$ such that $s \circ h = r \circ g$

$$\begin{tikzcd} M \arrow[d, "s"'] & T \arrow[l, "h"'] \arrow[d, "g"] \arrow[dl, "s\circ h"', "r\circ g"] \\ \hat{M} & S \arrow[l, "r"] \end{tikzcd}$$

Let $\bar{Z}_0$ be the lift g^*Z_0 of Z_0 on T via g. Then $\bar{Z}_0$ must be a *rational vector field* on T generating the foliation $\bar{\mathcal{F}}$ given by $g^*(r^*(\hat{\mathcal{F}})) = h^*\tilde{\mathcal{F}}$. On the other hand, $\bar{\mathcal{F}}$ is also generated by the rational vector field $\bar{Y}$ on T given by $h^*\tilde{Y}$, with $\tilde{Y} = \pi^*Y$. Hence there is a rational function $\bar{F}$ on T such that

$$\bar{Y} = \bar{F} \cdot \bar{Z}_0. \tag{6}$$

Remark 2. From the above construction we notice that:

• The map g is a composition of blowing-ups at a finite set $\Theta = \{\theta_i\}_{i=1}^s \subset S$ of regular points of Z_0. In fact $\Theta = r^{-1}(Sing(\hat{M}))$. The poles of $\bar{Z}_0$ are in $g^{-1}(\Theta)$ and they define a divisor $\mathcal{P} \subset T$ invariant by $\bar{\mathcal{F}}$. Hence $\bar{Z}_0$ is holomorphic on $T \setminus \mathcal{P}$. Note that in $T \setminus \mathcal{P}$, $\bar{Z}_0$ has only isolated zeroes.
• $\mathcal{P}$ is the exceptional divisor of g, $h(\mathcal{P})$ is the exceptional divisor of s and is $\tilde{\mathcal{F}}$-invariant. Then $h_{|T\setminus\mathcal{P}} : T \setminus \mathcal{P} \to M \setminus h(\mathcal{P})$ is a regular covering map.
• Let C_{θ_i} be the trajectory of Z_0 through θ_i. $\bar{Z}_0$ is a complete holomorphic vector field on $W \setminus \{g^{-1}(C_{\theta_i})\}_{i=1}^s$.

Lemma 4. *h is a birrational map.*

Proof. The set of components of the divisor $h(\mathcal{P})$ of the contraction s, that define curves in $\mathbb{C}^2$ after projection via $\pi_{|U}$ is or empty or an affine line L ([2, Lemma 6]). In the latter case Y is always of type $\mathbb{C}^*$ ([2, p. 445]). Thus there is Zariski open $W \subset T$ such that $\pi \circ h : W \to \mathbb{C}^2 \setminus \pi(E)$ is a regular covering. Therefore h is birational. ∎

We can project (6) by $\pi \circ h$ to obtain a decomposition $X = f \cdot Y = f \cdot F \cdot Z$, where Z is a rational vector field of type $\mathbb{C}$ which is complete outside a finite set of trajectories. Hence the restriction of $f \cdot F$ to each solution of Z must be constant, and $f \cdot F$ is a meromorphic first integral for X.

3. R-complete foliations with R of type $\mathbb{C}^*$. Let us assume that $\mathcal{F}$ is R-complete with R a rational map of type $\mathbb{C}^*$. One can assume that $\mathcal{F}$ is defined after the rational change H, (3), by the rational vector field H^*Y given in (4).

Proposition 1. *$\mathcal{F}$ has a multivaluated meromorphic first integral.*

Proof. If one takes v_0 with $c(v_0) \neq 0$, the trajectories of H^*Y except the horizontal ones and $\{u = 0\}$ are parameterized by maps $\sigma(w_0, t)$, where w_0 is a fixed point and σ is a multivaluated holomorphic map defined on $\mathbb{C}^* \times (\mathbb{C} \setminus \{c = 0\})$ of the form

$$\sigma(w, t) = (u(w, t), v(w, t)) = (we^{\int_{v_0}^t \frac{a(z)}{c(z)}\,dz}, t). \tag{7}$$

It is enough to extend the local solution through (w_0, v_0), with $w_0 \in \mathbb{C}^*$, of $1/c(v) \cdot Z$ by analytic continuation along paths in $\mathbb{C} \setminus \{c(v) = 0\}$. This map is defined as $\sigma(w_0, t)$ with σ equals (7) (see [6, Section 2]).

Let us take the one-form $\omega = [a(z)/c(z)]\, dz$ that appears in (7). It has a fraction expansion of the form

$$\omega = \left\{ s(z) + \sum_{j=0}^{r} \frac{A_1^j}{(z-\xi_j)} + \frac{A_2^j}{(z-\xi_j)^2} + \cdots + \frac{A_{r_j}^j}{(z-\xi_j)^{r_j}} \right\} dz, \tag{8}$$

where $s(z) \in \mathbb{C}[z]$, ξ_j are those roots of multiplicity r_j of the denominator of $a(z)/c(z)$ after simplifying $a(z)$ and $c(z)$, and $A_i^j \in \mathbb{C}$, for $1 \le i \le r_j$. If ξ_j is zero we assume that it is ξ_0. Otherwise $A_i^0 = 0$ and the sum of (8) begins from $j = 1$. Let us fix

$$\Gamma(z) = e^{\,\bar{s}(z)} \prod_{j=0}^{r} \Gamma_j(z) = e^{\,\bar{s}(z)} \prod_{j=0}^{r} e^{\lambda_1^j \log(z-\xi_j) + \frac{\lambda_2^j}{(z-\xi_j)} + \cdots + \frac{\lambda_{r_j}^j}{(z-\xi_j)^{r_j-1}}} \tag{9}$$

where $\bar{s}(z) = \int^z s(t)dt$, and $\lambda_1^j = A_1^j$ and $\lambda_i^j = A_i^j/(-i+1)$ for $2 \le i \le r_j$. If we introduce (8) in (7), after explicit integration of ω, one has that $\sigma(w, t)$ is of the form $(w \cdot \Gamma(t)/\Gamma(v_0),\, t)$. Then

$$F(u, v) = \frac{u}{\Gamma(v)} \tag{10}$$

is a first integral of H^*Y. Finally, we can express (10) in terms of x and y according to (3),

$$G(x, y) = \frac{x^{1/n}}{\Gamma(x^{m/n} \cdot (x^\ell y + p(x)))}, \tag{11}$$

and thus obtain a (multivalued) first integral of Y, and then of $\mathcal{F}$. ■

PROPOSITION 2. *The line $x = 0$ is invariant by $\mathcal{F}$.*

Proof. Let us suppose that $x = 0$ is not invariant. Each trajectory of Y through a non-singular point $(0, y_0)$, $y_0 \neq 0$, can be then locally parametrized by a map $t \mapsto \gamma(t) = (t, y(t))$, with t in a sufficiently small disk $\mathbb{D}$ and $y(0) = y_0$. In order to study the restriction of G to each of them we will consider the non-reduced parametrization $\gamma(t^{|n|})$.

a) Case $n > 0$**.** Let us take the function

$$q(t) = t^m (t^{n\ell} y(t^n) + p(t^n)),$$

It follows from (9) and (11) that

$$G \circ \gamma(t^n) = \Omega(t) \cdot \Delta(t), \tag{12}$$

where

$$\Omega(t) = t^{(1-m\lambda_1^0)} \cdot e^{-\frac{\lambda_2^0}{q(t)}} \cdots e^{-\frac{\lambda_{r_0}^0}{q(t)^{r_0-1}}} \tag{13}$$

is in general a *multivalued* holomorphic function in $\mathbb{D}^*$, and

$$\Delta(t) = \frac{e^{-\bar{s}(q(t)) - \lambda_1^0 \log(t^{n\ell} y(t) + p(t^n))}}{\prod_{j \ge 1} \Gamma_j(q(t))} \tag{14}$$

is a holomorphic in $\mathbb{D}$ with $\Delta(0) \neq 0$.

On the other hand, as $\gamma(\mathbb{D}^*)$ is contained in a trajectory of Y, we can determine $\delta(y_0) \in \mathbb{C}$ such that $\gamma(\mathbb{D}^*) \subset \{G = \delta(y_0)\}$. This implies that (12) must be constantly equal to $\delta(y_0)$, and hence $r_0 = 1$ and $1 - m\lambda_1^0 = 0$ in (13). Thus we can assume that $\Omega(t) \equiv 1$ and $G \circ \gamma(0) = \Delta(0) = \delta(y_0)$.

a.1) If $\ell > 0$, we know from (14) that the value

$$\Delta(0) = \frac{e^{-\bar{s}(0) - \lambda_1^0 \log(p(0))}}{\prod_{j=1}^{r} e^{\lambda_1^j \log(-\xi_j) + \frac{\lambda_2^j}{(-\xi_j)} + \cdots + \frac{\lambda_{r_j}^j}{(-\xi_j)^{r_j - 1}}}}$$

does not depend on y_0. Therefore (fixed the logarithmic branch) we may assume that $\delta(y_0) \equiv \delta = \Delta(0)$ for any y_0. In particular, there is an open set $\mathcal{N}$ of analytic dimension 2 containing $\{x = 0\} \cap \{Y \neq 0\}$ and such that $\mathcal{N} \setminus \{x = 0\} \subset \{G = \delta\}$, which gives us a contradiction.

a.2) If $\ell = 0$, by a simple inspection in (11), using $r_0 = 1$ and $1 - m\lambda_1^0 = 0$, we see that $y = 0$ must be invariant by Y. But this line can be assumed to be $x = 0$ after a symmetry $(x, y) \mapsto (y, x)$, contradicting again our assumptions.

b) **Case** $n < 0$. Let us take the function

$$\bar{q}(t) = \frac{t^{-n\ell} y(t^{-n}) + p(t^{-n})}{t^m}.$$

It follows from (9) and (11) that

$$G \circ \gamma(t^{-n}) = \tilde{\Omega}(t) \cdot \tilde{\Delta}(t), \tag{15}$$

where

$$\tilde{\Omega}(t) = t^{-(1 - m\sum_{j=0}^{r} \lambda_1^j)} \cdot e^{-\bar{s}(\bar{q}(t))} \tag{16}$$

is in general a *multivalued* holomorphic function in $\mathbb{D}^*$, and

$$\tilde{\Delta}(t) = \frac{e^{(-\sum_{j=0}^{r} \lambda_1^j \log(t^m(\bar{q}(t) - \xi_j)))}}{\prod_{j=0}^{r} e^{\lambda_2^j/(\bar{q}(t) - \xi_j)} \cdots e^{\lambda_{r_j}^j/(\bar{q}(t) - \xi_j)^{r_j - 1}}} \tag{17}$$

is holomorphic in $\mathbb{D}$ with $\tilde{\Delta}(0) \neq 0$. As $\gamma(\mathbb{D}^*)$ is contained in a trajectory of Y, we can determine $\delta(y_0) \in \mathbb{C}$ such that $\gamma(\mathbb{D}^*) \subset \{G = \delta(y_0)\}$. Then (15) must be constantly equal to $\delta(y_0)$, and hence $\bar{s}(z) \equiv 0$ (note that $\bar{s}(z)$ is zero when it is constant) and $1 - m\sum_{j=0}^{r} \lambda_1^j = 0$ in (16). Thus $\tilde{\Omega}(t) \equiv 1$ and $G \circ \gamma(0) = \tilde{\Delta}(0) = \delta_0$.

b.1) If $\ell > 0$, we know from (17) that the value

$$\tilde{\Delta}(0) = e^{(-\sum_{j=0}^{r} \lambda_1^j \log(p(0)))}$$

does not depend of y_0. So analogously to a.1), fixed the logarithmic branch, we may assume that $\delta(y_0) \equiv \delta = \tilde{\Delta}(0)$ for any y_0. In particular, there is an open set $\mathcal{N}$ of analytic dimension 2 containing $\{x = 0\} \cap \{Y \neq 0\}$ and such that $\mathcal{N} \setminus \{x = 0\} \subset \{G = \delta\}$, which gives us a contradiction.

b.2) If $\ell = 0$ we can proceed as in a.1). ∎

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