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LIOUVILLIAN FIRST INTEGRALS OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper we generalize to any dimension and codimension some theorems about existence of Liouvillian solutions or first integrals proved by M. Singer in *Liouvillian first integral of differential equations* (1992) for first order differential equations.

Introduction. In [15] M. Singer proved that order one differential equations with Liouvillian general solutions have Liouvillian first integrals and that existence of such a first integral forces the foliation given by the equation to be transversally affine. This last statement was extended to codimension one foliations and to other kinds of first integrals. One can look at [16] where H. Żołądek gives a proof of Singer's theorem using Khovanskii topological Galois theory.

In this paper we prove that for a foliation of any codimension with fixed independent variables, existence of Liouvillian general solution implies existence of Liouvillian first integrals but only existence of sufficiently many Liouvillian first integrals forces the foliation to have a transversal structure of solvable type.

Our main tool is the pseudogroup introduced by B. Malgrange [11] as a non-linear analogue of the Picard-Vessiot group. Relevant information is contained in the transversal part of the Lie algebra of the Malgrange pseudogroup. This quotient Lie algebra is called the Galois Lie algebra of the foliation. Both existence of a general Liouvillian solution and existence of sufficiently many Liouvillian first integrals imply solvability of the Galois Lie algebra but the latter implies finiteness of the dimension of this Lie algebra.

First order examples. From Liouville (see [14, 15]) one gets an example of a first order ordinary differential equation with a Liouvillian first integral but without Liouvillian

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general solutions. This equation is

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2}.$$

A Liouvillian first integral is $H(x, y) = y \exp(-y/x)$ but this equation has no Liouvillian general solution. If one changes the independent variable to $t = \frac{x}{y}$ the equation is $\frac{dy}{dt} = -\frac{y}{t^2}$ whose solutions are Liouvillian.

The very special foliation of Brunella [2], or the foliations of Lins-Neto's family [10] are examples of foliations with a Liouvillian first integral but without any birational system of dependent and independent variables t and z such that the foliation is given by a linear equation $\frac{dz}{dt} = a(t)z + b(t)$.

Second order example. From [12] one gets an example of a second order equation with Liouvillian general solution but only one Liouvillian first integral (i.e. any two Liouvillian first integrals are functionally dependent). This equation is

$$\frac{d^2u}{dx^2} = -\left(1 - \frac{1}{x}\right)\left(\frac{du}{dx}\right)^2.$$

It is equivalent to the system given by $\frac{d\ell}{dx} = \frac{1}{x}$ and $\frac{du}{dx} = \frac{1}{x-\ell}$. The Liouvillian field extensions are $\mathbb{C}(x) \subset \mathbb{C}(x, \ell = \log x) \subset \mathbb{C}(x, \ell, u = \int \frac{dx}{x-\ell})$. This equation has a Liouvillian first integral $H = \int (d\ell - \frac{dx}{x}) = \ell - \log x$. By computation of B. Malgrange [12], the Galois Lie algebra of this foliation is infinite dimensional. By theorem 6, there is no functionally independent second Liouvillian first integral. A second first integral is given by an integration with implicit parameter $c \in \mathbb{C}$:

$$K(x, \ell, u) = \int_{H(x,\ell,u)=c} \left(du - \frac{dx}{x - \ell}\right)$$

determined up to addition of an undetermined function of H.

1. Definitions

1.1. Foliations and equations. Let M be an affine irreducible smooth algebraic variety of dimension $q + n$ over $\mathbb{C}$ with ring of coordinate $\mathbb{C}[M]$, $\Omega^1(M)$ its $\mathbb{C}[M]$-module of 1-forms and $\Theta(M)$ its $\mathbb{C}[M]$-module of vector fields. The Lie bracket on $\Theta(M)$ is the $\mathbb{C}$-bilinear map $[\cdot,\cdot] : \Theta(M) \times \Theta(M) \to \Theta(M)$ defined by $[v, w] = vw - wv$ as derivations on $\mathbb{C}[M]$. To any submodule $F \subset \Theta(M)$ (resp. $N \subset \Omega^1(M)$) one associates its annihilator $N_F \subset \Omega^1(M)$ (resp. $F_N \subset \Theta(M)$). A foliation on M is a submodule $F \subset \Theta(M)$ such that $F_{N_F} = F$ and $[F, F] \subset F$.

From now on F is a foliation on M. Elements of N_F are called equations of the foliation, those of $\mathbb{C}(M) \otimes N_F$ over $\mathbb{C}[M]$ are rational equations and those of $\overline{\mathbb{C}(M)}^{alg} \otimes N_F$ algebraic equations. Equations satisfy $dN_F \subset N_F \wedge (\mathbb{C}(M) \otimes \Omega^1(M))$. Because of its reflexion property, N_F is torsion free and is characterized by its restriction to Zariski open subsets. This means that a $\mathbb{C}(M)$-basis of rational equations determines F. The dimension of this $\mathbb{C}(M)$-vector space is the codimension of F and is denoted by q. A subvariety W of M is F-invariant if $\mathbb{C}[W] \otimes F \subset \Theta(W)$. For such W, $\mathrm{codim}_M W \leq q$.

A dominant map $\pi : M \to B$ is called independent variables for F if $\mathbb{C}(M)\otimes\Omega^1(M) = \mathbb{C}(M)\otimes N_F \oplus \mathbb{C}(M)\otimes\Omega^1(B)$. Then F (or N_F) is a system of rational PDEs over B (with finite type solution space). In the case $\pi^* = \mathbb{C}[x_1,\dots x_n] \to \mathbb{C}[x_1,\dots x_n, y_1,\dots,y_q]$ a basis of rational equations can be written $dy_i - \sum_j E_i^j(x,y)dx_j$ for $1 \le i \le q$. Analytic dimension n subvarieties tangent to F are given by solutions of the system of PDEs

$$\frac{\partial y_i}{\partial x_j} = E_i^j(x,y), \qquad 1 \le i \le q, \quad 1 \le j \le n$$

satisfying formal integrability conditions

$$\frac{\partial}{\partial x_k}E_i^j + \sum_\ell \frac{\partial E_i^j}{\partial x_\ell}E_\ell^k = \frac{\partial}{\partial x_j}E_i^k + \sum_\ell \frac{\partial E_i^k}{\partial x_\ell}E_\ell^j,$$

$$1 \le i \le q, \quad 1 \le j \le n, \quad 1 \le k \le n.$$

Special type of foliations are given by rational maps $\varphi : M \dashrightarrow W$: $F_{d\varphi}$ has rational equations generated by $\varphi^*\Omega^1(W)$. When φ is explicitly given by a subfield I of $\mathbb{C}(M)$ then the foliation is denoted by F_{dI}.

1.2. First integrals and solutions. A differential ring is a ring $A \supset \mathbb{C}$ with a $\mathbb{C}$-derivation $d : A \to V_A$ taking values in a free A-module such that there is an extension $d : V_A \to \Lambda^2 V_A$ satisfying $d(av) = da\wedge v + adv$. One can extend d to $\Lambda^\bullet V_A$ by $d(v\wedge w) = dv\wedge w + (-1)^{\deg v} v\wedge dw$ and one gets $dd = 0$. To come back to the usual definition one chooses a basis $\underline{e}$ of V_A. Then $d = \sum \partial_i e_i$ for ∂_i's some A-valued derivations of A. Coordinates of $v \in \Lambda^d V_A$ are written $v = \sum v(\partial_{i_1},\dots,\partial_{i_d})e_{i_1}\otimes\dots\otimes e_{i_d}$. The existence of the extension of d means that the bracket on $\bigoplus A\partial_i$ defined by

$$dv(\partial_i,\partial_j) = v([\partial_i,\partial_j]) + \partial_i v(\partial_j) - \partial_j v(\partial_i), \quad \forall v \in V_A$$

takes values in $\bigoplus A\partial_i$. Then $\bigoplus A\partial_i$ is a subspace of $Der_{\mathbb{C}}(A,A)$ stable under Lie bracket: it is a $\mathbb{C}$-Lie subalgebra. The exterior derivative $d : \mathbb{C}[M] \to \Omega(M)$ gives to $\mathbb{C}[M]$ a differential structure and $(\mathbb{C}(M); d)$ is a differential extension.

A 1-form $\omega \in \Omega^1(A/\mathbb{C})$ gives a A-linear morphism from $Der_{\mathbb{C}}(A,A)$ to A. It can be extended by linearity to $Der_{\mathbb{C}}(A,V_A)$ by $\omega(\sum\partial_i e_i) = \sum\omega(\partial_i)e_i$. This definition does not depend on the basis.

A differential extension $(K; d_K)$ of $(A; d)$ is given by two inclusions $i : A \to K$ and $i' : V_A \to V_K$ such that $i'\circ d = d_K\circ i$ and i' is i-linear. An extension is strict if i' extends to an isomorphism of $K\otimes V_A$ to V_K.

Definition 1. A differential field extension $(K; d)$ of $(\mathbb{C}(M); d)$ is called *Liouvillian* if there exists a sequence of strict differential extensions $\mathbb{C}(x_1,\dots,x_n) = K_0 \subset K_1 \dots \subset K_{N-1} \subset K_N = K$ such that $K_i = K_{i-1}(F_i)$ with F_i algebraic over K_{i-1} or $dF_i \in V_{K_{i-1}}$ or $\frac{dF_i}{F_i} \in V_{K_{i-1}}$.

An element H of a differential extension $(K; d)$ of $\mathbb{C}[M]$ is a first integral of a foliation F if $dH \in K\otimes N_F$. The type of the $\mathbb{C}$-linear subspace IP of first integrals in K is the smallest integer q such that the map IP^{q+1} to $\Lambda^{q+1}V_K$ sending $(H_1,\dots,H_{q+1})$ to $\bigwedge dH_i$ is null.

Definition 2. Let F be a codimension q foliation on M. A differential field extension $(K; d)$ of $\mathbb{C}[M]$ such that the subfield of first integrals has type 'q' is called a field of complete integration.

Definition 3. Let F be a foliation on M and B be independent variables. An extension L of $\mathbb{C}(M)$ which is a strict differential extension $(L; d_L)$ of $(\mathbb{C}(B); d_B)$ with $\omega(d_L) = 0$ $\forall\omega \in N_F$ is called a general solution extension.

A particular solution is given by a general solution on an F-invariant subvariety of M.

Definition 4. Let F be a foliation on M. A leaf $\mathcal{L}$ of F is an immersed analytic submanifold such that $\forall x \in \mathcal{L},\ T_xL = F(x)$ and maximal for the inclusion.

From the Cauchy and Frobenius theorems complete integration fields, general solution fields and leaves through any point x where $N_F(x)$ has maximal rank exist. The subvariety $S(F)$ of M of points where N_F has not maximal rank is called the singular locus of F.

1.3. Malgrange pseudogroup and its Lie algebra. In this subsection, we outline the definition of Malgrange pseudogroup. Precise definitions and justifications can be found in [11, 12, 4, 5, 8]

The frame bundle of M is the proalgebraic variety RM on M of formal invertible maps $r : (\mathbb{C}^{n+q}, 0)^\wedge \to M$, i.e. whose completion $r^\wedge : (\mathbb{C}^{n+q}, 0)^\wedge \to (M, r(0))^\wedge$ is invertible where $(M, p)^\wedge$ denotes the formal neighborhood of p in M. From a commutative algebra view point, a frame is given by a $\mathbb{C}$-algebra morphism $r^\# : \mathbb{C}[M] \to \mathbb{C}[[x_1, \ldots, x_{n+q}]]$; if $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}[[x_1, \ldots, x_{n+q}]]$ and $J = (r^\#)^{-1}(\mathfrak{m})$ then the completion is the map $(r^\#)^\wedge : \varprojlim (\mathbb{C}[M]/J^n)_n \to \mathbb{C}[[x_1, \ldots, x_{n+q}]]$. It is a continuous $\mathbb{C}$-algebra morphism and it is required to be an isomorphism. The proalgebraic structure is given for each integer k by the projection on R_kM the space of k-jet of formal invertible maps.

Formal invertible maps and vector fields on M act on RM by target composition. The action on RM of a dynamical system on M is called its prolongation and is denoted by adding a R in front of its name. A foliation on M with equations N is prolonged on RM by the annihilator RN in $\Omega^1(RM)$ of prolongation of any vector field on M tangent to F_N. Let $I \subset \mathbb{C}(RM)$ be the subfield of rational first integrals of RN. By [9] generic leaves of the foliation F_{dI} are Zariski closures of leaves of F_{RN}.

The group Γ_{n+q} of formal biholomorphisms of $(\mathbb{C}^{n+q}, 0)$ as well as the Lie algebra $\Theta_{(\mathbb{C}^{n+q},0)}$ of formal vector fields act on RM by source composition. The frame bundle RM is a Γ_{n+q} principal space over M. Because action by source and by target composition commute, actions of Γ_{n+q} and $\Theta_{(\mathbb{C}^{n+q},0)}$ preserve F_{RN} and F_{dI}.

The space $AutM = \{\varphi : (M, a)^\wedge \to (M, b)^\wedge \text{ invertible} \mid (a, b) \in M \times M\}$ has a proalgebraic groupoid structure and its coordinate ring has a canonical differential structure. It acts on RM by target composition. A pseudogroup is a subgroupoid defined by a differential ideal. In particular $AutM$ is a pseudogroup. Then

- Malgrange pseudogroup, $Mal(F)$, is the pseudogroup of formal invertible maps $(M, a)^\wedge \to (M, b)^\wedge$ preserving I pointwise,
- Malgrange Lie algebra at $a \in M$, $mal(F)_a$, is given by all formal vector field on M at a preserving I pointwise.

These definitions are not the ones given by Malgrange in [11]. Equivalence of these definitions with the original ones is a consequence of [7, théorème 8.1]. An elementary proof can be found in [13, proposition 2.3.6].

It is straightforward that vector fields tangent to N are in the Malgrange Lie algebra. One defines the Galois Lie algebra of F at a to be the quotient $gal(F)_a = mal(F)_a/tan(F)_a$ where $tan(F)_a$ is the ideal of $mal(F)_a$ of formal vector fields tangent to F.

The Maurer-Cartan form on RM is $\Omega = TRM \to \Theta_{(\mathbb{C}^{n+q},0)}$ (see [8, ?]). It is defined by $\Omega(r,v) = r^*\alpha^{-1}(v)$ where $\alpha : \Theta_{(M,r(0))} \to T_rRM$ is the isomorphism given by prolongation of formal vector fields. Many properties of Ω are described in [8, 12]. One gets $\Omega(F_{dI}) = \mathfrak{g}$ and $\Omega(RF) = \mathfrak{h}$ where $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ a subalgebra of $\Theta_{(\mathbb{C}^{n+q},0)}$. A frame $(\mathbb{C}^{n+q},0)^\wedge \to (M,a)^\wedge$ at a generic point a of M allows us to identify $\mathfrak{h} \subset \mathfrak{g} \subset \Theta_{(\mathbb{C}^{n+q},0)}$ to $tan(F)_a \subset mal(F)_a \subset \Theta_{(M,a)}$ and $\mathfrak{g}/\mathfrak{h}$ to $gal(F)_a$.

Notice that F has a rational first integral if and only if for generic a in M, $mal(F)_a$ is not transitive, i.e. the map $\pi : mal(F)_a \to T_aM$ defined by $\pi(v) = v(0)$ is not onto.

2. Theorems

2.1. Statements

THEOREM 5. *Let E be a finite type system of PDE's satisfying formal integrability conditions. If E has a Liouvillian general solution field then its Galois Lie algebra is solvable.*

THEOREM 6. *Let F be a codimension q foliation on M. If F has a Liouvillian complete integration field then its Galois Lie algebra is solvable and finite dimensional.*

THEOREM 7. *If F is a foliation with solvable Galois Lie algebra then there exists a rational first integral or a finite dominant map $\widetilde{M} \dashrightarrow M$ and a rational 1-form $\Omega_/ : T\widetilde{M} \dashrightarrow gal$ such that*

- *gal is a regular algebraic transitive solvable Lie subalgebra of $\Theta_{(\mathbb{C}^q,0)}$,*
- *$d\Omega_/ = \Omega_/ \wedge \Omega_/$,*
- *$\pi \circ \Omega_/ : T\widetilde{M} \dashrightarrow \mathbb{C}^q$ gives q independent algebraic equations of F generating $\overline{\mathbb{C}(M)}^{alg} \otimes N_F$.*

Algebraic Lie subalgebras of $\Theta_{(\mathbb{C}^q,0)}$ are Lie subalgebras defined by a system of PDEs. Regularity means that 0 is a regular point of such a system. For instance $\mathbb{C}x_1\frac{\partial}{\partial x_1} \subset \Theta_{\mathbb{C}^2,0}$ is an algebraic but not regular Lie subalgebra. It is defined by

$$\left\{a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} \middle| b = 0, \frac{\partial a}{\partial x_2} = 0, x_1\frac{\partial a}{\partial x_1} - a = 0\right\}.$$

This system of PDEs is singular at 0.

2.2. Proofs. The pseudogroup $AutM$ is identified with the quotient of $RM \times RM$ by the diagonal action of Γ_{n+q} *via* the map $\lambda : RM \times RM \to AutM$ defined by $\lambda(r,s) = r \circ s^{-1}$. The foliation F can be prolonged in two ways on $RM \times RM$ by $R^tF = RF \oplus \{0\}$ or $R^sF = \{0\} \oplus RF$. These two foliations are Γ_{n+q}-invariant and thus λ-projectable on foliations on $AutM$ still denoted by R^tF and R^sF.

Along the open subset $Id_{M-S} = \{\lambda(r,r) | r \in R(M-S)\}$ of Id_M the identity subvariety of $AutM$, R^tF has no singularities. One can define $\mathcal{L}_{id}$ to be the union of leaves of R^tF going through points of Id_{M-S}. It is readily proved that $\mathcal{L}_{id}$ is a groupoid over $M-S$. Because of the stability under inversion, $\mathcal{L}_{id}$ is the union of leaves of R^sF.

LEMMA 8. *The Zariski closure* $\overline{\mathcal{L}_{id}}$ *is equal to* $Mal(F)$.

Proof. The Zariski closure $\overline{\mathcal{L}_{id}}$ is the smallest $R^sF \oplus R^tF$- invariant subvariety containing Id_M.

Let $i : RM \dashrightarrow W$ be a rational map defining the foliation F_{dI} and Z be the Zariski closure of the subvariety of $RM \times RM$ defined by $i(r) = i(s)$. One gets $\overline{\mathcal{L}_{id}} = \lambda(Z)$ and thus is a sub pseudogroup of $AutM$. By construction the elements of this pseudogroup are formal diffeomorphisms preserving I pointwise. ■

This lemma can be used to give a new proof of the following theorem:

THEOREM 9 ([4]). *If* $\varphi : M_1 \dashrightarrow M_2$ *is a rational dominant map and* F_1 *(resp.* F_2*) is a foliation on* M_1 *(resp.* M_2*) such that* F_1 *is* φ*-projectable on* F_2 *then one gets a dominant pseudogroup morphism*

$$\varphi_\star : (mal(F_1) \cap inv(F_\varphi)) \dashrightarrow mal(F_2).$$

The proof of this theorem will not be given here. It can be read in the following way: $mal(F_2)$ is a quotient of a Lie subalgebra of $mal(F_1)$. So the former is solvable as soon as the latter is.

Let $(K; d_K) \supset (\mathbb{C}(M); d)$ be a Liouvillian field extension and W be a geometrical model for K (i.e. $\mathbb{C}(W) = K$). Because d_K takes its values in $\Omega^1(M)$, one gets a morphism $\Omega^1(W) \to \Omega^1(M)$ whose kernel defines a foliation F_K on W (with M as independent variables). This foliation is defined by a sequence of rational equations $\omega_1, \ldots, \omega_N$ with $d\omega_i = 0 \mod (\omega_1, \ldots, \omega_{i-1})$. A direct computation (see [5]) shows that $gal(F_K)$ is solvable.

To prove theorem 5 one needs to apply the previous argument to $(K; d_K) \supset (\mathbb{C}(B), d)$. Then by definition the dominant map $W \dashrightarrow M$ maps F_K to F and theorem 9 gives the solvability of $gal(F)$.

One can prove the first part of theorem 6 in the same way. We apply previous argument to $(K; d_K) \supset (\mathbb{C}(M), d)$. Let V (resp. Z) be a model for the subfield of first integrals IP (resp. of rational first integrals). Consider the dominant map $\pi : W \dashrightarrow V$ and a section $s : Z \dashrightarrow V$ (possibly multiform but algebraic). Then $W_1 = \overline{\pi^{-1}s(Z)}$ is a subvariety of W whose projection on M is dominant. Let $F_K \cap W_1$ be the foliation induced on W_1. From solvability of $gal(F_K)$ and transversality of F_K and W_1 one gets solvability of $gal(F_K \cap W_1)$. This foliation projects on F and theorem 9 proves solvability of $gal(F)$.

To prove the second part of theorem 6 one considers the differential ring $\mathbb{C}(M)$ $\langle H_1, \ldots, H_q \rangle$ generated by q unknowns and the evaluation morphism $\mathbb{C}(M)\langle H_1, \ldots, H_q \rangle$ $\to K$ sending Hs to q independent first integrals. Element of Malgrange pseudogroups preserves the space of solutions of the system of PDEs given by the kernel I of the evaluation morphism. In formal local coordinate at a regular point a of F, one can choose independent variables x and transversal coordinates t. Transformations in Malgrange pseudogroup preserve the space of solutions of I. Because of finiteness of the transcendence degree of K over $\mathbb{C}(M)$ this space is finite dimensional. Moreover solutions are q functions of q variables t thus the space of transformations preserving x and the space of solutions must be finite dimensional. Taking infinitesimal transformations, one gets a Lie algebra isomorphic to $gal(F)_a$.

One proves the third theorem by taking a section $s : M \to RM$ of the frame bundle with values in the algebraic closure of a generic leaf of RF, i.e. a generic leaf of F_{dI}. This leaf dominates M if and only if there are no rational first integrals. Then by properties of Maurer-Cartan form, $s^*\Omega$ followed by the projection of $mal(F)$ on $gal(F)$ has the wanted properties. The section s may not exist but after a ramified covering.

3. Three corollaries and one remark

Corollary 10 (M. Singer [15]). *Let F be a codimension 1 foliation on M and B independent variables. If F has a Liouvillian general solution then it has a Liouvillian first integral. F has a Liouvillian first integral if and only if for ω an equation of F there is a closed 1-form α such that $d\omega = \omega \wedge \alpha$.*

Proof. This is an easy consequence of the theorems and of Lie classification of regular algebraic Lie subalgebras of $\Theta_{\mathbb{C},0}$ up to the choice of a formal coordinate. The only solvable one are

- $\{0\}$, there exists an exact ω;
- $\mathbb{C}\frac{\partial}{\partial x}$ there exists a closed ω on some ramified covering;
- $\mathbb{C}\frac{\partial}{\partial x} + \mathbb{C}x\frac{\partial}{\partial x}$ there exists a closed α on M satisfying the statement. ∎

Corollary 11. *Let F be a codimension 2 foliation on M. If F has a solvable Galois Lie algebra then there are algebraic equations ω, η of F such that there exist 1-forms α and β on a finite dominant $\widetilde{M}$ over M and $d\omega = \alpha \wedge \omega, d\alpha = 0$ and $d\eta = \beta \wedge \eta \mod \omega, d\beta = 0 \mod \omega$.*

Proof. This corollary can be proved by using theorem 7 and Lie classification of regular algebraic Lie subalgebras of $\Theta_{\mathbb{C}^2,0}$ up to the choice of a formal coordinate (see [1]). One gets that solvable Lie subalgebras of $\Theta_{\mathbb{C}^2,0}$ are subalgebra of $\mathbb{C}\frac{\partial}{\partial x} + \mathbb{C}x\frac{\partial}{\partial x} + \mathbb{C}\{x\}\frac{\partial}{\partial y} + \mathbb{C}\{x\}y\frac{\partial}{\partial y}$. Then ω is the coefficient to $\frac{\partial}{\partial x}$ in the form $\Omega_/$ given by theorem 7. If this coefficient is zero then F has a rational first integral H and $\omega = dH$. The form α is the coefficient of $x\frac{\partial}{\partial x}$, η the coefficient of any $\mathbb{C}\{x\}$ multiple of $\frac{\partial}{\partial y}$ and β the coefficient of any $\mathbb{C}\{x\}$ multiple of $y\frac{\partial}{\partial y}$. ∎

Corollary 12. *Let E be a finite type system of PDE's. If E has a Liouvillian general solution then its foliation F has at least one non-constant Liouvillian first integral.*

Proof. If $\pi \circ \omega$ is not onto then one gets a rational first integral. Now assume that it is onto. Let $\mathfrak{h}$ be the last non-zero derived subalgebra of gal and $\widetilde{\mathfrak{h}} = \{v \in gal \mid v \parallel \mathfrak{h}\}$. It is readily proved that $\widetilde{\mathfrak{h}}$ is an ideal of gal.

If $\pi(\mathfrak{h}) = \mathbb{C}^p \subsetneq \mathbb{C}^q$ then the quotient by the foliation defined by $\mathfrak{h}$ gives $\Omega_1 : T\widetilde{M} \to gal/\widetilde{\mathfrak{h}} \subset \Theta_{\mathbb{C}^{q-p},0}$.

If $\pi(\mathfrak{h}) = \mathbb{C}^q$ then $gal = \mathfrak{h} \ltimes \mathfrak{a}$ were $\mathfrak{a}$ is a solvable Lie subalgebra of $\mathfrak{gl}_q$. Then by the Lie theorem one can assume that the codimension one foliation given by level of x_1 is invariant. The quotient by this foliation gives $\Omega_2 : T\widetilde{M} \to \Theta_{\mathbb{C},0}$ whose image is a solvable Lie subalgebra. Lie classification gives a couple of 1-forms ω and α such that $d\omega = \alpha \wedge \omega, d\alpha = 0$. The first integral is $\int \exp^{(\int \alpha)} \omega$ on $\widetilde{M}$ and defines a Liouvillian function on M. ∎

Remark 13. Assume F has no rational first integral and $gal(F)$ is solvable. Because F_{dI} is Γ_{n+q}-invariant, a leaf L of this foliation is a G-principal bundle over a Zariski open subset of M for an algebraic regular subgroup G of Γ_{n+q}. The covering $\widetilde{M}$ needed to get a rational section of $\mathcal{L}$ over M has Galois group $G/\underline{G}$ where $\underline{G}$ is the connected component of the identity. The structure of Γ_{n+q} ensures that this quotient is isomorphic to $G_1/\underline{G_1}$ where G_1 is the subgroup of GL_q of the transversal to F linear parts of elements of G.

Up to enlarging the Lie algebra where the Maurer-Cartan form takes its values one can simplify the covering needed. For this purpose, one needs to know the group of connected components of maximal almost solvable subgroups of GL_q.

For instance if $q = 1$ no covering is needed to get the forms ω and α in Singer's theorem. If $q = 2$ and one looks at the equation $y'' = E(x, y)$ then $G_1 \subset SL_2(\mathbb{C})$ and either G_1 is finite and primitive and one gets two Liouvillian first integrals given by primitives of closed 1-form on a covering with Galois group G_1 either one can assume that $G_1/\underline{G_1} = \mathbb{Z}/2\mathbb{Z}$ and the equation has at least one Liouvillian first integral $\int \exp^{(\int \alpha)} \omega$ on a two sheets covering of M.

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