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INTEGRABILITY AND LIMIT CYCLES FOR ABEL EQUATIONS

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This article is dedicated to Michael Singer on the occasion of his 60th birthday

Abstract. Abel equations are among the most natural ordinary differential equations which have a Godbillon-Vey sequence of length 4. We show that the associated Poincaré mapping can be expressed by iterated integrals with three functions which are solutions of a system of partial differential equations.

1. Introduction. One of the simplest settings to investigate integrability of dynamical systems by algebraic methods is the case of 1-dimensional non-autonomous systems:

$$\frac{dy}{dx} = f(x, y).$$

Abel equations were first investigated and studied by Abel himself as natural extensions of Riccati equations. They are

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3.$$

Abel found several examples which are integrable [1]. Then this list was enriched by Liouville [21]. The classical contributions have been surveyed in [10].

Another motivation to investigate the integrability of Abel equations is that it relates to the integrability of the class of generalized Liénard equations. Change y into $1/y$, then the Abel equation transforms into

$$-\frac{dy}{y^2dx} = p(x)y^{-2} + q(x)y^{-3}$$
$$\frac{dy}{dx} = [-p(x)y - q(x)]/y,$$

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which defines the same foliation as the planar vector fields

$$\begin{aligned}\frac{dx}{dt} &= -y,\\ \frac{dy}{dt} &= p(x)y + q(x).\end{aligned} \tag{1.1}$$

Poincaré's center–focus problem asks for finding homogeneous polynomials $f(x,y)$ and $g(x,y)$ of degree d such that

$$\begin{aligned}\dot{x} &= y + f(x,y),\\ \dot{y} &= -x + g(x,y),\end{aligned} \tag{1.2}$$

displays a neighborhood of the origin filled with periodic orbits. In such a case, the origin is said to be a center. Writing the system in polar coordinates yields

$$\begin{aligned}\frac{dr}{dt} &= r^d A(\theta),\\ \frac{d\theta}{dt} &= 1 + r^{d-1}B(\theta),\end{aligned} \tag{1.3}$$

where A and B are homogeneous trigonometric polynomials. The associated foliation if defined by

$$\frac{dr}{d\theta} = \frac{r^d A(\theta)}{1 + r^{d-1}B(\theta)}. \tag{1.4}$$

Cherkas proposed to transform equation (3) by setting

$$\rho = \frac{r^{d-1}}{1 + r^{d-1}B(\theta)},$$

and he obtained the following (now called trigonometric Abel equation):

$$\frac{d\rho}{d\theta} = q(\theta)\rho^3 + p(\theta)\rho^2, \tag{1.5}$$

where $q(\theta) = -(d-1)f(\theta)g(\theta)$ and $p(\theta) = (d-1)f(\theta) - g'(\theta)$.

These findings motivated to study polynomial Abel equations (cf. [3])

$$\frac{dy}{dx} = q(x)y^3 + p(x)y^2, \tag{1.6}$$

and consider the following problems. Given two points $x_0 = 0$ and $x_1 = 1$,

(i) characterize the polynomials $q, p \in R[x]$ such that for all initial data $(0, y_0)$ the solution $y(x, y_0)$ of the equation satisfies $y(0, y_0) = y(1, y_0) = y_0$; or

(ii) count and locate the isolated solutions (called *limit cycles*) of equation (5) such that $y(0, y_0) = y(1, y_0) = y_0$.

In view of what precedes it is natural to call (i) *the center problem* for polynomial Abel equations, and (ii) *Hilbert's 16th problem* (sometimes also called in this context *Pugh's problem*) for polynomial Abel equations.

In fact this is much more than a mere analogy as can be understood by going to the complex field. Consider indeed a complex Abel equation (now (x, y) are complex variables)

$$\frac{dy}{dx} = q(x)y^3 + p(x)y^2, \tag{1.7}$$

where $(q,p) \in \mathbb{C}(x)$ are rational functions. Fix two points x_0 and x_1 in $\mathbb{C}$ and a path γ connecting them. Assume that the path γ avoids the movable singularities for an initial value $y_0 \in D(0,r)$, the disc centered at the origin of radius r. Then the Abel equation defines a Poincaré map $P_\gamma : y_0 \mapsto y_1$ using the solution $y(x,y_0)$ such that $y(0,y_0) = y_0$ and $y(1,y_0) = y_1$. In this general setting we are interested in finding *centers* (equations for which $P_\gamma = \mathrm{Id}$) or *isolated fixed points* of P_γ (equivalent to limit cycles for the equation).

We first recall what has been done in a series of articles in collaboration with M. Briskin, F. Pakovich, N. Roytvarf and Y. Yomdin.

2. Integrability of Abel equations. We denote by ω the 1-form associated to the Abel equation:

$$\omega = dy - [q(x)y^3 + p(x)y^2]dx.$$

In [3], we proved, in the real polynomial case:

PROPOSITION 2.1. *There is a unique analytic series*

$$\psi(x,y) = 1 + y\psi_1(x) + y^2\psi_2(x) + \dots + y^k\psi_k(x) + \dots$$

in y with polynomial coefficients in x so that $\psi_k(0) = 0$ which satisfies

$$d[\psi(x,y)\omega] = 0.$$

This analytic series is called an integrating factor of the polynomial Abel equation. Note that there exists then an analytic first integral $H(x,y)$ of the same type since

$$\psi(x,y)\{dy - [q(x)y^3 + p(x)y^2]dx\} = dH(x,y)$$

identifies as possible first integral

$$H(x,) = y + \dots y^{k+1}\psi_k(x)/(k+1) + \dots$$

In the same article [3], we also investigated the real trigonometric Abel equations:

$$\frac{dR}{d\theta} = q(\theta)R^3 + p(\theta)R^2, \tag{2.8}$$

If in analogy with the polynomial case, we look for an analytic series

$$\psi(R,\theta) = 1 + R\psi_1(\theta) + \dots + R^k\psi_k(\theta) + \dots$$

with polynomial coefficients so that

$$d[\psi(R,\theta)\omega] = 0,$$

this condition yields the recurrence relation:

$$\frac{d\psi_k(\theta)}{d\theta} = -(k+1)[p(\theta)\psi_{k-1}(\theta) + q(\theta)\psi_{k-2}(\theta)].$$

In deep contrast with the polynomial case, this equation does not necessarily have a polynomial trigonometric solution. At the kth step of the recurrence, this equation has a solution if and only if the constant term of the right-hand side is zero. If we impose that condition, then there is a solution $\psi_k(\theta)$ defined up to a constant and the construction goes on one step further. If at any steps, we impose the necessary condition, this yields a unique series $\psi(R,\theta)$ such that $\psi_k(R,0) = 0$. In [3] we proved

THEOREM 2.1. *The periodic real Abel equation has a unique analytic integrating factor if and only if the Abel equation has a center. In this case, the integrating factor is given by:*

$$\psi(R,\theta) = \exp \int_{\gamma_\theta} \frac{\partial f}{\partial R}(R,\phi)d\phi,$$

where γ_θ is the arc of solution to $\omega = 0$ from the initial point $(\theta = 0, R)$ to the final point $(\theta, R(\theta))$.

After this first article, many contributions were done (including for instance [4, 5, 6, 7, 11]) with an approach of perturbative theory from an analytic view point of the perturbed Abel equation:

$$\frac{dy}{dx} = q(x)y^3 + \epsilon p(x)y^2.$$

This leads us to algebraic moment theory and the complete solution of the moment conjecture by F. Pakovich [24, 25, 26, 27, 28]. A perturbative approach based on iterated integrals has also been proposed in [13, 14, 15].

In this article, we come back to a non-perturbative situation.

3. Godbillon-Vey sequence. The Godbillon-Vey sequence was introduced by Godbillon and J. Vey in a global context of foliation theory [18]. Their construction was initially more oriented to finding obstructions in cohomology classes of the forms built in the sequence. But later, the Godbillon-Vey sequence appears crucially also in a local context in relation to integrability in the article of B. Malgrange [23].

DEFINITION 3.1. Let ω be a 1-form. A Godbillon-Vey sequence for ω is a sequence of 1-forms ω_k defined inductively by:

$$d\omega = \omega \wedge \omega_1,$$
$$d\omega_1 = \omega \wedge \omega_2,$$
$$d\omega_k = \omega \wedge \omega_{k+1} + \sum_{1 \leq q \leq k} C_q^k \omega_q \wedge \omega_{k-q+1}.$$

Such a sequence satisfies

$$\alpha = dt + \sum_{k=0}^{+\infty} \frac{t^k}{k!}\omega_k,$$
$$d\alpha = \alpha \wedge \sum_k \frac{t^{k-1}}{(k-1)!}\omega_k.$$

Following G. Casale [8, 9], we say that ω has a Godbillon-Vey sequence of length p if $\omega_k = 0$, for all $k \geq p$. M. Singer proved

THEOREM 3.1. *A 1-form ω has a Liouvillian first integral if and only if it has a Godbillon-Vey sequence of length 2.*

This formulation of M. Singer's theorem is due to J.-P. Rollin and F. Thouzet [30]. See also the book of H. Żołądek ([31]) on p. 169. In [8], G. Casale proved

THEOREM 3.2. *A 1-form ω has a Godbillon-Vey sequence of length 3 if and only if it has a first integral of Riccati type.*

Abel equations are represented by the 1-form ω:

$$\omega = dy - f(x,y)dx,$$

where $f = q(x)y^3 + p(x)y^2$ is a polynomial of degree 3 in y. It is immediate to check that a Godbillon-Vey sequence for an Abel equation is

$$\begin{aligned} \omega_1 &= f'_y(x,y)dx, \\ \omega_2 &= f''_y(x,y)dx, \\ \omega_3 &= 6q(x)dx, \end{aligned} \tag{3.9}$$

and thus is of length 4. More generally, as the 1-form

$$\omega = dy - f(x,y)dx \tag{3.10}$$

is integrable, the 1-form

$$\alpha(x,y,t) = d(y+t) - f(x,y+t)dx \tag{3.11}$$

is integrable and its Taylor developement in t:

$$dt + dy - f(x,y)dx - tf'_y(x,y)dx - \frac{t^2}{2!}f''_y(x,y)dx - \frac{t^3}{3!}f'''_y(x,y)dx - \dots \tag{3.12}$$

provides a Godbillon-Vey sequence to the 1-form ω.

Note also that the integrating factor that we have described relates to the first Godbillon-Vey form ω_1 as

$$E(u,y_0) = \exp \int_{\gamma_u} \omega_1.$$

Note that the Godbillon-Vey sequence (3.9) is not necessarily of minimal length. For instance the example

$$\frac{dy}{dx} = cxy^3 + y^2,$$

studied in [16] has a Godbillon-Vey sequence of length 2 and thus according to M. Singer has a Liouvillian first integral.

4. The successive derivatives of the Poincaré mapping of an Abel equation. Let

$$\frac{dy}{dx} = f(x,y) = q(x)y^3 + p(x)y^2$$

be a complex analytic Abel equation defined on the whole space $\mathbb{C}^2$. Let γ be a fixed path in the complex plane $\mathbb{C}$ which avoids the fixed singularities of the equation. Assume γ starts at x_0 and ends at x_1. Let $x \in \gamma$ be an intermediary point. Denote γ_x the arc of the path γ which originates at x_0 and ends at x.

Consider the solution $y(x, y_0)$ obtained by integration along this fixed path with initial value $y(x_0, y_0) = y_0$. We consider the Poincaré mapping $L(y_0)$ which associates to y_0 the value $y(x_1, y_0)$. The aim of this section is to prove the existence of a closed formula which allows the computation of the ∞-jet of the Poincaré map L. This formula involves two functions which are related to the first two Godbillon-Vey forms.

The partial derivatives relative to the initial data y_0 satisfy

$$\frac{\partial}{\partial y_0}\left(\frac{dy(x,y_0)}{dx}\right) = f'_y(x, y(x,y_0))\,\frac{dy(x,y_0)}{dy_0}.$$

Since

$$\frac{dy(x_0,y_0)}{dy_0} = \mathrm{Id},$$

we have

$$\frac{dy(x,y_0)}{dy_0} = \exp\int_{\gamma_x} f'_y(u, y(u,y_0))\,du.$$

So that if we denote by $y_0 \mapsto L(y_0)$ the Poincaré mapping which associates to the initial data y_0 the point $y(x_1, y_0)$, we obtain

$$L'(y_0) = \exp\int_{\gamma} f'_y(u, y(u,y_0))du.$$

So that, using the integrating factor

$$E(x,y_0) = \exp\int_{\gamma_x} f'_y(u, y(u,y_0))du,$$

the first derivative is

$$L'(y_0) = E(x_1, y_0).$$

The above expression yields

$$L''(y_0) = \int_{\gamma}\Big\{f''_y(u,y(u,y_0))\exp\int_{\gamma_u}[f'_y(v,y(v,y_0))dv]du\Big\}\exp\int_{\gamma} f'_y(u,y(u,y_0))du.$$

Denote eventually

$$D(u,y_0) = E(u,y_0)f''_y(u, y(u,y_0)),$$

which gives

$$L''(y_0) = E(x_1,y_0)\int_{\gamma} D(u,y_0)du.$$

Note now that there is a closed formula which expresses the derivatives of $E(u, y_0)$ and $D(u, y_0)$ relative to y_0. It reads

$$\begin{aligned} E'(u,y_0) &= E(u,y_0)\int_{\gamma_u} D(v,y_0)dv, \\ D'(u,y_0) &= D(u,y_0)\int_{\gamma_u} D(v,y_0)dv + 6q(u)E^2(u,y_0). \end{aligned} \tag{4.13}$$

This formula allows to express all the successive derivatives of the return map in terms of complex iterated integrals depending only in $D(u, y_0)$ and $E(u, y_0)$. For instance the third derivative is

$$\begin{aligned} L'''(y_0) = E(x_1,y_0)\Big\{\Big[\int_{\gamma} D(u,y_0)du\Big]^2 + \int_{\gamma} D(u,y_0)\Big[\int_{\gamma_u} D(v,y_0)dv\Big]du \\ + 6\int_{\gamma} E(u,y_0)^2 q(u)du\Big\}. \end{aligned} \tag{4.14}$$

In the real case, this formula becomes:

$$L'''(y_0) = E(x_1, y_0)\Big\{\frac{3}{2}\Big[\int_\gamma D(u, y_0)du\Big]^2 + 6\int_\gamma E(u, y_0)^2 q(u)du\Big\}. \tag{4.15}$$

This formula was already derived in the real case by N. Lloyd ([22]) in the trigonometric case. Later it was also discussed and used by A. Gasull and J. Llibre [17]. It shows that if q is constant and positive, $L'''(y_0)$ never vanishes and hence that the Poincaré mapping has at most 3 fixed points (hence the Abel equations have at most 3 limit cycles). A. Lins Neto [20] gave another proof of this fact. The initial proof of Lloyd extends immediately to the real polynomial case.

This formula seems crucial to proving finiteness results in the real case. Two interesting questions emerge. One is: what can be deduced from this formula in the case when q is no longer constant? The other is: what can be deduced from it in the complex case?

In the case of a real variable u, the mixed first-order integro-differential system (4.10) yields that $E(u, y_0)$ is a solution of a PDE in the independent variables (u, y_0). In that case,

$$\int_0^u D(v, y_0)dv = \frac{\partial E}{\partial y_0}(u, y_0)/E(u, y_0), \tag{4.16}$$

then

$$\frac{\partial}{\partial u}\left[\frac{\partial E}{\partial y_0}(u, y_0)/E(u, y_0)\right] = D(u, y_0), \tag{4.17}$$

and then

$$\begin{aligned}&\frac{\partial^2}{\partial y_0 \partial u}[\frac{\partial E}{\partial y_0}(u, y_0)/E(u, y_0)]\\ &\quad= \frac{\partial}{\partial u}[\frac{\partial E}{\partial y_0}(u, y_0)/E(u, y_0)]\frac{\partial E}{\partial y_0}(u, y_0)/E(u, y_0) + 6q(u)E(u, y_0)^2.\end{aligned} \tag{4.18}$$

Alternatively, this PDE can be written as a partial differential system:

$$\begin{aligned}\frac{\partial E}{\partial y_0} &= E.D,\\ \frac{\partial D}{\partial u} &= F,\\ \frac{\partial F}{\partial y_0} &= F.D + 6q(u)E^2.\end{aligned} \tag{4.19}$$

5. Some representative example. The following Abel equation has been studied in [16]:

$$\frac{dy}{dx} = \frac{1}{4}xy^3 + y^2.$$

With $y = v/x$, this equation yields:

$$\frac{dv}{dx} = \frac{1}{4x}(v^3 + 4v^2 + 4v),$$

which separates. This provides the Liouvillian first integral forecast by M. Singer's theorem and gives the solution $y(x)$ corresponding to the initial data y_0 as the solution to:

$$\frac{y}{xy+2}\mathrm{e}^{\frac{2}{xy+2}} = \frac{\mathrm{e}}{2}y_0.$$

The equation for finding the fixed points of the Poincaré's mapping ($y(1) = y(0) = y$) is

$$\frac{2}{y+2}\mathrm{e}^{\frac{2}{y+2}} = \mathrm{e}.$$

If we change $y = \frac{2\xi}{1-\xi}$, this yields

$$1 - \xi = \mathrm{e}^{\xi}.$$

We write $\xi = x + \mathrm{i}y$, and derive the two equations

$$1 - x = \mathrm{e}^{x}\cos y,$$
$$-y = \mathrm{e}^{x}\sin y.$$

This gives

$$F(y) = 1 + \log\left(-\frac{\sin y}{y}\right) + \frac{y}{\tan y} = 0.$$

Then we note that as $y \to (2n+1)\pi$, $F(y) \to +\infty$ and that as $y \to (2n+2)\pi$, $F(y) \to -\infty$. There is at least one solution (and in fact a single one) in the interval. The Poincaré mapping has thus infinitely many fixed points, i.e. local solutions $y^j(x)$ at the origin, $j = 1, \ldots, n$, and paths s^j from 0 to 1, such that each $y^j(x)$ being analytically continued along s^j satisfies $y(0) = y(1)$. A more detailed analysis of this example can be done using properties of the so-called Lambert function (see for instance [12]).

We showed that the above fixed points “sit on different leaves" of the Poincaré mapping. In other words, although the equality $y(0) = y(1)$ is satisfied for a large number of the initial values y_0, it is realized on more and more complicated continuation paths from a to b. Accordingly, we may ask to what extent this property remains valid for the Poincaré mapping of a general polynomial Abel equation. We hope that the PD system (4.16) displayed in this article could be useful in this direction of investigation.

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