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LINEAR DIFFERENTIAL EQUATIONS AND HURWITZ SERIES

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This paper is dedicated to Professor Michael Singer on his sixtieth birthday

Abstract. In this article, we study solutions of linear differential equations using Hurwitz series. We first obtain explicit recursive expressions for solutions of such equations and study the group of differential automorphisms of the solutions. Moreover, we give explicit formulas that compute the group of differential automorphisms. We require neither that the underlying field be algebraically closed nor that the characteristic of the field be zero.

1. Conventions and basics. Throughout, all rings are commutative with identity, and all differential rings are ordinary (i.e., possess a single derivation, which is often suppressed from the notation). Also, $\mathbf{N}$ will denote the natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbf{Q}$ the field of rational numbers. Unless otherwise noted, k will denote a field. If V is a vector space over k and $X \subset V$, then $\mathrm{Span}_k X$ will denote the k-subspace of V spanned by X. Let R be a differential ring and let $y_1, y_2, \ldots, y_n \in R$. We denote the Wronskian of $y_1, y_2, \ldots, y_n$ by $w(y_1, y_2, \ldots, y_n)$. The set of all $n \times n$ matrices and $n \times n$ invertible matrices over a field k will be denoted by $M(n, k)$ and $GL(n, k)$ respectively. For $A \in M(n, k)$, we denote the centralizer of A in $M(n, k)$ by $C_k(A) := \{T \in M(n, k) | AT = TA\}$. Finally, for any $m, n \in \mathbf{N}$, δ_n^m will denote the Kronecker delta, i.e., $\delta_n^m = 1$ if $m = n$ and $\delta_n^m = 0$ if $m \neq n$.

From [1] we recall that for any commutative ring R with identity, the *ring of Hurwitz series* over R, denoted by HR, is defined as follows. The elements of HR are sequences $(a_n) = (a_0, a_1, a_2, \ldots)$, where $a_n \in R$ for each $n \in \mathbf{N}$. Let $(a_n), (b_n) \in HR$. Addition in HR is defined termwise, i.e.,

$$(a_n) + (b_n) = (c_n), \quad \text{where} \quad c_n = a_n + b_n$$

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for all $n \in \mathbf{N}$. The (Hurwitz) product of (a_n) and (b_n) is given by

$$(a_n) \cdot (b_n) = (c_n), \quad \text{where} \quad c_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}$$

for all $n \in \mathbf{N}$. We recall from [1] that if $\mathbf{Q} \subseteq R$, then $HR \cong R[[t]]$ via the mapping $(a_n) \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$.

Moreover, HR is a differential ring with derivation $\partial_R : HR \to HR$ given by

$$\partial_R((a_0, a_1, a_2, \ldots)) = (a_1, a_2, a_3, \ldots).$$

We will often write ∂ in place of ∂_R. We have, as in [2], for any $j \in \mathbf{N}$, the additive mapping $\pi_j : HR \to R$ defined by $\pi_j((a_n)) = a_j$.

In [1] it was shown that H is a functor from **Comm** (the category of commutative rings with identity) to **Diff** (the category of ordinary differential rings) which is the right adjoint to the functor $U : \mathbf{Diff} \to \mathbf{Comm}$ that “forgets” the derivation d of a differential ring (R, d). This can be expressed as follows.

PROPOSITION 1.1. *For any differential ring (R, d) and any ring S, there is a natural bijection between the sets of morphisms*

$$\mathbf{Comm}(R, S) \cong \mathbf{Diff}((R, d), (HS, \partial_S)).$$

In particular, for any ring homomorphism $f : R \to S$, there is a unique differential ring homomorphism

$$\tilde{f} : (R, d) \to (HS, \partial_S) \quad \textit{given by} \quad \tilde{f}(r) = (f(r), f(d(r)), f(d^2(r)), \ldots).$$

2. Linear homogenous differential operators. Throughout this section, let k be a field of any characteristic and let Hk be the differential ring of Hurwitz series over k. Let $h_0, \ldots, h_{n-1} \in Hk$ and consider the monic linear homogeneous differential operator

$$L : Hk \to Hk$$

defined for any $h \in Hk$ by

$$L(h) = \partial^n(h) + \sum_{i=0}^{n-1} h_i \partial^i(h).$$

We are interested in solutions to $L(h) = 0$ in Hk. To this end, let $V = \{h \in Hk \mid L(h) = 0\}$. We see from Corollary 4.3 of [2] that for any $c_0, c_1, \ldots, c_{n-1} \in k$, there exists a unique $y \in V$ such that $\pi_j(y) = c_j$ for $j = 0, 1, \ldots, n-1$.

PROPOSITION 2.1. *Let $h_0, h_1, \ldots, h_{n-1} \in Hk$, and let L be the linear homogeneous differential operator on Hk defined for any $h \in Hk$ by*

$$L(h) = \partial^n(h) + \sum_{i=0}^{n-1} h_i \partial^i(h).$$

Then V is an n-dimensional k-vector space.

Proof. Since $L : Hk \to Hk$ is a k-linear operator, it is clear that $V = \ker(L)$ is a k-vector space, so it remains to prove that $\dim_k V = n$. To see this, we define a mapping $T : k^n \to V$ as follows: If $\underline{a} = (a_1, \ldots, a_n) \in k^n$, then $T(\underline{a})$ is the unique solution in Hk

to $L(h) = 0$ such that $\pi_i(T(\underline{a})) = a_{i+1}$ for $i = 0, \ldots, n-1$ by [2, Corollary 4.3]. It is clear that T is a k-vector space isomorphism, from which the result follows. ∎

It follows that Hk has the following "completeness" property: Any n^{th} order monic linear homogeneous ordinary differential equation with coefficients in Hk has a complete set of n linearly independent solutions in Hk.

This can be done more generally as follows. Let A denote any commutative ring with identity, let $h_0, h_1, \ldots, h_{n-1} \in HA$ and $c_0, c_1, \ldots, c_{n-1} \in A$. As before, consider the linear homogeneous differential operator L defined on HA for any $h \in HA$ by $L(h) = \partial^n(h) + \sum_{i=0}^{n-1} h_i \partial^i(h)$. We know from Corollary 4.3 of [2] that for any $c_0, c_1, \ldots, c_{n-1} \in A$, there is a unique solution $y \in HA$ to $L(h) = 0$ such that $\pi_i(y) = c_i$ for each $i = 0, 1, \ldots, n-1$. We now give a constructive method for finding solutions to $L(h) = 0$ in HA.

Proposition 2.2. *Let A be a commutative ring with identity, let $h_i \in HA$ and let $c_i \in A$ for $i = 0, \ldots, n-1$. Let L be the linear homogeneous differential operator defined on HA for any $h \in HA$ by*

$$L(h) = \partial^n(h) + \sum_{i=0}^{n-1} h_i \partial^i(h).$$

The unique solution $y \in HA$ to $L(h) = 0$ such that $\pi_i(y) = c_i$ for each $i = 0, 1, \ldots, n-1$ is given by

$$\pi_i(y) = c_i, i = 0, 1, \ldots, n-1$$

and

$$\pi_{n+m}(y) = -\sum_{i=0}^{n-1} \sum_{j=0}^{m} \binom{m}{j} \pi_j(h_i) \pi_{m-j+i}(y), m \in \mathbf{N}.$$

Proof. Clearly $y \in HA$ given by the above prescription is unique, and y satisfies the initial conditions $\pi_i(y) = c_i, i = 0, 1, \ldots, n-1$ by definition, so we must only show that $L(y) = 0$. This means we must show that for each $r \in \mathbf{N}$, $\pi_r(L(y)) = 0$. Now we have

$$\begin{aligned}
\pi_r(L(y)) &= \sum_{i=0}^{n-1} \pi_r(h_i \partial^i(y)) + \pi_r(\partial^n(y)) \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{r} \binom{r}{j} \pi_j(h_i) \pi_{r-j}(\partial^i(y)) + \pi_r(\partial^n(y)) \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{r} \binom{r}{j} \pi_j(h_i) \pi_{r-j+i}(y) + \pi_{r+n}(y) \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{r} \binom{r}{j} \pi_j(h_i) \pi_{r-j+i}(y) + \Big(-\sum_{i=0}^{n-1} \sum_{j=0}^{r} \binom{r}{j} \pi_j(h_i) \pi_{r-j+i}(y) \Big) \\
&= 0. \blacksquare
\end{aligned}$$

The following corollary gives a very simple description of the solutions in the case that the coefficients of the equation are constants.

Corollary 2.3. *Let A be a commutative ring with identity, let $a_0, \ldots, a_{n-1} \in A$ and let $c_0, \ldots, c_{n-1} \in A$. Let L be the linear homogeneous differential operator defined on HA*

for any $h \in HA$ by

$$L(h) = \partial^n(h) + \sum_{i=0}^{n-1} a_i \partial^i(h).$$

The unique solution $y \in HA$ to $L(h) = 0$ such that $\pi_i(y) = c_i$ for each $i = 0, 1, \ldots, n-1$ is given by

$$\pi_i(y) = c_i, i = 0, 1, \ldots, n-1$$

and

$$\pi_{n+m}(y) = -\sum_{i=0}^{n-1} a_i \pi_{m+i}(y), m \in \mathbf{N},$$

or more simply,

$$y_{n+m} = -\sum_{i=0}^{n-1} a_i y_{m+i}, m \in \mathbf{N}.$$

Proof. Since the $h_i = a_i$ are constants, we have $\pi_j(h_i) = a_i$ if $j = 0$ and $\pi_j(h_i) = 0$ if $j \geq 1$. Therefore the only nonzero term in the inner sum is the $j = 0$ term. From this the result follows. ∎

Corollary 2.3 shows that, in the case of constant coefficients, the solutions in Hk to $L(y) = 0$ are *linearly recursive sequences.*

3. Linear homogeneous differential equations with constant coefficients. As before, let k be a field of any characteristic and let Hk be the differential ring of Hurwitz series over k. For any $\beta \in k$, the element $\exp(\beta) = (1, \beta, \beta^2, \ldots, \beta^n, \ldots) \in Hk$ is called the *exponential* of β. Note that for any $c \in k$, $c\exp(\beta)$ is the unique solution in Hk to the differential equation $\partial(y) - \beta y = 0$ with initial condition $y(0) = c$. The following result is immediate.

LEMMA 3.1. *Let $\alpha, \beta \in k$. Then*

(a) $\exp(\alpha + \beta) = \exp(\alpha)\exp(\beta)$.
(b) $\exp(0) = 1$.
(c) *For each $\beta \in k$,* $\exp(\beta)$ *is invertible in Hk, and* $\exp(-\beta) = \exp(\beta)^{-1}$.

From [2] we recall the divided powers $x^{[i]}$ in Hk, for $i \in \mathbf{N}$, defined by $x^{[i]} = (\delta_n^i)$, so that

$$x^{[0]} = 1_{Hk}, \quad x^{[1]} = x = (0, 1, 0, 0, \ldots, 0, \ldots), \quad x^{[2]} = (0, 0, 1, 0, \ldots),$$

etc. Using the natural topology on Hk and the divided powers $x^{[i]}$, we have $\exp(\beta) = \sum_{i=0}^{\infty} \beta^i x^{[i]} = \sum_{i=0}^{\infty} (\beta x)^{[i]}$. We will denote $\exp(\beta)$ by $e^{\beta x}$.

Let V be a k-subspace of Hk that is closed under the derivation ∂ which we denote by $'$. We denote the group of all k-differential automorphisms of V by $G(V|k)$. That is, $G(V|k) := \{\sigma \in \mathrm{Aut}_k V \mid \sigma(v)' = \sigma(v') \text{ for all } v \in V\}$. We will sometimes denote $G(V|k)$ by $G(L|k)$ if V is the full set of solutions of a linear homogeneous differential equation $L(y) = 0$.

3.1. Computing the group $G(V|k)$. Let $y \in Hk$, $y' = \alpha y$ where $\alpha \in k$ and let $\pi_0(y) = c$. Then from Corollary 4.3 of [2], it follows that $y = ce^{\alpha x}$. Thus $V = \{ce^{\alpha x}|c \in k\}$ forms a full set of solutions of the equation $y' = \alpha y$. Let $\sigma \in G(V|k)$ and note that $\sigma(e^{\alpha x})' = \alpha\sigma(e^{\alpha x})$. Thus $\sigma(e^{\alpha x})$ is also a non-zero solution of $y' = \alpha y$. Therefore there is some $c_\sigma \in k^*$ such that $\sigma(e^{\alpha x}) = c_\sigma e^{\alpha x}$. Also note that $\sigma\tau(e^{\alpha x}) = c_\tau\sigma(e^{\alpha x}) = c_\tau c_\sigma e^{\alpha x} = c_\sigma c_\tau e^{\alpha x} = \tau\sigma(e^{\alpha x})$. Thus $G(V|k) \hookrightarrow (k^*, \times)$ by $\sigma \mapsto c_\sigma$ is an injective group homomorphism. Moreover, for any $c \in k^*$, we may define $\sigma_c : V \to V$ by $\sigma_c(e^{\alpha x}) = ce^{\alpha x}$. It is clear that σ_c is a k-differential automorphism of V. Thus $G(V|k) \cong (k^*, \times)$.

More generally, for $\alpha \in k$, consider the k-subspace V_t of Hk defined by

$$V_t = \text{Span}_k\{z_0, z_1, \ldots, z_t\},$$

where $z_j = x^{[j]}e^{\alpha x}$ for $j = 0, 1, \cdots, t$. It can be shown that $\{z_0, z_1, \ldots, z_t\}$ is linearly independent over k, since $\mathrm{w}(z_0, z_1, \ldots, z_t) = z_0^{t+1} = e^{(t+1)\alpha x}$, which is invertible in Hk. One can construct a linear homogeneous differential equation over k whose full set of solutions equals V_t, namely

$$L_t(y) = \frac{w(y, z_0, z_1, \ldots, z_t)}{w(z_0, z_1, \ldots, z_t)} = 0.$$

THEOREM 3.2. *Let $\alpha \in k$ and let $z_j = x^{[j]}e^{\alpha x}$ for $j = 0, 1, \cdots, t$. Let $V_t = \mathrm{Span}_k\{z_0, z_1, \ldots, z_t\}$. Let $Z_t := \begin{pmatrix} z_0, & z_1, & \cdots, & z_t \end{pmatrix}$ and let I_t be the identity matrix of order t. Let $\mathfrak{u}_t := \begin{pmatrix} 0 & I_t \\ 0 & 0 \end{pmatrix} \in M(t+1, k)$ if $t \geq 1$, and $\mathfrak{u}_0 = 0 \in k$. Then*

(a) $Z_t' = Z_t(\alpha I_{t+1} + \mathfrak{u}_t)$.
(b) $C_k(I_{t+1} + \mathfrak{u}_t) = C_k(\mathfrak{u}_t) = \mathrm{Span}_k\{I_t, \mathfrak{u}_t, \mathfrak{u}_t^2, \ldots, \mathfrak{u}_t^t\}$.
(c) *Under the basis $z_0, z_1, \ldots, z_t$,*

$$G(V_t|k) \cong \begin{cases} C_k(\mathfrak{u}_t) \cap GL(t+1, k) & \text{if } \alpha \neq 0; \\ C_k(\mathfrak{u}_t) \cap U(t+1, k) & \text{if } \alpha = 0; \end{cases}$$

where $U(t+1, k)$ is the group of all upper triangular matrices in $GL(t+1, k)$ with 1 on the main diagonal.

Proof. Since $z_i' = \alpha z_i + z_{i-1}$ for all $i \geq 1$ and $z_0' = \alpha z_0$, it follows that $Z_t' = Z_t(\alpha I_{t+1} + \mathfrak{u}_t)$.

A straightforward computation proves (b).

Let $\sigma \in G(V_t|k)$. Since $\sigma(z_j) \in V_t$, there is an element $C_\sigma \in GL(t+1, k)$ such that $\sigma(Z_t) = Z_t C_\sigma$. Therefore

$$\sigma(Z_t') = \sigma(Z_t)(\alpha I_{t+1} + \mathfrak{u}_t) = Z_t C_\sigma(\alpha I_{t+1} + \mathfrak{u}_t).$$

On the other hand, $\sigma(Z_t)' = (Z_t C_\sigma)' = Z_t(\alpha I_{t+1} + \mathfrak{u}_t)C_\sigma$. Since $\sigma(Z_t)' = \sigma(Z_t')$, we obtain that $(\alpha I_{t+1} + \mathfrak{u}_t)C_\sigma = C_\sigma(\alpha I_{t+1} + \mathfrak{u}_t)$, which is true if and only if $C_\sigma \mathfrak{u}_t = \mathfrak{u}_t C_\sigma$. Thus there is an injective group homomorphism $\phi : G(V_t|k) \to C_k(\mathfrak{u}_t) \cap GL(t+1, k)$ given by $\phi(\sigma) = C_\sigma$. Moreover, if $\alpha = 0$ then $z_0 = 1$ and therefore $\sigma(z_0) = z_0$. It then follows that $\phi(G(V_t|k)) \subseteq C_k(\mathfrak{u}_t) \cap U(t+1, k)$ if $\alpha = 0$.

To prove that ϕ is surjective in each of the cases $\alpha = 0$ and $\alpha \neq 0$, we first note that for any $c_0, c_1, \ldots, c_t \in k$ and $0 \leq j \leq t$

$$\Big(\sum_{i=0}^{j} c_{j-i} z_i\Big)' = c_j z_0' + \sum_{i=1}^{j} c_{j-i} z_i' = \alpha c_j z_0 + \sum_{i=1}^{j} c_{j-i}(\alpha z_i + z_{i-1})$$
$$= \sum_{i=0}^{j} \alpha c_{j-i} z_i + \sum_{i=1}^{j} c_{j-i} z_{i-1}. \tag{1}$$

Let $C = \sum_{i=0}^{t} c_i \mathfrak{u}_t^i$ with $c_0 \neq 0$ when $\alpha \neq 0$ and $c_0 = 1$ if $\alpha = 0$. Then $C \in C_k(\mathfrak{u}_t) \cap GL(t+1, k)$ when $\alpha \neq 0$ and $C \in C_k(\mathfrak{u}_t) \cap U(t+1, k)$ when $\alpha = 0$. Then we may define an automorphism σ_C of the k-vector space V_t by $\sigma_C(Z) = ZC$, that is, $\sigma_C(z_j) = \sum_{i=0}^{j} c_{j-i} z_i$. Now from Equation (1) it can be seen that σ_C is a k-differential automorphism of V. Thus

$$G(V_t|k) \cong C_k(\mathfrak{u}_t) \cap GL(t+1, k) \quad \text{or} \quad G(V_t|k) \cong C_k(\mathfrak{u}_t) \cap U(t+1, k)$$

depending on whether $\alpha \neq 0$ or $\alpha = 0$ respectively. ∎

The following theorem is an immediate consequence of Theorem 3.2.

THEOREM 3.3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_r \in k$ be distinct elements. For $t = 1, 2, \ldots, r$, let $V_t = Span_k\{z_{j,t} |\ 0 \leq j \leq m_t\}$ and $V = \bigoplus_{t=1}^{r} V_t$, where $z_{j,t} = x^{[j]} e^{\alpha_t x}$. Let $Z := z_{0,1}, \ldots$ $\ldots, z_{m_1,1}, \ldots, z_{0,r}, \ldots, z_{m_r,r}$, I_{m_t} be the identity matrix of order m_t and $\mathfrak{u}_t := \left(\begin{smallmatrix} 0 & I_{m_t} \\ 0 & 0 \end{smallmatrix}\right)$ $\in M(m_t + 1, k)$. Then*

(a) *$Z' = Z(S + \mathfrak{u})$, where*

$$S = \begin{pmatrix} \alpha_1 I_{m_1+1} & 0 & \cdots & 0 \\ 0 & \alpha_2 I_{m_2+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_r I_{m_r+1} \end{pmatrix}, \qquad \mathfrak{u} = \begin{pmatrix} \mathfrak{u}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{u}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathfrak{u}_r \end{pmatrix}.$$

(b) *We have the following group isomorphism*

$$G(V|k) \cong \bigoplus_{t=1}^{r} G(V_t|k).$$

Since $z_{0,1}, \ldots, z_{m_1,1}, \ldots, z_{0,r}, \ldots, z_{m_r,r}$ is a basis for V, each of the groups $G(V_t|k)$ can be computed using Theorem 3.2.

3.2. Non-algebraically closed fields. Let k be a field and let $\bar{k}$ denote its algebraic closure. Let $a_0, \ldots, a_{n-1} \in k$ and consider the monic linear homogeneous differential operator $L(y) = y^{(n)} + \sum_{i=0}^{n-1} a_i y^{(i)}$. Then from Proposition 2.1, we know that there are k-linearly independent elements $y_1, y_2, \ldots, y_n \in Hk$ such that $L(y_i) = 0$ for each i. Let $Y := (y_1, \ldots, y_n)$ and note that y_i' is also a solution of $L(y) = 0$ for each i. Thus, there is a matrix $B \in M(n, k)$ such that

$$Y' = YB. \tag{2}$$

Considering the differential operator $L(y)$ over the field $\bar{k}$, the characteristic polynomial of the operator $L(y)$ splits into a product of linear factors. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the distinct roots of the characteristic polynomial in $\bar{k}$. Let $Z, S, \mathfrak{u}$ be as in Theorem 3.3. Then it can be shown that $V(Z,\bar{k}) := \mathrm{Span}_{\bar{k}}\{z_1,\dots,z_n\}$ is the set of all solutions of $L(y)=0$ in $H\bar{k}$. Let $V(Y,k) := \mathrm{Span}_k\{y_1,\dots,y_n\}$ and note that $V(Y,k)\subset V(Z,\bar{k})$ and since $w(y_1,\dots,y_n)\neq 0$, $y_1,y_2,\dots,y_n$ remain linearly independent over $\bar{k}$. Thus $V(Z,\bar{k}) = V(Y,\bar{k})$. Let $\phi : V(Y,\bar{k}) \to V(Z,\bar{k})$ be a map of $\bar{k}$-vector spaces such that ϕ maps the ordered basis Y to the ordered basis Z. Then there is a matrix $T_\phi \in GL(n,\bar{k})$ such that $YT_\phi = Z$. Applying ϕ to Equation (2), we obtain

$$Z' = Y'T_\phi = YBT_\phi = ZT_\phi^{-1}BT_\phi.$$

Thus we see that $T_\phi^{-1}BT_\phi = S+\mathfrak{u}$. In particular, $S+\mathfrak{u}$ is the Jordan normal form of B. From the above discussion, we derive the following result.

PROPOSITION 3.4. *Let k be a field and let $L(y) = y^{(n)} + \sum_{i=0}^{n-1} a_i y^{(i)}$, where $a_i\in k$ for each i. Let $\alpha_1,\dots,\alpha_r$ be the distinct roots of the characteristic polynomial of $L(y)=0$ in $\bar{k}$. Let $Z,S,\mathfrak{u}$ be as in Theorem 3.3 and let $A := S+\mathfrak{u}$. Let $y_1,y_2,\dots,y_n\in Hk$ be k-linearly independent solutions of $L(y)=0$ in Hk, $Y := (y_1,\dots,y_n)$ and let $Y' = YB$ for some $B\in M(n,k)$. Then*

(a) *$Span_k\{B^i|0\le i\le n-1\} = Span_k\{B^i|0\le i\le\infty\}$, where $B^0 := I$.*
(b) *$C_k(B) = Span_k\{B^i|0\le i\le n-1\}$.*

Proof. Since $Y' = YB$, we have $B^n = -\sum_{i=0}^{n-1} a_iB^i$. To prove (a), it is enough to show that $\{I,B,\cdots,B^{n-1}\}$ is linearly independent over $\bar{k}$. Suppose that $b_0,\dots,b_{n-1}\in\bar{k}$ and $\sum_{i=0}^{n-1} b_iB^i = 0$. Then we have $\sum_{i=0}^{n-1} b_iYB^i = 0$, which implies $\sum_{i=0}^{n-1} b_iY^{(i)} = 0$. Let $G(y) := \sum_{i=0}^{n-1} b_iy^{(i)}$ and note that $G(y_j) = 0$ for each $j=1,2,\dots,n$. Since the order of $G(y)$ is less than n, we obtain that $b_i = 0$ for all i. Thus (a) is proved.

To show (b), it suffices to consider the case when $r=1$. We know from Theorem 3.2 that $C_{\bar{k}}(A) = \mathrm{Span}_{\bar{k}}\{I,\mathfrak{u},\mathfrak{u}^2,\dots,\mathfrak{u}^{n-1}\}$. Let $T\in GL(n,\bar{k})$ such that $YT = Z$. Then since $C_{\bar{k}}(B) = TC_{\bar{k}}(A)T^{-1}$, we see that $C_{\bar{k}}(B)$ is a $\bar{k}$-vector space of dimension n. As noted in the proof of (1), $\{I,B,\cdots,B^{n-1}\}$ is $\bar{k}-$linearly independent, so it follows that $C_{\bar{k}}(B) = \mathrm{Span}_{\bar{k}}\{I,B,\cdots,B^{n-1}\}$. Now clearly $C_k(B)\supseteq \mathrm{Span}_k\{I,B,\cdots,B^{n-1}\}$. To see that $C_k(B)\subseteq \mathrm{Span}_k\{I,B,\cdots,B^{n-1}\}$, let $X = \alpha_0I+\alpha_1B+\cdots+\alpha_{n-1}B^{n-1}$, where $\alpha_i\in\bar{k}$ for $i=0,\dots,n-1$, and let E be the smallest Galois extension of k containing $\alpha_0,\dots,\alpha_{n-1}$. Let $\sigma\in\mathrm{Gal}(E|k)$ and extend σ to an automorphism of $M(n,E)$ in the usual way. Then $\sigma(B) = B$ and $\sigma(X) = X$, and it follows that $\sigma(\alpha_i) = \alpha_i$ and hence $\alpha_i\in k$ for each $i=0,1,\dots,n-1$, so that $X\in\mathrm{Span}_k\{I,B,\cdots,B^{n-1}\}$. ■

THEOREM 3.5. *Let k be a field, $\bar{k}$ be its algebraic closure and let $L(y) = y^{(n)}+\sum_{i=0}^{n-1} a_iy^{(i)}$, where $a_i\in k$ for each i. Let $\alpha_1,\dots,\alpha_r$ be the distinct roots of the characteristic polynomial of $L(y)=0$ in $\bar{k}$. Let $Z,S,\mathfrak{u}$ be as in Theorem 3.3 and let $A := S+\mathfrak{u}$. Let $y_1,y_2,\dots,y_n\in Hk$ be k-linearly independent solutions of $L(y)=0$ in Hk, $Y := (y_1,\dots,y_n)$ and let $Y' = YB$ for some $B\in M(n,k)$.*

Then,

(a) *if $a_0\neq 0$, with respect to the basis Y,*

$$G(V|k)\cong C_k(B)\cap GL(n,k) = Span_k\{B^i|0\le i\le n-1\}\cap GL(n,k),$$

(b) *if* $a_0 = 0$, *with respect to the basis* Y,

$$G(V|k) \cong GL(n,k) \cap T \begin{pmatrix} \mathcal{G}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{G}_r \end{pmatrix} T^{-1},$$

where $T \in GL(n, \bar{k})$ *such that* $YT = Z$, $\mathcal{G}_i = C_{\bar{k}}(\mathfrak{u}_i) \cap U(m_i + 1, \bar{k})$ *for at most one* i *and in that case* $\alpha_i = 0$ *and* $\mathcal{G}_t = C_{\bar{k}}(\mathfrak{u}_t) \cap GL(m_t + 1, k)$ *for all other* t.

Proof. Let $T \in GL(n,k)$ such that $YT = Z$. Since $C_{\bar{k}}(B) = TC_{\bar{k}}(A)T^{-1}$, with respect to the basis Y, we obtain

$$G(V|\bar{k}) \cong TC_{\bar{k}}(A)T^{-1} \cap GL(n, \bar{k}).$$

Since $\{y_1, \ldots, y_n\}$ is linearly independent over both k and $\bar{k}$, it follows that

$$G(V|k) \cong TC_{\bar{k}}(A)T^{-1} \cap GL(n, k).$$

The rest of the proof follows from Theorem 3.3 and Proposition 3.4. ■

Remark. From Theorem 3.2(b), we see that $C_{\bar{k}}(u_t)$ is a commutative linear algebraic group for each $t, 1 \leq t \leq r$. Then it follows that $G(V|k)$ is a commutative linear algebraic group as well. Also, from Theorem 3.5, we see that the condition that k be algebraically closed is not needed.

4. Examples. Let k be a field of any characteristic. In the following examples, we will compute the group $G(V|k)$.

Example 1. Consider the second order operator $L(y) = y''$. Let $Y = (1, x)$ and note that $V = \mathrm{Span}_k Y$ consists of all solutions of the equation $y'' = 0$. Also note that

$$Y' = YA,$$

where $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Applying Theorem 3.3, we obtain that $G(V|k) \cong U(2,k)$.

Example 2. Consider the differential operator

$$L(y) = y'' - y' - y.$$

Let $Y = (y_1, y_2)$, where $y_1 = (1, 0, 1, 1, 2, 3, \ldots)$ and $y_2 = (0, 1, 1, 2, 3, 5, \ldots)$ are Fibonacci sequences. Then it can be seen that $V = \mathrm{Span}_k\{y_1, y_2\}$ consists of all solutions of $L(y) = 0$. Since $a_0 = -1$, from Equation (1), we obtain that

$$G(V|k) \cong C_k(B)$$

with respect to the basis Y, where $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. One can directly compute the centralizer of B and obtain

$$C_k(B) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha + \beta \end{pmatrix} \in GL(2,k) | \alpha, \beta \in k \right\} = \mathrm{Span}_k\{I, B\} \cap GL(2,k).$$

A note on initial conditions. Let $a_0, \ldots, a_{n-1} \in k$ and consider the monic linear homogeneous differential operator $L(y) = y^{(n)} + \sum_{i=0}^{n-1} a_i y^{(i)}, y \in Hk$. Let $y_1, y_2, \ldots, y_n \in Hk$ be k-linearly independent elements such that $L(y_i) = 0$ and that $\pi_{i-1}(y_j) = \delta_i^j$ for each

$i, j = 1, 2, \ldots, n$. Let $Z := (z_1, z_2, \ldots, z_n)$, where $\{z_1, \ldots, z_n\}$ is set of linearly independent solutions of $L(y) = 0$ in $H\bar{k}$. Let $Y = (y_1, y_2, \ldots, y_n)$ and $Z = (z_1, z_2, \ldots, z_n)$. Then it follows, from the uniqueness of solutions subject to initial conditions, that for

$$T := \begin{pmatrix} \pi_0(z_1) & \pi_0(z_2) & \cdots & \pi_0(z_n) \\ \pi_1(z_1) & \pi_1(z_2) & \cdots & \pi_1(z_n) \\ \vdots & \vdots & \vdots & \vdots \\ \pi_{n-1}(z_1) & \pi_{n-1}(z_2) & \cdots & \pi_{n-1}(z_n) \end{pmatrix}$$

we have $YT = Z$. This observation along with Theorem 3.5 enables us to compute the group $G(V|k)$ with respect to the basis Y.

Example 3. Consider the operator $L(y) = y''' - 3y'' + 3y' - y$ and let $Y = (y_1, y_2, y_3)$ be linearly independent solutions of $L(y) = 0$ with initial conditions $\pi_{i-1}(y_j) = \delta_i^j$ for each $i, j = 1, 2$ and 3. Let $B = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{smallmatrix}\right)$ and note that $Y' = YB$. From Proposition 3.4, it follows that

$$G(V|k) \cong C_k(B) \cap GL(3, k) = \operatorname{Span}_{\bar{k}}\{I, B, B^2\} \cap GL(3, k).$$

It is also possible to realize the group as a full set of solutions for a system of linear equations over k. Note that

$$C_{\bar{k}}(\mathfrak{u}_2) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \in GL(3, \bar{k}) \right\}$$

and that $YT = Z$ for $T = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{smallmatrix}\right)$. Thus we have

$$\begin{aligned} G(V|k) &\cong C_k(B) = TC_{\bar{k}}(\mathfrak{u}_2)T^{-1} \cap GL(3, k) \\ &= \left\{ \begin{pmatrix} a - b + c & b - 2c & c \\ c & -b + a - 2c & c + b \\ c + b & -3b - 2c & c + 2b + a \end{pmatrix} | a, b, c \in k, a \neq 0 \right\}. \end{aligned}$$

References

[1] W. F. Keigher, *On the ring of Hurwitz series*, Comm. Algebra 25 (1997), 1845–1859.

[2] W. F. Keigher and F. L. Pritchard, *Hurwitz series as formal functions*, J. Pure Appl. Algebra 146 (2000), 291–304.