ALGEBRAIC METHODS IN DYNAMICAL SYSTEMS
BANACH CENTER PUBLICATIONS, VOLUME 94
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

ON THE PROBLEM OF BAER AND KOLCHIN IN THE PICARD-VESSIOT THEORY

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Abstract. We present the history of the development of Picard-Vessiot theory for linear ordinary differential equations. We are especially concerned with the condition of not adding new constants, pointed out by R. Baer. We comment on Kolchin's condition of algebraic closedness of the subfield of constants of the given differential field over which the equation is defined. Some new results concerning existence of a Picard-Vessiot extension for a homogeneous linear ordinary differential equation defined over a real differential field K with real closed field of constants F are also mentioned.

1. Introduction. Sophus Lie was the first to transpose the ideas of E. Galois concerning algebraic equations to differential equations. He considered local groups of transformations associated to a given equation. But in general differential equations do not admit non-trivial groups of symmetries, so these ideas give a rather weak analogy with Galois theory. At the end of the 19th century Émile Picard announced his theory for linear ordinary differential equations, which is now known as Picard-Vessiot theory. There were also attempts to build an infinite dimensional theory by Jules Drach at the end of the 19th century. Unfortunately his work contains gaps and mistakes. Vessiot tried to improve the work of Drach, but still there were some ambiguities. A new idea of infinite dimensional differential Galois theory can be found in papers by Hiroshi Umemura from the 90's of the 20th century (see [U]).

2010 *Mathematics Subject Classification*: 12Hxx, 01A05.
Key words and phrases: differential Galois theory, Picard-Vessiot theory.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc94-0-14

In the 50's of the 20th century Ellis Kolchin developed Picard-Vessiot theory in a rigorous form (see [Ko1]). He worked on Ritt's results in differential algebra and used the language of algebraic geometry introduced by André Weil. His theory was built under the assumption that the subfield of constants of the base differential field is algebraically closed. We should also mention the contribution of Abraham Seidenberg (see [Se1], [Se2]). A development of Galois theory in the non-linear case with an approach different from Umemura's and based on the theory of foliations is due to Bernard Malgrange (see [Ma]).

2. The beginning—Picard and Vessiot. The first attempt to associate to a homogeneous linear differential equation a group, in a similar manner as the Galois group of algebraic equation, was accomplished by C. E. Picard and his Ph.D. student E. Vessiot. Picard and Vessiot considered a homogeneous linear differential equation in a so–called *derivative domain of rationality*, which was an extension of a field of constants by a finite number of functions in one variable, closed under differentiation. Apart from constants and functions in the variable x a derivative domain of rationality contains indeterminates and their derivatives. A *rational differential function* is a rational function in x and some indeterminates $y_1, \dots, y_n$.

Let $y_1(x), \dots, y_n(x)$ be a fundamental system of a homogeneous linear differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_n y = 0 \tag{1}$$

The *numerical value* of a rational differential function

$$t(x, y_1, \dots, y_n, y_1', \dots, y_n', \dots)$$

is the function

$$t(x, y_1(x), \dots, y_n(x), y_1'(x), \dots, y_n'(x), \dots)$$

in the variable x obtained by substituting the function $y_i(x)$ into the indeterminate y_i (and the same for derivatives).

A *rationality group* of the equation (1) is the group $G \subset \mathrm{GL}(n)$ of linear transformations satisfying the following two dual conditions:

(i) every rational differential function t with numerical value in the rationality domain is *numerically invariant* under the action of the group G;

(ii) for any rational differential function t in the relevant rationality domain numerically invariant under the action of the group G, its numerical value belongs to the rationality domain.

Consider a function $V(x)$ which is the numerical value of a rational differential function in $(y_1, \dots, y_n, y_1', \dots, y_n', \dots)$ and such that each y_i (as well as its derivatives) is a rational differential function in $(V, V', V'', \dots)$, i.e.

$$y_i = Y_i(V, V', V'', \dots),\ (i = 1, \dots, n),$$

where Y_i is a rational differential function in the rational differential domain. Since the derivatives of y_i of order at least n can be eliminated via equation (1), the derivatives of V depend on n^2 functions, so V satisfies a differential equation of order $\leq n^2$, we call such an equation *resolvent*. We can take $V = \sum_{ij} u_i y_j$, with $u_1, \ldots, u_m$ rational functions (the *Picard resolution*).

Let $\Gamma^0 \supset \Gamma^1 \supset \Gamma^2 \supset \ldots$ be the composition sequence of $\Gamma = \Gamma^0$, the connected component of the neutral element in G. For each k there is a minimal order of resolution of a function invariant under Γ^{k-1}. Extending the rationality domain successively by general solutions of these minimal resolvents decreases the rationality group to identity and solves the equation.

A group is called *integrable* if it has a series of normal subgroups with dimensions dropping down by one and ending with a subgroup of dimension one. In his thesis Vessiot gave the following version of the Galois correspondence.

THEOREM 2.1. *A homogeneous linear differential equation is solvable by quadratures if and only if its rationality group is integrable.*

3. The problem of Reinhold Baer. The theory of Picard and Vessiot was developed further by their followers who tried to make it more rigorous. One of the main obstacles was the vagueness of the notion of derivative domain of rationality. This definition described by E. Vessiot can be found in [M], §41, page 156. In his works Loewy ([L]) used the name *Rationalitätsbereich*, which was introduced by Kronecker for finite field extensions (especially number fields). The difficulty is that we equip a field with the operation of differentiating which is "non-commutative" with respect to algebraic operations.

The proper notion of differential field was systematically studied by Baer (in [Ba]). He gave a necessary and sufficient condition for the existence of a differential field structure for a given algebraic field with a given subfield as the field of constants and established the existence of an extension of a given field that is closed with respect to integration.

Picard-Vessiot theory was developed as a generalization of the classical Galois theory. One important step was finding a counterpart of the notion of splitting field. Let us note that a differential field extension generated by a fundamental system of solutions of a linear differential equation is not uniquely determined in general. The reason is that such an extension may introduce new constants. Baer ([Ba]) posed the problem of finding an extension which does not introduce new constants. We can find a note on this matter in Felix Klein's book *Vorlesungen über hypergeometrische Funktion* (see [Kle], pp. 332–333).

4. Kolchin theory. A first systematic and rigorous formulation of Galois theory for linear homogeneous differential equations was developed by Kolchin. Kolchin took advantages of differential algebra theory developed by J. F. Ritt. One of the main ingredients of Kolchin's theory is the notion of a Picard-Vessiot extension. An extension of differential fields $K \subset L$ is a *Picard-Vessiot* extension iff L is generated over K, as a differential field, by a fundamental system of solutions of a homogeneous linear differential equation defined over K and the fields of constants of L and K coincide.

Kolchin proved the existence of a Picard-Vessiot extension for a given linear differential operator under the assumption that the field of constants of K is algebraically closed.

5. Example of A. Seidenberg. As mentioned above, Kolchin's Picard-Vessiot theory works under the assumption that the subfield of constants C_F of the given differential field F of characteristic zero over which we define the homogeneous linear ordinary differential equation considered is *algebraically closed*. One may ask whether this condition can be weakened. Below we present an example of A. Seidenberg motivated by this question (see [Se1], section 6 or [Ko1], chapter 6, ex. 1). We can find a footnote concerning this example in the paper *On the theory of Picard-Vessiot extensions* by Marvin P. Epstein (see [Ep]). In Epstein's words:

> *A. Seidenberg has constructed an example to show that, unlike the case when C is algebraically closed, it is generally impossible to choose $(\eta_1, \ldots, \eta_n)$ so that $D = C$.*

Here C denotes the subfield of constants of the base field F, $(\eta_1, \ldots, \eta_n)$ a fundamental system of solutions of the considered equation and D the subfield of constants of the extension $F\langle \eta_1, \ldots, \eta_n \rangle$.

Armand Borel also comments on this example in his article contained in *Selected Works of Ellis Kolchin* (see [Bor]) in the following way:

> *This is under our standing assumption that C_F is algebraically closed (of char. 0). If not, then Seidenberg has produced an equation such that $C_E \neq C_F$ for all differential field extensions E generated over F by a fundamental set of solutions of that equation.*

Maybe that was the reason for the lack of an existence theorem for a Picard-Vessiot extension even in the case $C_F = \mathbb{R}$. Seidenberg indeed gives an example of a differential field F with constant field $\mathbb{R}$ and a homogeneous linear differential equation defined over F such that any extension of F which contains a nontrivial solution of the equation adds new constants. But as one can observe by analysing this example, the base field F is not a *real field* (it contains -1 as a sum of squares), see [BCR]. Let us analyse this example step by step (see also [CK], example 4.6.1).

EXAMPLE 5.1. Let us consider the field of real numbers $\mathbb{R}$ with trivial derivation, i.e. $\forall x \in \mathbb{R} : x' = 0$. Let a be any solution of the equation

$$4x^2 + x'^2 = -1 \tag{2}$$

such that $a' \neq 0$. We set $F = \mathbb{R}\langle a \rangle$.

STEP 1: We will prove that $C_F = \mathbb{R}$. We observe that $F = \mathbb{R}\langle a \rangle = \mathbb{R}(a, a')$. Since $f(X) = X^2 + 4a^2 + 1 \in \mathbb{R}(a)[X]$ is irreducible, then in fact $F = \mathbb{R}(a)[a']$, where a is transcendent over $\mathbb{R}$ and a' is algebraic of degree 2 over $\mathbb{R}(a)$. Let us take $p = \alpha + \beta a' \in \mathbb{R}(a)[a']$, where $\alpha, \beta \in \mathbb{R}(a)$. If p is a constant, then

$$p' = (\alpha + \beta a')' = \frac{d\alpha}{da} a' + \frac{d\beta}{da} a'^2 + \beta a'' = 0.$$

By equation (2) we have that $a'^2 = -4a^2 - 1$. By differentiating this equality and taking into account $a' \neq 0$ we obtain $a'' = -4a$. Hence

$$\frac{d\alpha}{da} a' - (4a^2 + 1)\frac{d\beta}{da} - 4a\beta = 0.$$

Now $a' \notin \mathbb{R}(a)$, so $\frac{d\alpha}{da} = 0$. This means that $\alpha \in \mathbb{R}$ and

$$(4a^2+1)\frac{d\beta}{da} + 4a\beta = 0.$$

If $\beta = 0$, then we are done. Assume that $\beta \neq 0$. Let us write $\beta = (4a^2+1)^k \frac{\gamma}{\delta}$, where $k \in \mathbb{Z}$ and $\gamma, \delta \in \mathbb{R}[a]$ such that $4a^2+1$ does not divide them. We substitute β to the equation above and obtain that

$$(4a^2+1)^k 8ka\frac{\gamma}{\delta} + (4a^2+1)^{k+1}\frac{1}{\delta^2}(\frac{d\gamma}{da}\delta - \gamma\frac{d\delta}{da}) + 4a(4a^2+1)^k\frac{\gamma}{\delta} = 0.$$

Multiplying by δ^2 we get

$$4a(2k+1)\gamma\delta + (4a^2+1)(\frac{d\gamma}{da}\delta - \gamma\frac{d\delta}{da}) = 0.$$

But this means that $4a^2+1$ divides γ or δ. We have a contradiction. So $\beta = 0$ and $p = \alpha \in \mathbb{R}$.

Let us consider

$$y'' + y = 0, \tag{3}$$

the linear differential equation defined over F. Let η be a nonzero solution of (3). We denote $u = \frac{\eta'}{\eta}$ and consider the differential field $K = F\langle u\rangle$.

STEP 2: We will prove that $\mathbb{R} \subsetneq C_K$. We observe that u satisfies the Riccati equation $z' = -z^2 - 1$. If $1 + u^2 = 0$, then $u = \pm i$ and it is a new constant. So we assume that $1 + u^2 \neq 0$. This means that $\eta^2 + \eta'^2 \neq 0$. Let us see that $v = \eta^2 + \eta'^2$ and $w = a\eta^2 + a'\eta\eta' - a\eta'^2$ are constants. Indeed

$$v' = 2\eta'(\eta + \eta'') = 0$$

and

$$w' = 2a\eta'(\eta - \eta'') + a'\eta(\eta + \eta'') + a''\eta\eta' = (a'\eta - 2a\eta')(\eta + \eta'') = 0.$$

We denote $c = \frac{w}{v}$. If $c \notin \mathbb{R}$, then it is a new constant. So let us assume that $c \in \mathbb{R}$. The equation $cv = w$, i.e.

$$c(\eta^2 + \eta'^2) = a\eta^2 + a'\eta\eta' - a\eta'^2$$

is equivalent to

$$(c+a)u^2 - a'u + (c-a) = 0.$$

So u is a root of the quadratic polynomial given above and the discriminant of this polynomial is $\Delta = a'^2 - 4(c^2 - a^2) = -(1+4c^2)$. Now since $c \in \mathbb{R}$, then $\sqrt{\Delta} = i\sqrt{1+4c^2}$, which implies that $i \in K$ is a new constant.

6. Real case. Many interesting and significant results concerning differential algebra for real fields can be found in papers by Michael Singer, Tobias Dyckerhoff, Thomas Grill, Manfred Knebusch and Marcus Tressl. But, as mentioned before, an existence theorem for Picard-Vessiot extension even in the case $C_K = \mathbb{R}$ has not been proved before. Our observation on Seidenberg's example was a motivation for studying homogeneous linear differential equations defined over a real field. We refer the reader to [BCR] for the theory of real fields. Our results are contained in [So2]. Below we present our main theorem.

THEOREM 6.1. *Let K be a real differential field whose field of constants C_K is real closed. Consider a homogeneous linear ordinary differential equation*

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1 Y' + a_0 Y = 0, \tag{4}$$

where $a_i \in K$ for $i \in \{0, 1, \ldots, n-1\}$.

Then there exists a Picard-Vessiot extension of K for equation (4), which moreover is a real field.

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