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ON THE ENVELOPE OF A VECTOR FIELD

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To Michael Singer

Abstract. Given a vector field X on an algebraic variety V over $\mathbb{C}$, I compare the following two objects: (i) the envelope of X, the smallest algebraic pseudogroup over V whose Lie algebra contains X, and (ii) the Galois pseudogroup of the foliation defined by the vector field $X + d/dt$ (restricted to one fibre t = constant). I show that either they are equal, or the second has codimension one in the first.

1. Introduction. This note is a simple exercise in the "non-linear differential Galois theory". I refer for this theory to [Ma2], or [Ma1] (but this last paper is written in an analytic context, and one should make the translation analytic $\to$ algebraic to recover the situation of [Ma2]).

Let X be a complex algebraic variety (= a reduced scheme of finite type over $\mathbb{C}$), which I will suppose irreducible. As usual in this subject, I work birationnally, i.e. I can replace freely X by an open dense Zariski subvariety. Therefore, I can suppose X affine, non-singular, and even a finite étale covering $X \xrightarrow{p} U$ of $U = \mathbb{C}^n - Z$, Z a closed hypersurface. In that situation, I denote by x a (closed) point of X, and by $(x_1, \dots, x_n)$ the coordinates of $p(x) \in U$. I call these data "étale coordinates" on X.

Let ξ be a vector field on X; in étale coordinates, one has $\xi = \sum a_i \frac{\partial}{\partial x_i}$, $a_i \in \mathbb{C}[X]$ (= a_i regular over X). Recall the following definition (see loc. cit.).

Definition 1.1. The envelope $E(\xi)$ of ξ is the smallest (algebraic) pseudogroup on X whose Lie algebra $\operatorname{Lie} E(\xi)$ contains ξ as solution.

Now, to ξ is associated naturally a differential equation, which, in étale coordinates, is written $\frac{dx_i}{dt} = a_i(x)$. Instead of this equation, it is equivalent to consider, on $X \times \mathbb{C}$, the foliation $\{\omega_i = dx_i - a_i\, dt\}$ (the Frobenius condition is obviously satisfied here). This

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foliation is also defined by the vector field $\xi+\frac{\partial}{\partial t}$, or any of its multiples. To this foliation is associated its the *Galois pseudogroup*, which is the smallest pseudogroup on $X\times\mathbb{C}$ whose Lie algebra contains as solutions the vectors tangent to the leaves, i.e. the multiples of $\xi+\frac{\partial}{\partial t}$. Take a point $a\in\mathbb{C}$; by restriction to $X\times\{a\}$, we obtain a pseudogroup on X, independent of a (because $\xi+\frac{\partial}{\partial t}$ is fixed by translations in t). I will call this restriction, by abuse of terminology, the "Galois pseudogroup of ξ". I will denote it by $G(\xi)$, and its D-Lie algebra will be denoted by $\operatorname{Lie} G(\xi)$.

A natural question is the following: what is the relation between $E(\xi)$ and $G(\xi)$? *A priori*, they should not be very different. Before giving the general result, I will give a few very simple examples.

2. Examples

(i) $X=\mathbb{C}$, $\xi=\frac{\partial}{\partial x}$. Of course, $E(\xi)$ is the group of translations G_a over $\mathbb{C}$, more precisely the pseudogroup whose solutions are the translations, i.e. the pseudogroup $x\mapsto\bar{x}$, $\frac{d\bar{x}}{dx}=1$. On the other hand, to determine $G(\xi)$, we must look at the foliation $\{dx-dt\}$ of $\mathbb{C}^2$. This foliation admits the first integral $x-t$. Therefore, the Galois pseudogroup of this foliation is given by $\bar{x}-\bar{t}=x-t$. Setting $\bar{t}=t=a$, we find that $G(\xi)$ reduces to the identity.

(ii) $X=\mathbb{C}$ (or $\mathbb{C}^*$), $\xi=x\frac{\partial}{\partial x}$. The envelope is the pseudogroup associated to G_m, with equation (on $x\bar{x}\neq 0$) $x\frac{d\bar{x}}{dx}=\bar{x}$, or $\frac{d\bar{x}}{\bar{x}}=\frac{dx}{x}$. On the other hand, the foliation is given by $dx=x\,dt$, or better by the closed form $\frac{dx}{x}-dt$. The corresponding pseudogroup is given by $\frac{d\bar{x}}{\bar{x}}-d\bar{t}=\frac{dx}{x}-dt$. Setting $t=\bar{t}=a$, we find $\frac{d\bar{x}}{\bar{x}}=\frac{dx}{x}$; in other words we have $E(\xi)=G(\xi)$.

(iii) Take more generally $X=\mathbb{C}^n$, and take for ξ a linear vector field $\xi=\sum a_{ij}x_j\frac{\partial}{\partial x_i}$, $a_{ij}\in\mathbb{C}$. I write $A=(a_{ij})$, and I identify ξ and A. To state the result, I need a few conventions. If G is an algebraic subgroup of $G\ell(n)$ over $\mathbb{C}$, I will identify G with the pseudogroup $\widetilde{G}$ on $\mathbb{C}^n$ whose solutions are the transformations of $G(\mathbb{C})$ (cf. [Ma1]). Similarly, I identify the Lie algebra $\operatorname{Lie} G$ with the D-Lie algebra $\operatorname{Lie}\widetilde{G}$.

Note that all the (closed) subpseudogroups of $\widetilde{G\ell(n)}$ are of the form $\widetilde{G}$ for a suitable G (this result, easy, can be left to the reader).

Now, let $A=S+N$, $[S,N]=0$, be the standard decomposition of A into a semisimple and nilpotent part. The result is the following

PROPOSITION 2.1. *If ξ is semisimple, then $E(\xi)=G(\xi)$. If $\xi=S+N$, $N\neq 0$, then $G(\xi)=E(S)$. One has $E(\xi)\supset G(\xi)$, and* $\operatorname{Lie} E(\xi)=\operatorname{Lie} G(\xi)+\mathbb{C}\xi$ *(or $\mathbb{C}N$, it is equivalent). In particular, $G(\xi)$ has codimension one in $E(\xi)$.*

The proof can be left to the reader (work directly, or use the general results of the next sections). Just a few comments.

a) With the identification made above, $E(\xi)$ is simply the smallest algebraic subgroup of $G\ell(n)$ over $\mathbb{C}$ whose Lie algebra contains $\xi=A$. Its determination is essentially classical: use the Jordan normal form. The crucial point is given by the linear relations over $\mathbb{Q}$ of the eigenvalues of A.

b) $G(\xi)$ is the Galois group of the system $\frac{dx_i}{dt} = \sum a_{ij}x_j$, in the sense of the usual linear theory [Ko], [vP-Si] (cf. loc. cit.). Therefore, the determination is also classical (again use the Jordan normal form).

(iv) $X = \mathbb{C}^2$, $\xi = xy\frac{\partial}{\partial x}$.

First method, elementary. To find $E(\xi)$, one writes the flow of ξ, i.e. the solutions of $\frac{d\bar{x}}{dt} = \bar{x}\bar{y}$, $\frac{d\bar{y}}{dt} = 0$ with the initial conditions $(x, y, t = 0)$. One has $\bar{x} = xe^{yt}$, $\bar{y} = y$.

One fixes $t = a \in \mathbb{C}$, and one looks at the differential equations of $\bar{x}$ and $\bar{y}$ in terms of x, y, independently of a.

One has $\frac{\partial \bar{x}}{\partial x} = \frac{\bar{x}}{x}$, $\frac{\partial \bar{x}}{\partial y} = a\bar{x}$; to have an equation independent of a, one replaces the second equation by $d\left[\frac{1}{\bar{x}}\frac{\partial \bar{x}}{\partial y}\right] = 0$.

To obtain the corresponding infinitesimal equations, one writes $\bar{x} = x+\varepsilon u$, $\bar{y} = y+\varepsilon v$, $\varepsilon^2 = 0$. One finds $v = 0$, $x\frac{\partial u}{\partial x} = u$, $d\left[\frac{1}{x}\frac{\partial u}{\partial y}\right] = 0$. The solutions are $C_1 x\frac{\partial}{\partial x} + C_2 xy\frac{\partial}{\partial x}$.

To find $G(\xi)$, we must write the flow in a slightly different way, i.e. write $\bar{x}$, $\bar{y}$ at time $\bar{t}$ with initial conditions x, y at time t. This gives $\bar{x} = xe^{y(\bar{t}-t)}$, $\bar{y} = y$, therefore $\frac{\partial \bar{x}}{\partial x} = \frac{\bar{x}}{x}$, $\frac{\partial \bar{x}}{\partial y} = (\bar{t} - t)\bar{x}$. By restriction to $\bar{t} = t = a$, this gives $\frac{\bar{x}}{\partial x} = \frac{\bar{x}}{x}$, $\frac{\partial \bar{x}}{\partial y} = 0$, then $\bar{x} = c\bar{x}$, $\bar{y} = y$, $c \in \mathbb{C}$. The solutions of $\operatorname{Lie} G(\xi)$ are $cx\frac{\partial}{\partial x}$, $c \in \mathbb{C}$.

Second method. The preceding method has two inconveniences. First, it is not obvious *a priori* that the equation obtained really defines pseudogroups (the verification, here easy, is only made *a posteriori*). Second, the method is very particular to equations which can be integrated explicitly, and does not generalize much.

I will give another method, which is similar to the one used in [Ma2], Chap. IV. The vector field ξ is the Hamiltonian field of $h = y$ for the symplectic form $\sigma = \frac{1}{xy}dx \wedge dy$. Therefore the calculation of $G(\xi)$ is a special case of loc. cit., §IV.5. I just give the result.

The foliation is given by $\{dy,\ dx - xy\,dt\}$, with the first integral y. If we replace $dx - xy\,dt$ by $\omega = \frac{dx}{x} - y\,dt - t\,dy$, we get $d\omega = 0$. Therefore, the pseudogroup (in x, y, t) is obtained by fixing y and ω. By restriction to $t = a$, we obtain that $G(\xi)$ is defined by fixing y and $\frac{dx}{x}$; this is equivalent to the result obtained by the first method.

To find $E(\xi)$ is a little more difficult. We will see later the following result: take $\varphi \in \mathbb{C}(t)$, $\varphi \neq 0$, and denote by $G(\xi, \varphi)$ the restriction to $t = a$ (for a generic a) of the Galois pseudogroup of the *foliation* of $X \times \mathbb{C}$ defined by $\xi + \frac{1}{\varphi}\frac{\partial}{\partial t}$. Then $G(\xi, \varphi) \subset E(\xi)$, with equality for φ "sufficiently general" (see §6, Prop. 6.2).

The foliation is defined by $\{dy, \omega\}$, $\omega = \frac{dx}{x} - y\varphi\ dt$. One works as in loc. cit.

One has $d\omega = dy \wedge \omega_1$, $\omega_1 = -\varphi\ dt$, and $d\omega_1 = 0$. One lifts these equations in a suitable frame bundle, by defining $\widetilde{\omega} = \omega + u\ dy$, $\widetilde{\omega}_1 = \omega_1 - du$, $u \in \mathbb{C}$. One has again $d\widetilde{\omega} = dy \wedge \widetilde{\omega}_1$, $d\widetilde{\omega}_1 = 0$. Now, $(dy, \widetilde{\omega}, \widetilde{\omega}_1)$ give a prolongation of the foliation to $X \times \mathbb{C}_t \times \mathbb{C}_u$, and an "admissible pseudogroup" is obtained if one fixes y, $\widetilde{\omega}$, $\widetilde{\omega}_1$. It will be the Galois pseudogroup of the prolongation if the calculation is "minimal", i.e. if the class of ω_1 in the relative de Rham cohomology of $X \times \mathbb{C}_t/\mathbb{C}_y$ is not zero. This will be the case if $\varphi\ dt$ is not exact i.e. $\varphi\ dt \neq d\psi$, $\psi \in \mathbb{C}(t)$. Suppose that this is the case: then, fixing $t = a$, one finds that the prolongation of $G(\xi, \varphi)$ to $X \times \mathbb{C}_u$ is obtained by fixing y, $\frac{dx}{x} + u\ dy, du$.

I leave to the reader to go down to X, and to verify that the result is the same as the one given above.

REMARK. The same method permits, more generally, by a suitable modification of [Ma2], §IV.5, to calculate the envelope of a symplectic integrable vector field. I leave this to people who are interested.

3. First integrals. As in §1, let X be a complex algebraic variety. I denote by $\mathbb{C}[X]$ (resp. $\mathbb{C}(X)$) its regular (resp. rational) functions. Let ξ be a vector field on X. I denote by K the field of first integrals of ξ, i.e. the subfield of $\mathbb{C}(X)$ annihilated by ξ. Similarly, I consider the vector field $\xi + \frac{\partial}{\partial t}$ on $X \times \mathbb{C}$, and I denote by $L \subset \mathbb{C}(X \times \mathbb{C})$ its field of first integrals. Obviously, $K \subset L$. One has the following result.

PROPOSITION 3.1. *The following statements are equivalent:*

(i) One has $L \neq K$.
(ii) There exists $f \in \mathbb{C}(X)$ *with* $\xi f = 1$. *Furthermore* $L = K(t - f)$.

The proof was suggested to me by a remark of J. A. Weil.

(ii)⇒(i) is trivial. Now, let us suppose that $\xi + \frac{\partial}{\partial t}$ admits a first integral $g \in \mathbb{C}(X \times \mathbb{C})$ depending effectively on t; one can write $g = \frac{P}{Q}$, $P, Q \in \mathbb{C}(X)[t]$, relatively prime (as polynomials in t). One has $(\xi + \frac{\partial}{\partial t})P/P = (\xi + \frac{\partial}{\partial t})Q/Q = c$, with $c \in \mathbb{C}(X)[t]$. Looking at the degrees in t, one shows that, actually, $c \in \mathbb{C}(X)$.

Let $P = a_0 + a_1 t + \cdots + a_n t^n$, $a_n \neq 0$. From $(\xi + \frac{\partial}{\partial t})P = cP$, one deduces $\xi a_n = c a_n$; therefore $(\xi + \frac{\partial}{\partial t})\,(\frac{P}{a_n}) = 0$. Therefore $\overline{P} = \frac{P}{a_n}$ is a first integral of ξ. The same result holds for Q.

Now, note that $\frac{\partial}{\partial t}$ commutes with $\xi + \frac{\partial}{\partial t}$; therefore the $(\frac{\partial}{\partial t})^k \overline{P}$ are also first integrals. Taking $k = n - 1$, we get a first integral of the form $t - f$.

Finally, we must prove that $L = K(t - f)$. The preceding results show that it is sufficient to consider the first integrals which are polynomial in t. If $R = a_0 + \cdots + a_n t^n$, $a_n \neq 0$, is such a first integral, a_n is a first integral by the preceding calculation. Now, replace R by $R - a_n(t - f)^n$ and proceed by recurrence.

EXAMPLE 3.2. Take the vector field $\xi = (x + y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, corresponding to the Jordan matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. One has $\xi(\frac{x}{y}) = 1$. A similar result holds for $\xi = S + N$, with $N \neq 0$ (notations of §2, iii)). On the other hand, I leave it to the reader to prove that $\xi f = 1$ has no solution if $\xi = S$.

REMARK 3.3. If, instead of $\xi + \frac{\partial}{\partial t}$, we take $\xi + ct\frac{\partial}{\partial t}$, $c \in \mathbb{C}$, the same method gives the following result. Denote again by K (resp. L) the field of first integrals of ξ (resp. $\xi + ct\frac{\partial}{\partial t}$). Then, the following statements are equivalent

(i) $L \neq K$.
(ii) Let $\Lambda = \{k \in \mathbb{Z};\ \exists a \in \mathbb{C}(X),\ \text{with } \xi a + cka = 0\}$. Λ is obviously a subgroup of $\mathbb{Z}$. Then $\Lambda \neq \{0\}$. In that case, let $\ell > 0$ be the generator of Λ, and let $a \in \mathbb{C}(X)$ satisfy $\xi a + c\ell a = 0$. Then $at^\ell \in L$, and $L = K(at^\ell)$.

4. Prolongation

(i) Let, as before, $E(\xi)$ be the envelope of ξ. I recall a method to describe it, given in [Ca2]. For $k \geq 0$, let $R_k(X)$, or R_k, be the space of k-frames on X, i.e. of invertible k-jets $(\mathbb{C}^n, 0) \to X$ ($n = \dim X$). It is a principal bundle (or "torsor") on the group Γ_k of invertible k-jets $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Let $X \xrightarrow{p} U$, $U \subset \mathbb{C}^n$, and $(x; x_1, \ldots, x_n)$ étale coordinates on X as in §1. Then the coordinates on $R_k(U)$ are $(x_{i,\alpha})$, $1 \leq i \leq n$, $|\alpha| \leq k$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, with $x_{i,0} = x_i$, $\det(x_{i,j}) \neq 0$ and $R_k(X) = X \times_U R_k(U)$. On $\varprojlim R_k(U)$, put

$$D_i = \sum_{j,d} x_{j,d+\varepsilon_i} \frac{\partial}{\partial x_{j,\alpha}} \quad \text{and} \quad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n};$$

denote by the same letters the lifting of these operators to $R_k(X)$. Now, the vector field $\xi = \sum a_i \frac{\partial}{\partial x_i}$ on X has a canonical lifting $R_k \xi$ on $R_k(X)$. This lifting is fixed by Γ_k; and, in étale coordinates,

$$R_k \xi = \xi + \sum_{1 \leq |\alpha| \leq k} D^\alpha a_i \frac{\partial}{\partial x_{i,\alpha}}$$

(see e.g. [Ol]). Then the pseudogroup $E(\xi)$ on X is defined by a collection of closed subvarieties $Z_k \subset J_k^*(X)$, $J_k^*(X)$ the space of invertible jets of order k from X to X. (Maybe after restricting X) Z_k is a subgroupoid of $J_k^*(X)$. This is equivalent to giving a $\widetilde{Z}_k$, equivalence relation in R_k stable by Γ_k (see [Ma1] or [Ma2]). After restricting again X, such an equivalence relation is given by a quotient $R_k \xrightarrow{\pi} S_k$, and one has $\widetilde{Z}_k = R_k \times_{S_k} R_k$ (see references in loc. cit.).

But, practically by definition, $\widetilde{Z}_k$ is the smallest equivalence relation on R_k for which $R_k \xi$ is vertical, i.e. tangent to the fibers of π. In terms of first integrals, this means the following: if K_k is the subfield of $\mathbb{C}(R_k)$ of first integrals, then K_k is the field of functions $\mathbb{C}(S_k)$. Then, $E(\xi) = \{Z_k\}$ can be described by the successive first integrals of the $R_k \xi$ (observe that a first integral of $R_k \xi$ is also one for $R_\ell \xi$, $\ell \geq k$).

(ii) We will now give a similar description of $G(\xi)$. *A priori*, we could do the same thing with X replaced by $X \times \mathbb{C}$, and ξ replaced by the family of all the multiples of $\xi + \frac{\partial}{\partial t}$. But, as explained in [Ma2] (see I.6 to I.8), it is sufficient to work with the *transverse frame bundles* of the foliation. Choosing $X \times \mathbb{C} \to \mathbb{C}$ as a transverse projection, this transverse frame bundle is identified with $R_k \times \mathbb{C}$.

The prolongation of the foliation to $R_k \times \mathbb{C}$ is given by the vector field $R_k \xi + \frac{\partial}{\partial t}$: to prove that, it is sufficient to prove that the corresponding differential equations $\frac{dx_{i,\alpha}}{dt} = D^\alpha a_i$ coincide with the variational equation of order k (cf. loc. cit. I.8), which is obvious.

Let now $Z'_k \subset J_k^*(X \times \mathbb{C})$ be the equations of order k of the Galois pseudogroups of the foliation $\{\xi + \frac{\partial}{\partial t}\}$. We have a description of Z'_k similar to that one of Z_k. Call I_k the subfield of $\mathbb{C}(R_k \times \mathbb{C})$ of first integrals of $R_k + \frac{\partial}{\partial t}$. Choosing a general $a \in \mathbb{C}$, this field can be identified with its restriction to $\mathbb{C}(R_k) = \mathbb{C}(R_k \times \{a\})$; and Z'_k is described by L_k as Z_k is by K_k.

(iii) Now, we are in a position to apply the results of §3, by just replacing ξ by $R_k \xi$ and X by R_k. We obtain the following result:

Theorem 4.1. *(i) We have always $G(\xi) \subset E(\xi)$.*

(ii) If $G(\xi) \not\subset E(\xi)$, there exists a $k \geq 0$ such that $L_k \neq K_k$. In that case, there exists an $f \in \mathbb{C}(R_k)$ satisfying $R_k(\xi)f = 1$, and one has $L_k = K_k(f)$. If this is true for k, it is also true for $\ell \geq k$, with the same function f.

(I wrote $K_k(f)$ instead of $K_k(t-f)$; this is equivalent, f being transcendental over K; otherwise, it would be a first integral by a classical lemma; see e.g. [Ro]).

Examples. In the linear case (Example 2(iii)) the dichotomy occurs already for $k = 0$. But this is not always the case. For instance, if $\xi = xy\frac{\partial}{\partial x}$ the equation $\xi f = 0$ has no solution, but the equation $(R_1\xi)f = 1$ has one.

Explicitly, denoting by $(x, y; x_1, y_1, x_2, y_2)$ the coordinates on $R_1(X)$, with here $X = \mathbb{C}^2$, we have

$$R_1\xi = xy\frac{\partial}{\partial x} + (x_1y + xy_1)\frac{\partial}{\partial x_1} + (x_2y + xy_2)\frac{\partial}{\partial x_2}, \text{ and } f = \frac{x_1}{xy_1}.$$

This reflects the fact that (by both methods) we had to make a prolongation to order one to calculate $G(\xi)$.

Remark 4.2. With $\xi + ct\frac{\partial}{\partial t}$ instead of $\xi = \frac{\partial}{\partial t}$, starting from Remark 3.3 and arguing as in §4, we get the following result (with obvious notations, similar to those of Theorem 4.1).

(i) One has $G(\xi, ct\frac{\partial}{\partial t}) \subset E(\xi)$.
(ii) If $G(\xi, ct\frac{\partial}{\partial t}) \neq E(\xi)$, there exists a $k \geq 0$ such that $L_k \neq K_k$, L_k the field of first integrals of $\xi + ct\frac{\partial}{\partial t}$. Then $0 \neq \Lambda = \{k \in \mathbb{Z} \mid \exists a \in \mathbb{C}(R_k), R_k\xi a + cka = 0\}$; let $\ell > 0$ be the generator of Λ, and let $a \in \mathbb{C}(R_k)$ satisfy $R_\ell\xi a + c\ell a = 0$. Then $L_k = K_k(at^\ell)$ $[= K_k(a)]$, and the same is true for all $m \geq k$.

5. Lie algebras. Consider again $\xi + \frac{\partial}{\partial t}$. It remains to analyze the relations between $\operatorname{Lie} G(\xi)$ and $\operatorname{Lie} E(\xi)$ when $G(\xi) \neq E(\xi)$.

For that purpose, we have to recall briefly the relation between Lie pseudogroups and their Lie algebras (cf. [Ma1] or [Ma2]). Let X be a smooth $\mathbb{C}$-variety, $T = T_X$ its tangent bundle, and $\mathcal{O}_X$ (resp. Ω_X) the sheaf of regular functions (resp. 1-forms) on X. The ingredients are as follows.

(i) Denote by J_kT the space of k-jets of sections of T, and by $J_k^*(X)$ the groupoid of k-jets of invertible maps from X to X. Then J_kT is canonically isomorphic to the normal bundle along the identity of $J_k^*(X)$.
(ii) Denote by D_k the sheaf of linear differential operators of order $\leq k$ on X, and put $D = \bigcup D_k$. Then D_k is an $\mathcal{O}_X$-bimodule, and $J_k(T)$ is the vector bundle associated to $D_k \otimes_{\mathcal{O}_X} \Omega_X$ by the *contravariant* correspondence "vector bundles" $\leftrightarrow$ "coherent sheaves" (cf. e.g. [Gr]). In particular, the sheaf $\underline{J_kT}$ of sections of J_kT is the dual over $\mathcal{O}_X$ of $D_k \otimes_{\mathcal{O}_X} \Omega_X$.
(iii) Let $R_k \xrightarrow{\pi} X$ the frame bundle of X of order k and $T(R_k)$ its tangent bundle. Then the sections of J_kT are canonically isomorphic to the sections of $T(R_k)$ stable by Γ_k (definition in §4). Localizing over X, one gets an isomorphism of sheaves $\underline{J_kT} \sim [\pi_*\underline{T(R_k)}]^{\Gamma_k}$. Of course, one has also a similar result for the fibers over a point $a \in X$.

Denoting by ρ the map $\underline{J_k T} \to \pi_* \underline{T(R_k)}$, one has, in particular, $\rho(j_k\xi) = R_k\xi$ for a vector field ξ on X. I leave to the reader to give the explicit expression in étale coordinates of ρ, using the expression of R_k given in §4.

Now, let $Z = \{Z_k\}$ be a pseudogroup on X, with Z_k a closed subvariety of $J_k^*(X)$. Restricting X if necessary, we can suppose that all the Z_k are smooth, and the maps $Z_k \to Z_\ell$ $(0 \le \ell \le k)$ are smooth and surjective. Let $L_k = \operatorname{Lie} Z_k$ be the normal bundle of the identity on Z_k. Then L_k is a vector subbundle of $J_k T$, and the sections of its dual L_k^* are a quotient L_k^* of $D_k \otimes_{\mathcal{O}} \Omega^1$; the collection of the L_k^* is a D-module L^* (similarly, for each $k \ge 0$, the first prolongation $p_1 L_k$ contains L_{k+1}).

Let now $\widetilde{Z}_k$ be the equivalence relation on R_k corresponding to Z_k. Then the description of L_k in terms of $\widetilde{Z}_k$ is the following: we take (locally on X) the vector fields on R_k which are tangent to the equivalence classes of $\widetilde{Z}_k$, and are Γ_k-invariant (note that $\widetilde{Z}_k$ is stable by Γ_k). If $\widetilde{Z}_k$ is given by a projection $R_k \xrightarrow{\pi} S_k$, this means that we take the vector fields on R_k which are Γ_k-invariant and tangent to the fibers of π.

To apply this to our situation we need one more definition. Let ξ be the given vector field on X. To ξ we can associate $D_k\xi$, i.e. $D_k \otimes_{\mathcal{O}} \Omega^1/P$, P the sub-$\mathcal{O}_X$-module annihilating ξ. Outside of the singularities of ξ, the dual over $\mathcal{O}_X$ is the rank one bundle $J_k\xi$ generated by $j_k\xi$ over X. We denote the direct limit of $D_k\xi$ by $D\xi$, and the inverse limit of $J_k\xi$ by $J\xi$. Then, the theorem is the following.

THEOREM 5.1. *If $G(\xi) \neq E(\xi)$, then* $\operatorname{Lie} E(\xi) = \operatorname{Lie} G(\xi) \oplus J\xi$.

It is sufficient to prove this for every k. Write $E(\xi) = \{Z_k\}$, $Z_k \subset J_k^*(X)$, and similarly $G(\xi) = \{Z'_k\}$; suppose $Z'_k \neq Z_k$. As, by definition, Z_k is the smallest subgroupoid of $J_k^*(X)$ whose Lie algebra contains $j_k\xi$, we have $j_k\xi \in \operatorname{Lie} Z_k$, $j_k\xi \notin \operatorname{Lie} Z'_k$.

On the other hand, the description of $\operatorname{Lie} G(\xi)$ and $\operatorname{Lie} E(\xi)$ in terms of first integrals, and the results of §4 show that $\operatorname{Lie} Z'_k$ has codimension one in $\operatorname{Lie} Z_k$, as vector bundles on X. Therefore, $\operatorname{Lie} Z_k = \operatorname{Lie} Z'_k \oplus J_k\xi$. This proves the theorem.

This result explains what we have obtained in the examples: either ξ is a solution of $\operatorname{Lie} G(\xi)$, and $G(\xi) = E(\xi)$, or the solutions of $\operatorname{Lie} E(\xi)$ are obtained by adding ξ to the solutions of $\operatorname{Lie} G(\xi)$.

REMARK 5.2. The same result holds for $\frac{\partial}{\partial t}$ replaced by $ct\frac{\partial}{\partial t}$, with the same proof. I omit the details.

6. Generalization. I just sketch the results. They are based on the following beautiful result by Rosenlicht [Ro].

THEOREM 6.1. *Let ξ be a vector field on X as before, and take $\varphi \in \mathbb{C}(t)$, $\varphi \neq 0$. Denote by η the vector field $\xi + \frac{1}{\varphi}\frac{\partial}{\partial t}$ on $X \times \mathbb{C}$. Let K (resp. L) be the subfield of $\mathbb{C}(X)$ (resp. $\mathbb{C}(X \times \mathbb{C})$ of first integrals of ξ (resp. η). Then $K = L$ unless φ has one of the following forms:*

(i) $\varphi = \psi'$, $\psi \in \mathbb{C}(t)$.
(ii) $\varphi = c\frac{\psi'}{\psi}$, $\psi \in \mathbb{C}(t)$.

Consider now the Galois pseudogroup of the *foliation* defined by η, and denote by $G(\xi,\varphi)$, *its restriction* to $t=a$, for *a general value* $a\in\mathbb{C}$. Using the arguments of prolongation of §4 and the preceding theorem, we get the following result.

PROPOSITION 6.2. *If φ does not belong to the exceptional cases (i) or (ii), then $G(\xi,\varphi)=E(\xi)$.*

This is precisely the result mentioned in Example 2(iv).

It remains to analyze the exceptional cases. Suppose we are in case (i). Then we remark that the map $X\times\mathbb{C}\to X\times\mathbb{C}$, $(x,t)\mapsto(x,s=\psi(t))$ maps η to the vector field $\xi+\frac{\partial}{\partial s}$. From results by Casale on the behavior of the Galois pseudogroup under projections, it follows that $G(\xi,\varphi)=G(\xi,1)$ (see [Ca1]). Therefore, we are reduced to a case already studied.

Of course, if $f\in\mathbb{C}(R_k)$ satisfies $(R_k\,\xi)f=1$, then $f-\psi$ is a first integral of $R_k\,\xi+\frac{1}{\varphi}\frac{\partial}{\partial t}$, or, equivalently, a differential invariant of η. But the preceding result shows that, actually, $(f-\psi)$ generates the "new" differential invariants of η.

The case $\varphi=c\frac{\psi'}{\psi}$ is treated similarly. I omit the details.

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