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PICARD-VESSIOT THEORY IN GENERAL GALOIS THEORY

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Abstract. We give a transparent proof that difference Picard-Vessiot theory is a part of the general difference Galois theory. We apply the proof to iterative q-difference Picard-Vessiot theory to show that Picard-Vessiot theory for iterative q-difference field extensions is in the scope of the general Galois theory of Heiderich. We also show that Picard-Vessiot theory is commutative in the sense that studying linear difference-differential equations, no matter how twisted the operators are, we cannot encounter quantification of the Galois groupoid.

1. Introduction. Morikawa proved in [12], Section 3 that our general difference Galois theory includes difference Picard-Vessiot theory as a particular case. As his proof does not seem accessible, at least for our taste, our aim is to give a transparent proof, dependent on few fundamental principles so that it would work also for other generalized Picard-Vessiot theories (Theorem 3.20 in 3.1). Namely, we expect that we could apply this proof, without any change to the Picard-Vessiot theories to show that these Picard-Vessiot theories are in the scope of the general Galois theory of D-module fields of Heiderich [7], [8]. Hardouin's Picard-Vessiot theory [5] for iterative q-difference equations is one of such instances.

We show in 3.2 how our proof works for the Picard-Vessiot theory of differential equations.

In 3.3 we apply the proof to the Picard-Vessiot theory of iterative q-difference fields (Theorem 3.50).

In general setting, for a given difference-differential algebra A of certain type, the generalized universal Taylor morphism

$$\iota : A \to R_A$$

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is a difference-differential algebra morphism where R_A is a non-commutative ring. See for example Proposition 3.44. The Galois hull that is an algebraic counterpart of groupoid, is constructed in the non-commutative algebra R_A.

We show that Picard-Vessiot theory is commutative (Proposition 3.52). Namely, for a Picard-Vessiot extension, however non-commutative the ring of operators may be, the Galois hull is a commutative algebra showing that we cannot encounter the quantum groupoid in studying linear difference-differential equations.

So the challenging question is to ask what happens when the Galois hull is not commutative. Does a non-commutative Galois hull describe a quantification of the Galois groupoid? Give an example of equations with non-commutative Galois hull. See Proposal 4.7 at the end of the note.

We use the notation and convention of Morikawa [12]. So we are in *characteristic* 0. When we speak of rings without mentioning commutativity, they are commutative except for Lie algebras.

2. Review of our previous paper

2.1. Notation. Let us recall basic notation. Let (R, σ) be a difference ring so that $\sigma : R \to R$ is an endomorphism of a ring R. When there is no danger of confusion, we simply speak of the difference ring R without referring to the endomorphism σ. We often have to talk however about the abstract ring R that we denote by $R^\natural$. For a ring S, we denote by $F(\mathbb{N}, S)$ the ring of functions on the set

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

taking values in the ring S. For a function $f \in F(\mathbb{N}, S)$, we define the shifted function $\Sigma f \in F(\mathbb{N}, S)$ by

$$(\Sigma f)(n) = f(n+1) \quad \text{for every } n \in \mathbb{N}.$$

Hence the shift operator

$$\Sigma : F(\mathbb{N}, S) \to F(\mathbb{N}, S)$$

is an endomorphism of the ring $F(\mathbb{N}, S)$ so that $(F(\mathbb{N}, S), \Sigma)$ is a difference ring.

2.2. Universal Euler morphism. Let (R, σ) be a difference ring and S a ring. An Euler morphism is a difference morphism

$$(R, \sigma) \to (F(\mathbb{N}, S), \Sigma). \tag{1}$$

Given a difference ring (R, σ), among the Euler morphisms (1), there exists a universal one. In fact, for an element $a \in R$, we define the function $u[a] \in F(\mathbb{N}, R^\natural)$ by

$$u[a](n) = \sigma^n(a) \quad \text{for } n \in \mathbb{N}.$$

Then the map

$$\iota : (R, \sigma) \to (F(\mathbb{N}, R^\natural), \Sigma) \quad a \mapsto u[a]$$

is the universal Euler morphism (Proposition 2.5, [12]).

2.3. Galois hull $\mathcal{L}/\mathcal{K}$. Let $(L,\sigma)/(k,\sigma)$ be a difference field extension such that the abstract field $L^\natural$ is finitely generated over the abstract base field $k^\natural$. We construct the Galois hull $\mathcal{L}/\mathcal{K}$ in the following manner.

We take a mutually commutative basis

$$\{D_1, D_2, \dots, D_d\}$$

of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$ of $k^\natural$-derivations of the abstract field $L^\natural$. So we have

$$[D_i, D_j] = D_iD_j - D_jD_i = 0 \quad \text{for } 1 \le i,\, j \le d.$$

Now we introduce a partial differential structure on the abstract field $L^\natural$ using the derivations $\{D_1, D_2, \dots, D_d\}$. Namely we set

$$L^\sharp := (L^\natural, \{D_1, D_2, \dots, D_d\})$$

that is a partial differential field. Similarly we define a differential structure on the ring $F(\mathbb{N}, L^\natural)$ of functions taking values in $L^\natural$ by considering the derivations $\{D_1, D_2, \dots, D_d\}$. In other words, we work with the differential ring $F(\mathbb{N}, L^\sharp)$. So the ring $F(\mathbb{N}, L^\natural)$ has difference-differential structure defined by the shift operator Σ and the set $\{D_1, D_2, \dots, D_d\}$ of derivations. Since there is no danger of confusion of the choice of the difference operator Σ, we denote this difference-differential ring by

$$F(\mathbb{N}, L^\sharp).$$

We have the universal Euler morphism

$$\iota : L \to F(\mathbb{N}, L^\natural) \tag{2}$$

that is a difference morphism. We add further the $\{D_1, D_2, \dots, D_d\}$-differential structure on $F(\mathbb{N}, L^\natural)$ or we replace the target space $F(\mathbb{N}, L^\natural)$ of the universal Euler morphism (2) by $F(\mathbb{N}, L^\sharp)$ so that we have

$$\iota : L \to F(\mathbb{N}, L^\sharp).$$

In the definition below, we work in the difference-differential ring $F(\mathbb{N}, L^\sharp)$ with difference operator Σ and differential operators $\{D_1, D_2, \dots, D_d\}$. We may identify $L^\sharp$ with the set of constant functions on $\mathbb{N}$. Namely,

$$L^\sharp = \{f \in F(\mathbb{N}, L^\sharp) \mid f(0) = f(1) = f(2) = \cdots \in L^\sharp\}.$$

Hence $L^\sharp$ is a difference-differential sub-field of the difference-differential ring $F(\mathbb{N}, L^\sharp)$. The action of the shift operator on $L^\sharp$ being trivial, the notation is adequate. Similarly, we set

$$k^\sharp := \{f \in F(\mathbb{N}, L^\sharp) \mid f(0) = f(1) = f(2) = \cdots \in k \subset L^\sharp\}.$$

So both the shift operator and the derivations act trivially on $k^\sharp$ and so $k^\sharp$ is a difference-differential sub-field of $L^\sharp$ and hence of the difference-differential algebra $F(\mathbb{N}, L^\sharp)$.

Definition 2.1. The *Galois hull* $\mathcal{L}/\mathcal{K}$ is a difference-differential sub-algebra of $F(\mathbb{N}, L^\sharp)$, where $\mathcal{L}$ is the difference-differential sub-algebra generated by the image $\iota(L)$ and $L^\sharp$ and $\mathcal{K}$ is the sub-algebra generated by the image $\iota(k)$ and $L^\sharp$. So $\mathcal{L}/\mathcal{K}$ is a difference-differential algebra extension with difference operator Σ and derivations $\{D_1, D_2, \dots, D_d\}$.

2.4. Universal Taylor morphism. The differential counterpart of Euler morphism is Taylor morphism. Let

$$(R, \{\partial_1, \partial_2, \dots, \partial_d\})$$

be a partial differential ring. So $\partial_i : R \to R$ are mutually commutative derivations. For a ring S, the power series ring

$$\left(S[[X_1, X_2, \dots, X_d]], \left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \dots, \frac{\partial}{\partial X_d}\right\}\right)$$

gives us an example of partial differential ring.

A Taylor morphism is a differential morphism

$$(R, \{\partial_1, \partial_2, \dots, \partial_d\}) \to \left(S[[X_1, X_2, \dots, X_d]], \left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \dots, \frac{\partial}{\partial X_d}\right\}\right). \tag{3}$$

For a differential algebra $(R, \{\partial_1, \partial_2, \dots, \partial_d\})$, among Taylor morphisms (3), there exists the universal one ι_R given below.

Definition 2.2. The *universal Taylor morphism* is a differential morphism

$$\iota_R : (R, \{\partial_1, \partial_2, \dots, \partial_d\}) \to \left(R^\natural[[X_1, X_2, \dots, X_d]], \left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \dots, \frac{\partial}{\partial X_d}\right\}\right) \tag{4}$$

defined by the formal power series expansion

$$\iota_R(a) = \sum_{n \in \mathbb{N}^d} \frac{1}{n!} \partial^n a \, X^n$$

for an element $a \in R$, where we use the standard multi-index notation.

Namely, for $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$,

$$|n| = \sum_{i=1}^{d} n_i, \qquad \partial^n = \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_d^{n_d},$$

$$n! = n_1! n_2! \cdots n_d!, \qquad X^n = X_1^{n_1} X_2^{n_2} \cdots X_d^{n_d}.$$

See Proposition (1.4) in Umemura [17].

2.5. The functor $\mathcal{F}_{L/k}$ of infinitesimal deformations. For the partial differential field $L^\sharp$, we have the universal Taylor morphism

$$\iota_{L^\sharp} : L^\sharp \to L^\natural[[W_1, W_2, \dots, W_d]] = L^\natural[[W]], \tag{5}$$

where we replaced the variables X in (4) by W's for a notational reason. The universal Taylor morphism (5) gives a difference-differential morphism

$$F(\mathbb{N}, L^\sharp) \to F(\mathbb{N}, L^\natural[[W_1, W_2, \dots, W_d]]). \tag{6}$$

Restricting the morphism (6) to the difference-differential sub-algebra $\mathcal{L}$ of $F(\mathbb{N}, L^\sharp)$, we get a difference-differential morphism $\mathcal{L} \to F(\mathbb{N}, L^\natural[[W_1, W_2, \dots, W_d]])$ that we denote by ι. So we have the difference-differential morphism

$$\iota : \mathcal{L} \to F(\mathbb{N}, L^\natural[[W_1, W_2, \dots, W_d]]). \tag{7}$$

Similarly for every $L^\natural$-algebra A, thanks to the differential morphism

$$L^\natural[[W]] \to A[[W]]$$

we have the canonical difference-differential morphism

$$\iota : \mathcal{L} \to F(\mathbb{N}, A[[W_1, W_2, \ldots, W_d]]). \tag{8}$$

We define the functor

$$\mathcal{F}_{L/k} : (Alg/L^\natural) \to (Sets)$$

from the category $(Alg/L^\natural)$ of $L^\natural$-algebras to the category $(Sets)$ of sets, by associating to an $L^\natural$-algebra A the set of infinitesimal deformations of the canonical morphism (7). So

$$\begin{aligned}\mathcal{F}_{L/k}(A) = \{f : \mathcal{L} \to F(\mathbb{N}, A[[W_1, W_2, \ldots, W_d]]) \mid f \text{ is a difference-differential}\\ \text{morphism congruent to the canonical morphism } \iota \text{ modulo nilpotent elements}\\ \text{such that} f = \iota \text{ when restricted to the sub-algebra } \mathcal{K}\}.\end{aligned}$$

See Definition 2.13 in [12] for a rigorous definition.

2.6. Group functor Inf-gal (L/k) **of infinitesimal automorphisms.** The Galois group in our Galois theory is the group functor

$$\text{Inf-gal}\,(L/k) : (Alg/L^\natural) \to (Grp)$$

defined by

$$\begin{aligned}\text{Inf-gal}\,(L/k)(A) = \{f : \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \to \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \mid f \text{ is a difference-differential}\\ \mathcal{K} \otimes_{L^\sharp} A[[W]]\text{-automorphism continuous with respect to the } W\text{-adic topology}\\ \text{and congruent to the identity modulo nilpotent elements}\}\end{aligned}$$

for an $L^\natural$-algebra A. See Definition 2.19 in [12].

Then the group functor Inf-gal (L/k) operates on the functor $\mathcal{F}_{L/k}$ in such a way that the operation (Inf-gal $(L/k), \mathcal{F}_{L/k}$) is a principal homogeneous space (Theorem 2.20, [12]).

3. Picard-Vessiot theory is a part of general Galois theory

3.1. Difference Picard-Vessiot theory. As we study Picard-Vessiot extensions, we further assume that the endomorphism $\sigma : R \to R$ of a difference ring (R, σ) is an automorphism and that for a base difference field (k, σ), the field C_k of constants is algebraically closed. We consider a linear difference system

$$\sigma(y) = Ay \tag{9}$$

with $A \in \mathrm{GL}_n(k)$ and $y = (y_{ij})$ is an unknown invertible $(n \times n)$-matrix.

Let $X = (X_{ij})$ be an $(n \times n)$-matrix of indeterminates X_{ij} over k. We extend the automorphism σ of the field k to the automorphism of the k-algebra $k[X_{ij}, (\det X)^{-1}]$ by setting $\sigma(X) = AX$. We denote the extended automorphism of the algebra $k[X_{ij}, (\det X)^{-1}]$ also by σ so that

$$(k[X_{ij}, (\det X)^{-1}], \sigma)/(k, \sigma)$$

is a difference ring extension. For a maximal σ-invariant ideal I of the difference ring $k[X_{ij}, (\det X)^{-1}]$, we call the quotient difference ring $k[X_{ij}, (\det X)^{-1}]/I$ a Picard-Vessiot ring of the linear difference equation (9) over k. We know that Picard-Vessiot rings are determined up to isomorphism ([3], [14]). The total Picard-Vessiot ring L is the total ring of fractions of the Picard-Vessiot ring.

We call the group $\mathrm{Gal}\,(L/k)$ of k-automorphisms of the ring L commuting with the action of σ the difference Galois group of equation (9) over the field k. The following theorems are well-known.

Theorem 3.1. *The difference Galois group* $\mathrm{Gal}\,(L/k)$ *has a natural structure of reduced linear algebraic group over* C.

See Theorem 1.13 of [14].

Theorem 3.2. *Let* L/k *be the total Picard-Vessiot ring. Then there exist idempotents* $e_0, e_1, \ldots, e_{t-1} \in L$ *with* $e_i e_j = 0$ *for* $0 \le i \ne j \le t-1$ *such that*

(i) $L = L_0 \oplus L_1 \oplus \cdots \oplus L_{t-1}$ *where* $L_i = Le_i$,
(ii) $\sigma(e_i) = e_{i+1 \mod t}$,
(iii) $(L_0, \sigma^t)/(k, \sigma^t)$ *is a Picard-Vessiot field.*

Moreover there exists an exact sequence

$$0 \to \mathrm{Gal}\,(L_0/k) \to \mathrm{Gal}\,(L/k) \to \mathbb{Z}/t\mathbb{Z} \to 0 \tag{10}$$

of algebraic groups over C.

See Corollary 1.16 and Corollary 1.17 of [14]. By exact sequence (10), we get an isomorphism

$$\mathrm{Lie}\,(\mathrm{Gal}\,(L_0/k)) \cong \mathrm{Lie}\,(\mathrm{Gal}\,(L/k))$$

of C-Lie algebras. As we are interested in the Lie algebra of the Galois group, we may assume that the Picard-Vessiot ring R is a domain so that the total Picard-Vessiot ring L is a field. Replacing the base field by its algebraic closure in L, we may assume that the base field k is algebraically closed in L.

Namely we consider a difference field extension $(L, \sigma)/(k, \sigma)$, called a Picard-Vessiot field extension, such that

(1) $L = k(z_{ij})_{1 \le i,\, j \le n}$,
(2)
$$\sigma(Z) = AZ \tag{11}$$
with $A \in \mathrm{GL}_n(k)$, $Z = (z_{ij}) \in \mathrm{GL}_n(L)$, and such that
(3) $C_L = C_k$.

We take a mutually commutative basis $D_1, D_2, \ldots, D_d$ of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/k^\natural)$ of derivations. So as we assume that the base field k is algebraically closed in L, we have

$$\{c \in L \,|\, D_i(c) = 0 \text{ for all } 1 \le i \le d\} = k.$$

We show that after an extension of the field of constants the Lie algebra of the Galois group $\mathrm{Gal}\,(L/k)$ in Picard-Vessiot theory coincides with the Lie algebra of the infinitesimal Galois group $\text{Inf-gal}\,(L/k)$ (Theorem 3.20). To this end, we need several lemmas. We treat sub-algebras of the ring of functions on $\mathbb{N}$ that is not a domain. We notice, however, that since C is algebraically closed in L, for a domain R over C, the tensor product

$$S = L \otimes_C R$$

is a domain and a fortiori every sub-algebra of S is a domain. Hence we can speak of its field of fractions. Similarly for the tensor product $k[Z, (\det Z)^{-1}] \otimes_C R$.

Now we work in the difference-differential ring $F(\mathbb{N}, L^\natural[[W]])$ and we identify the difference-differential ring $F(\mathbb{N}, L^\sharp)$ with its image in $F(\mathbb{N}, L^\natural[[W]])$. Let

$$\iota_W : L^\sharp \to L^\natural[[W_1, W_2, \ldots, W_d]]$$

be the universal Taylor morphism. We identify the elements through the morphisms

$$L^\sharp \to F(\mathbb{N}, L^\sharp) \to F(\mathbb{N}, L^\natural[[W]]).$$

So if we take an element $a \in L^\sharp$, by the first morphism we identify a with the constant function on $\mathbb{N}$ taking the constant value $a \in L^\sharp$ and finally by the second arrow, we identify it with the constant function on $\mathbb{N}$ taking the constant value $\iota_W(a) \in L^\natural[[W]]$. As derivations $\mathrm{Der}(L^\natural/k^\natural)$ operate trivially on k, for an element $a \in k$, $\iota_W(a) = a \in k^\natural \subset L^\natural$ so that we have

$$k^\sharp = k^\natural \subset L^\natural \subset L^\natural[[W]] \subset F(\mathbb{N}, L^\natural[[W]]).$$

Since the universal Euler morphism $\iota = \iota_L$ is a difference morphism, the image

$$\iota(Z) = (\iota(z_{ij})) \in \mathrm{GL}_n(F(\mathbb{N}, L^\sharp)) = F(\mathbb{N}, \mathrm{GL}_n(L^\sharp))$$

of the matrix Z by the universal Euler morphism $\iota : L \to F(\mathbb{N}, L^\sharp)$ satisfies

$$\Sigma(\iota(Z)) = \iota(A)\iota(Z) \tag{12}$$

by (11), $\Sigma : F(\mathbb{N}, L^\sharp) \to F(\mathbb{N}, L^\sharp)$ being the shift operator. We set

$$B := \iota(Z)(Z^\sharp)^{-1} \in \mathrm{GL}_n(F(\mathbb{N}, L^\sharp)) = F(\mathbb{N}, \mathrm{GL}_n(L^\sharp)), \tag{13}$$

where $Z^\sharp = (z^\sharp_{ij}) \in \mathrm{GL}_n(L^\sharp) \subset \mathrm{GL}_n(F(\mathbb{N}, L^\sharp)) = F(\mathbb{N}, \mathrm{GL}_n(L^\sharp))$ or the constant function on $\mathbb{N}$ taking the value $Z^\sharp \in \mathrm{GL}_n(L^\sharp)$. It follows from (12) and (13) that

$$\Sigma(B) = \iota(A)B. \tag{14}$$

Lemma 3.3. *The matrix $B \in F(\mathbb{N}, \mathrm{GL}_n(L^\sharp))$ is in $F(\mathbb{N}, \mathrm{GL}_n(k^\sharp))$.*

Proof. We prove the assertion that $B(l) \in \mathrm{GL}_n(k^\sharp)$ for all $l \in \mathbb{N}$ by induction on l. It follows from (13) that $B(0) = \mathrm{I}_n \in \mathrm{GL}_n(k^\sharp)$. Since $\iota(A)(m) = \sigma^m(A) \in \mathrm{GL}_n(k^\sharp)$ for every $m \in \mathbb{N}$, if the assertion is proved for $j \le l$, the assertion for $l+1$ follows from (14). ∎

Remark 3.4. By (12) and (13), we can explicitly write the function B. Namely, for $l \in \mathbb{N}$, the value $B(l)$ of the function $B \in F(\mathbb{N}, \mathrm{GL}_n(L^\sharp))$ is given by

$$B(l) = \begin{cases} \mathrm{I}_n, & \text{for } l = 0, \\ \sigma^{l-1}(A)\sigma^{l-2}(A)\cdots\sigma(A)A \in \mathrm{GL}_n(k^\sharp), & \text{for } l \ge 1. \end{cases} \tag{15}$$

As we explained in 2.3, in the construction of Galois hull $\mathcal{L}$ in general difference Galois theory, we start from the universal Euler morphism

$$\iota : L \to F(\mathbb{N}, L^\sharp). \tag{16}$$

It seems, however, more adequate to take a model of the field extension L/k. So we replace the difference field L by the difference ring $k[Z, (\det Z)^{-1}]$ of finite type over k and the field $k(Z, (\det Z)^{-1})$ of fractions is the difference field L. Namely we start from the restriction of the universal Euler morphism ι to the difference sub-algebra $k[Z, (\det Z)^{-1}]$ of L. We denote the restriction also by ι so that we have

$$\iota : k[Z, (\det Z)^{-1}] \to F(\mathbb{N}, L^\sharp).$$

If the reader prefers logical conformity, please replace $k[Z, (\det Z)^{-1}]$ by L in the sequel. The replacement does not affect the arguments.

LEMMA 3.5. *In the difference-differential ring $F(\mathbb{N}, L^\sharp)$, the sub-ring*

$$\iota(k[Z, (\det Z)^{-1}]).L^\sharp$$

coincides with the sub-ring $\iota(k)[B, (\det B)^{-1}].L^\sharp$.

Proof. We get by (13)

$$\begin{aligned}\iota(k[Z, (\det Z)^{-1}]).L^\sharp &= \iota(k)[\iota(Z), \iota((\det Z)^{-1})].L^\sharp \\ &= \iota(k)[BZ^\sharp, (\det BZ^\sharp)^{-1}].L^\sharp = \iota(k)[B, (\det B)^{-1}].L^\sharp. \ \blacksquare\end{aligned}$$

LEMMA 3.6. *The sub-ring $\iota(k[Z, (\det Z)^{-1}]).L^\sharp$ is closed under the shift operator Σ and the derivations D_i for $1 \le i \le d$. So it is a difference-differential sub-ring of $F(\mathbb{N}, L^\sharp)$.*

Proof. Since both $\iota(k[Z, (\det Z)^{-1}])$ and $L^\sharp$ are closed under the shift operator Σ, the sub-ring $\iota(k[Z, (\det Z)^{-1}]).L^\sharp$ is closed under the shift operator. To see that the difference ring $\iota(k[Z, (\det Z)^{-1}]).L^\sharp$ is closed under the derivations D_i for $1 \le i \le d$, we have to show by Lemma 3.5 that the ring $\iota(k)[B, (\det B)^{-1}].L^\sharp$ is closed under the derivations. The sub-ring $L^\sharp$ is evidently differentially closed and $\iota(k)[B, (\det B)^{-1}]$ is differentially trivial by Lemma 3.3. ■

We need to slightly modify the definition of the Galois hull $\mathcal{L}$.

DEFINITION 3.7. The Galois hull $\mathcal{L}$ of the Picard-Vessiot extension L/k is the difference-differential subring of $F(\mathbb{N}, L^\sharp)$ generated by $\iota(k[Z, (\det Z)^{-1}])$ and $L^\sharp$.

We replace the Picard-Vessiot extension L/k by the model $k[Z, (\det Z)^{-1}]/k$. If we pursue a model of groupoid, or a model of the ring extension $\mathcal{L}/\mathcal{K}$, maybe we must also replace the partial differential field $L^\sharp$ by a partial differential k-sub-algebra S of $L^\sharp$ finite type over k the field of fractions of which is equal to $L^\sharp$. We do not touch this problem in this note.

LEMMA 3.8. *We have*

$$\mathcal{L} = \iota(k[Z, (\det Z)^{-1}]).L^\sharp = \iota(k)[B, (\det B)^{-1}].L^\sharp. \tag{17}$$

Proof. The ring $\mathcal{L}$ is by definition the difference-differential sub-algebra of $F(\mathbb{N}, L^\sharp)$ generated by $\iota(k[Z, (\det Z)^{-1}])$ and $L^\sharp$. The lemma follows from Lemma 3.5. ■

We need linear disjointness theorems for difference and differential cases. The theorems are well-known but as we use them repeatedly in a fundamental way, we give a proof for the difference version.

LEMMA 3.9. *Let (R, σ) be a difference ring and M a difference sub-field. Then the field M and C_R of constants of R are linearly disjoint over C_M.*

Proof. Assume that the lemma were false. Let $c_1, c_2, \dots, c_d$ be elements of C_R that are linearly independent over C_M but linearly dependent over M. We choose the elements c_i's so that the number d is minimum. So there exist elements $m_1, m_2, \dots, m_d$ of M such that

$$m_1 c_1 + m_2 c_2 + \cdots + m_d c_d = 0, \tag{18}$$

where at least one of the m_i's is not 0. Since M is a field, we have $d \geq 2$ and reordering the elements c_i's if necessary, we may assume $m_1 \neq 0$. Multiplying (18) by m_1^{-1}, we may assume $m_1 = 1$ so that

$$c_1 + m_2 c_2 + \cdots + m_d c_d = 0. \tag{19}$$

Applying σ to (19), we get

$$c_1 + \sigma(m_2) c_2 + \cdots + \sigma(m_d) c_d = 0. \tag{20}$$

Subtracting equation (20) from equation (19), we get

$$(m_2 - \sigma(m_2)) c_2 + \cdots + (m_d - \sigma(m_d)) c_d = 0.$$

Since c_2, $c_3, \dots, c_d$ are linearly independent over C_M, by the minimality of d, the elements $c_2, c_3, \dots, c_d$ are linearly independent over M. Hence we have

$$m_2 - \sigma(m_2) = m_3 - \sigma(m_3) = \cdots = m_d - \sigma(m_d) = 0$$

Therefore $m_1 = 1, m_2, m_3, \dots, m_d$ are constants. This together with (18) contradicts the fact that $c_1, c_2, \dots, c_d$ are linearly independent over C_M. ∎

LEMMA 3.10. *In the ring $\mathcal{L} = \iota(k[Z, (\det Z)^{-1}]).L^\sharp$, $\iota(k[Z, (\det Z)^{-1}])$ and $L^\sharp$ are linearly disjoint over C. So we have a difference isomorphism*

$$k[Z, (\det Z)^{-1}] \otimes_C L^\natural \simeq \mathcal{L}.$$

In particular, the ring $\mathcal{L}$ and hence the sub-ring $\iota(k)[B, (\det B)^{-1}]$ of $\mathcal{L}$ are domains.

Proof. We are interested in the difference structure of the difference-differential ring $\mathcal{L}$ and forget its differential structure. If the difference sub-ring $\iota([Z, (\det Z)^{-1}])$ of $\mathcal{L}$ were a field, we could apply Lemma 3.9 to the difference ring $(\mathcal{L}, \Sigma)$ and $\iota(k[Z, (\det Z)^{-1}])$ to conclude that $\iota(k[Z, (\det Z)^{-1}])$ and $C_{\mathcal{L}} = L^\sharp$ are linearly disjoint over C. To remedy the situation, we replace the difference ring $k[Z, (\det Z)^{-1}]$ by the difference field L that is the field of fractions of $k[Z, (\det Z)^{-1}]$. We introduce the difference-differential sub-ring $\mathcal{L}^*$ of $F(\mathbb{N}, L^\sharp)$ generated by $\iota(L)$ and $L^\sharp$. The above argument allows us to show

$$\mathcal{L}^* = \iota(L).L^\sharp.$$

We can apply Lemma 3.9 to the difference ring $(\mathcal{L}^*, \Sigma)$ and the difference sub-field $\iota(L)$ to conclude that $\iota(L)$ and $C_{\mathcal{L}^*} = L^\sharp$ are linearly disjoint over $C_L = C$. Since $\mathcal{L} \subset \mathcal{L}^*$, this implies the assertion of the lemma. ∎

Hiroshi Saito pointed out the following lemma.

LEMMA 3.11. *Let $(F(\mathbb{N}, M), \Sigma)$ be the difference ring of functions on $\mathbb{N}$ with values in a field M. Let R be a difference sub-domain of $(F(\mathbb{N}, M), \Sigma)$ containing the field M of* *constant functions. If Σ induces an automorphism of the domain R, then the field $Q(R)$ of fractions of the difference domain R has a natural structure of difference field and we have*

$$C_{Q(R)} = M,$$

denoting by $C_{Q(R)}$ the field of constants of the difference field $Q(R)$.

Proof. Since Σ induces an automorphism of R, the automorphism Σ extends naturally to an automorphism of the field $Q(R)$ of fractions of R. The inclusion $M \subset C_{Q(R)}$ follows from the assumption. Conversely, let $f, g \in (F(\mathbb{N}, M), \Sigma)$ with $g \neq 0$ such that f/g is Σ-invariant. So for $l \in \mathbb{N}$, we have

$$\Sigma^l \left(\frac{f}{g}\right) = \frac{\Sigma^l(f)}{\Sigma^l(g)} = \frac{f}{g}$$

and hence

$$\Sigma^l(f)g - f\Sigma^l(g) = 0. \tag{21}$$

Comparing the value at $m \in \mathbb{N}$ in (21), we get

$$f(m + l)g(m) - f(m)g(m + l) = 0 \tag{22}$$

for every $m \in \mathbb{N}$. Namely, we have (22) for every l, $m \in \mathbb{N}$. Since $g \neq 0$, there exists an $s \in \mathbb{N}$ such that $g(s) \neq 0$. We set $c := f(s)/g(s) \in M$ that is the constant function c and show $f/g = c \in Q(R)$. We have to show $f - cg = 0$ which is equivalent to

$$f(t) - g(t)c = 0 \tag{23}$$

for every $t \in \mathbb{N}$. Equation (23) is equivalent to

$$f(t) - \left(\frac{f(s)}{g(s)}\right) g(t) = 0,$$

which is a consequence of (22). ∎

LEMMA 3.12. *The field $C_{Q(\iota(k)[B,(\det B)^{-1}].k^\sharp)}$ of difference constants of the difference field $Q(\iota(k)[B, (\det B)^{-1}].k^\sharp)$ of fractions of the difference domain $\iota(k)[B, (\det B)^{-1}].k^\sharp$ is $k^\sharp$.*

Proof. In fact it is sufficient to apply Lemma 3.11 to $\iota(k)[B, (\det B)^{-1}].k^\sharp \subset F(\mathbb{N}, k^\sharp)$. ∎

The argument of the proof of Lemma 3.12 allows us to show the following result.

COROLLARY 3.13. *The field of constants of the difference field $(Q(\mathcal{L}), \Sigma)$ is $L^\sharp$.*

LEMMA 3.14. *In the difference ring $(\mathcal{L}, \Sigma)$, the difference sub-ring $\iota(k)[B, (\det B)^{-1}].k^\sharp$ and $L^\sharp$ are linearly disjoint over $k^\sharp$. So we have a difference isomorphism*

$$(\iota(k)[B, (\det B)^{-1}].k^\sharp) \otimes_{k^\sharp} L^\natural \simeq \mathcal{L}.$$

Proof. If the difference sub-ring $\iota(k)[B, (\det B)^{-1}].k^\sharp$ were a field, we could argue as in the proof of Lemma 3.10. So we work in the difference field $Q(\mathcal{L})$ of fractions and replace the difference ring $\iota(k)[B, (\det B)^{-1}].k^\sharp$ by its field $Q(\iota(k)[B, (\det B)^{-1}].k^\sharp)$ of fractions. Now the lemma follows from Lemma 3.12 and Corollary 3.13. ∎

PROPOSITION 3.15. *We have a difference-differential isomorphism*

$$(\iota(k)[B, (\det B)^{-1}].k^\sharp) \otimes_{k^\sharp} L^\natural \simeq k[Z, (\det Z)^{-1}] \otimes_C L^\natural.$$

Proof. We get a difference isomorphism

$$(\iota(k)[B, (\det B)^{-1}].k^{\sharp}) \otimes_{k^{\sharp}} L^{\natural} \simeq k[Z, (\det Z)^{-1}] \otimes_C L^{\natural}.$$

by Lemmas 3.10 and 3.14. Since the derivations D_i , $1 \leq i \leq d$ operate trivially on both sides. The difference isomorphism is a difference-differential isomorphism. ■

We also need later the differential version of Lemma 3.9.

LEMMA 3.16. *Let $(R, \{\partial_1, \partial_2, \ldots, \partial_d\})$ be a partial differential ring and M a partial differential sub-field of R. Then the sub-rings M and C_R are linearly disjoint over C_M, where C_R (resp. C_L) denotes the ring (resp. field) of constants of R (resp. L).*

REMARK 3.17. We use in the final step a more general linear disjointness theorem.

Let $\sigma_\alpha : R \to R$, $\alpha \in I$ be a set of endomorphisms of a ring R indexed by a set I and $\partial_\beta : R \to R$, $\beta \in J$ a set of differential operators on the ring R indexed by a set J. So $(R, \{\sigma_\alpha\}_{\alpha \in I}, \{\partial_\beta\}_{\beta \in J})$ is a difference-differential ring. Let M be a difference-differential sub-field of L. Then, in the ring R, the field M and the ring

$$C_R = \{c \in R \,|\, \sigma_\alpha(c) = c,\ \partial_\beta(c) = 0 \text{ for every } \alpha \in I \text{ and for every } \beta \in J\}$$

of difference-differential constants of the ring R are linearly disjoint over C_M.

We have three sub-rings $\iota(k[Z, (\det Z)^{-1}])$, $L^{\natural}$, $L^{\sharp}$ isomorphic to L as abstract rings, in $F(\mathbb{N}, L^{\natural}[[W]])$ that we should not confuse.

PROPOSITION 3.18. *The difference-differential sub-ring*

$$\mathcal{L}.L^{\natural} = \iota(k[Z, (\det Z)^{-1}]).L^{\natural}.L^{\sharp}$$

generated by $\iota(k[Z, (\det Z)^{-1}]).L^{\natural}$ and $L^{\sharp}$ in $F(\mathbb{N}, L^{\natural}[[W]])$ is isomorphic to

$$(\iota(k)[B, (\det B)^{-1}] \otimes_C L^{\natural}) \otimes_{k^{\sharp}} L^{\sharp}$$

as difference-differential rings.

Proof. The sub-ring

$$\iota(k[Z, (\det Z)^{-1}]).L^{\natural}.L^{\sharp}$$

of $F(\mathbb{N}, L^{\sharp}[[W]])$ generated by $\iota(k[Z, (\det Z)^{-1}]).L^{\natural}$ and $L^{\sharp}$ coincides with $\mathcal{L}.L^{\natural}$ by Lemma 3.8 and hence is closed under the shift operator Σ and the derivations $\partial/\partial W_i$'s. We also have

$$\iota(k[Z, (\det Z)^{-1}]).L^{\natural}.L^{\sharp} = (\iota(k)[B, (\det B)^{-1}]).L^{\natural}.L^{\sharp}$$

by Lemma 3.8 in such a way that all the sub-rings $\iota(k)[B, (\det B)^{-1}]$, $L^{\natural}$ and $L^{\sharp}$ are difference-differential invariant. In other words they are difference-differential rings. We are going to show that this decomposition gives the desired isomorphism thanks to the linear disjointness theorems. In fact, in the differential ring

$$F(\mathbb{N}, L^{\natural}[[W_1, W_2, \ldots W_d]], \{\partial/\partial W_1, \partial/\partial W_2, \ldots, \partial/\partial W_d\}),$$

we apply Lemma 3.16 to the differential ring $(\iota(k)[B, (\det B)^{-1}].L^{\natural}).L^{\sharp}$ and to the differential sub-field $L^{\sharp}$ to conclude that $C_{(\iota(k)[B, (\det B)^{-1}].L^{\natural}).L^{\sharp}} = \iota(k)[B, (\det B)^{-1}].L^{\natural}$ and $L^{\sharp}$ are linearly disjoint over $k^{\natural}$. So we have a differential isomorphism

$$\iota(k)[B, (\det B)^{-1}].L^{\natural}L^{\sharp} \simeq (\iota(k)[B, (\det B)^{-1}].L^{\natural}) \otimes_{k^{\natural}} L^{\sharp}. \tag{24}$$

We notice here that since the sub-rings are also difference invariant, isomorphism (24) is also a difference isomorphism so that (24) is a difference-differential isomorphism. The tensor product in (24) taken over the field k algebraically closed in L by assumption, is a domain and consequently $\iota(k[Z,(\det Z)^{-1}]).L^\natural.L^\sharp = (\iota(k)[B,(\det B)^{-1}].L^\natural)L^\sharp$ is a domain. Therefore we can speak of its field of fractions.

Now we show that $\iota(k)[B,(\det B)^{-1}].k^\natural$ and $L^\natural$ are linearly disjoint over $k^\natural$. Indeed, we work in the difference field $Q(\iota(k)[B,(\det B)^{-1}].L^\natural)$ of fractions of $\iota(k)[B,(\det B)^{-1}].L^\natural$. The argument of Lemma 3.12 allows us to show that the field of constants

$$C_{Q(\iota(k)[B,(\det B)^{-1}].L^\natural)}$$

of the difference field $Q(\iota(k)[B,(\det B)^{-1}].L^\natural)$, that is the field of fractions of the difference ring $\iota(k)[B,(\det B)^{-1}].L^\natural$, is $L^\natural$. We apply Lemma 3.9 to the difference field $Q((\iota(k)[B,(\det B)^{-1}].L^\natural)$ and the difference sub-field $Q((\iota(k)[B,(\det B)^{-1}].k^\natural)$ to conclude that the difference field $Q(\iota(k)[B,(\det B)^{-1}].k^\natural)$ and $L^\natural$ are linearly disjoint over $k^\natural$. Hence we have a difference isomorphism

$$\iota(k)[B,(\det B)^{-1}].L^\natural \simeq (\iota(k)[B,(\det B)^{-1}].k^\natural) \otimes_{k^\natural} L^\natural. \tag{25}$$

Since derivations operate trivially in all the difference ring involved in (25), the difference isomorphism (25) is also a differential isomorphism. Now the proposition follows from (24), (25) and Proposition 3.15. ∎

Remark 3.19. Under the assumption of Proposition 3.18, by working in the difference-differential algebra $F(\mathbb{N}, A[[W]])$ for an $L^\natural$-algebra A, the argument of the proof of the proposition allows us to show the difference-differential isomorphism

$$\iota(k[Z,(\det Z)^{-1}]).A.L^\sharp \simeq (L \otimes_C A) \otimes_{k^\natural} L^\sharp.$$

Theorem 3.20. *Let $L = k(z_{ij})_{1\le i,j\le n}/k$ be a Picard-Vessiot field. Then we have an isomorphism*

$$\mathrm{Lie}\,(\mathrm{Inf\text{-}gal}\,(L/k)) \simeq \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^\natural$$

of $L^\natural$-Lie algebras.

Proof. Let $f \in \mathrm{Lie}(\mathrm{Gal}(L/k)) \otimes_C L^\natural$ so that

$$f : k[Z,(\det Z)^{-1}] \otimes_C L^\natural[\epsilon] \to k[Z,(\det Z)^{-1}] \otimes_C L^\natural[\epsilon] \quad (\epsilon^2 = 0) \tag{26}$$

is a $k \otimes_c L^\natural[\epsilon]$-difference isomorphism that is an infinitesimal deformation of the identity. Tensoring (26) with $L^\sharp$ over $k^\sharp$, we get a difference isomorphism

$$f' : L \otimes_{L^\natural} L^\natural[\epsilon] \otimes_{k^\sharp} L^\sharp \to L \otimes_{L^\natural} L^\natural[\epsilon] \otimes_{k^\sharp} L^\sharp. \tag{27}$$

Then by Lemma 3.15, Proposition 3.18 and Remark 3.19, we get a difference-differential isomorphism

$$\mathcal{L}.L^\natural[\epsilon] \simeq L \otimes_{L^\natural} L^\natural[\epsilon] \otimes_{k^\sharp} L^\sharp. \tag{28}$$

So by (27) and (28), we get the composition map

$$\tilde{f} : \mathcal{L} \to \mathcal{L}.L^\natural[\epsilon] \simeq L \otimes_{L^\natural} L^\natural[\epsilon] \otimes_{k^\sharp} L^\sharp \xrightarrow{f'} L \otimes_{L^\natural} L^\natural[\epsilon] \otimes_{k^\sharp} L^\sharp \simeq \mathcal{L}.L^\natural[\epsilon] \to F(\mathbb{N}, L^\natural[\epsilon][[W]]) \tag{29}$$

giving an infinitesimal deformation of the canonical map

$$\mathcal{L} \to F(\mathbb{N}, L^\natural[[W]]).$$

In sequence (29), the morphisms except for f' are canonical. So $\tilde{f} \in \mathcal{F}_{L/k}(L^\natural[\epsilon])$. We denote by $\hat{f}$ the element in Inf-gal $(L/k)(L^\natural[\epsilon])$ corresponding to $\tilde{f}$. So we have a group morphism and hence an $L^\natural$-Lie algebra morphism

$$\mathrm{Lie}\,(\mathrm{Gal}(L/k)) \otimes_C L^\natural \to \text{Inf-gal}\,(L/k)(L^\natural[\epsilon]), \quad f \mapsto \hat{f}.$$

Conversely let $g \in \mathrm{Lie}(\text{Inf-gal}\,(L/k))$. So g defines an infinitesimal deformation

$$g : \mathcal{L} \to F(\mathbb{N}, L^\natural[\epsilon][[W]]).$$

It follows from Remark 3.17 that in the difference-differential ring $\mathcal{L}L^\natural$, $\mathcal{L}$ and $L^\natural$ are linearly disjoint over $k^\natural$. So we have a difference-differential isomorphism

$$\mathcal{L}.L^\natural \simeq \mathcal{L} \otimes_{k^\natural} L^\natural.$$

Therefore we get a morphism

$$\mathcal{L}.L^\natural \simeq \mathcal{L} \otimes_{k^\natural} L^\natural \to F(\mathbb{N}, L^\natural[\epsilon][[W]]).$$

So by Proposition 3.18, we have an infinitesimal deformation over $L^\sharp \otimes_C L^\natural$

$$\tilde{g} : k[Z, (\det Z)^{-1}] \otimes_C L^\natural \to F(\mathbb{N}, L^\natural[[\epsilon]][[W]]) \tag{30}$$

so that $\tilde{g}$ is a a difference-differential morphism. Since the derivations D_i operate trivially on L by definition and hence $D_i(Z) = 0$ for every $1 \le i \le d$, morphism (30) induces an infinitesimal deformation

$$\tilde{g} : k[Z, (\det Z)^{-1}] \otimes_C L^\natural \to F(\mathbb{N}, L^\natural[\epsilon])$$

that is a difference morphism. So $\tilde{g}$ arises from an infinitesimal deformation

$$k[Z, (\det Z)^{-1}] \otimes_C L^\natural[\epsilon] \to k[Z, (\det Z)^{-1}] \otimes_C L^\natural[\epsilon]$$

that is a difference automorphism. So we construct an $L^\natural$-Lie algebra morphism

$$\mathrm{Lie}(\text{Inf-gal}\,(L/k)) \to \mathrm{Lie}(\mathrm{Gal}(L/k)) \otimes_C L^\natural, \quad g \mapsto \hat{g}. \tag{31}$$

Since the mappings (30) and (31) are mutually inverse, we have proved the Theorem. ∎

3.2. Differential Picard-Vessiot theory

3.2.1. *Review of general differential Galois theory ([18], [19], [20] and [13]).* General differential Galois theory is formulated as in the difference case explained in Section 1.

For a differential ring (R, ∂) so that $\partial : R \to \mathcal{R}$ is a derivation, we denote the abstract ring R by $R^\natural$. Let $(L, \partial_L)/(k, \partial_k)$ be a differential field extension. So L/k is a field extension and $\partial_L : L \to L$ and $\partial_k : k \to k$ are derivations such that ∂_L is an extension of ∂_k. We have the universal Taylor morphism

$$\iota : L \to L^\natural[[X]]. \tag{32}$$

We assume that the abstract field $L^\natural$ is finitely generated over $k^\natural$. We choose a mutually commutative basis $D_1, D_2, \ldots, D_d$ of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/k^\natural)$ of $k^\natural$-derivations of the field $L^\natural$. We introduce the partial differential field $L^\sharp := (L^\natural, \{D_1, D_2, \ldots, D_d\})$. Just as we introduced the difference-differential ring $F(\mathbb{N}, L^\sharp)$ in the difference case, we consider the partial differential field

$$L^\sharp[[X]] := \left(L^\natural[[X]], \left\{ \frac{d}{dX}, D_1, D_2, \ldots, D_d \right\} \right),$$

where the derivations $D_1, D_2, \ldots, D_d$ operate through the coefficients. We regard the universal Taylor morphism (32) as

$$\iota : L \to L^\sharp[[X]]. \tag{33}$$

We denote by $L^\sharp$ the partial differential field of constant power series so that

$$L^\sharp := \Big\{ \sum_{i=0}^{\infty} a_i X^i \in L^\sharp[[X]] \,|\, a_i = 0 \text{ for } i \geq 1 \Big\}.$$

Hence $L^\sharp$ is a partial differential sub-field of $L^\sharp[[X]]$.

DEFINITION 3.21. The Galois hull $\mathcal{L}/\mathcal{K}$ is a partial differential algebra extension in the partial differential ring $L^\sharp[[X]]$, where $\mathcal{L}$ is the partial differential sub-algebra generated by the image $\iota(L)$ and $L^\sharp$ in $L^\sharp[[X]]$ and $\mathcal{K}$ is the partial differential sub-algebra generated by the image $\iota(k)$ and $L^\sharp$ in $L^\sharp[[X]]$.

We have the universal Taylor expansion

$$\iota_{L^\sharp} : L^\sharp \to L^\natural[[W_1, W_2, \ldots, W_d]] = L^\natural[[W]]. \tag{34}$$

The morphism (34) gives a differential morphism

$$\Big(L^\sharp[[X]], \Big\{\frac{d}{dX}, D_1, D_2, \ldots, D_d\Big\}\Big) \\ \to \Big(L^\natural[[W_1, W_2, \ldots, W_d]][[X]], \Big\{\frac{d}{dX}, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_d}\Big\}\Big). \tag{35}$$

Restricting the differential morphism (35) to the differential sub-algebra $\mathcal{L}$, we get a canonical differential algebra morphism

$$\iota : \mathcal{L} \to L^\natural[[W, X]]. \tag{36}$$

For an $L^\natural$-algebra A, we have the partial differential morphism

$$L^\natural[[W, X]] \to A[[W, X]] \tag{37}$$

induced by the structural morphism $L^\natural \to A$. The composition of morphisms (36) and (37) gives the canonical morphism

$$\iota : \mathcal{L} \to A[[W, X]].$$

The infinitesimal deformation functor

$$\mathcal{F}_{L/k} : (Alg/L^\natural) \to (Sets)$$

is defined by setting

$$\mathcal{F}_{L/k}(A) = \{f : \mathcal{L} \to A[[W, X]] \,|\, f \text{ is a partial differential morphism} \\ \text{congruent to the canonical morphism } \iota \text{ modulo nilpotent elements} \\ \text{such that the restriction to } \mathcal{K} \text{ coincides with the canonical morphism } \iota\}.$$

Our Galois group is a group functor

$$\text{Inf-gal}\,(L/k) : (Alg/L^\natural) \to (Grp)$$

associating to an $L^\natural$-algebra A the automorphism group

$$\begin{aligned}\text{Inf-gal}\,(L/k)(A) = \{f : \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \to \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \,|\, f \text{ is a differential} \\ \mathcal{K}\hat{\otimes}_{L^\sharp} A[[W]]\text{-automorphism continuous with respect to the } W\text{-adic topology} \\ \text{and congruent to the identity modulo nilpotent elements}\}.\end{aligned}$$

The group functor Inf-gal (L/k) operates on the functor $\mathcal{F}_{L/k}$ in such a manner that the operation (Inf-gal $(L/k), \mathcal{F}_{L/k})$ is a principal homogeneous space.

REMARK 3.22. In Umemura [18], a different definition of Inf-gal (L/K) and of $\mathcal{F}_{\mathcal{L}/\mathcal{K}}$ is adopted. See page 106 in [18]. Namely we considered the field $Q(\mathcal{L})$ of fractions of the ring $\mathcal{L}$ defined above so that $Q(\mathcal{L})$ is a differential sub-field of the Laurent series ring

$$L^\natural[[W, X]][W^{-1}, X^{-1}]$$

We then studied the infinitesimal deformations of $Q(\mathcal{L})$ in the Laurent series ring

$$A[[W, X]][W^{-1}, X^{-1}]$$

for an $L^\natural$-algebra A. This definition agrees with the definition above thanks to Lemma 4.5 on page 105 in [18].

3.2.2. *Application of our proof to differential Picard-Vessiot theory.* Let $(L, \partial)/(k, \partial)$ be a differential Picard-Vessiot extension. So the field of constants C_L coincides with the field C_k of constants of k and C_k is algebraically closed. We may assume that the base field $k^\natural$ is algebraically closed in $L^\natural$. There exist matrices $(z_{ij}) \in \mathrm{GL}_n(L)$ and $A \in \mathrm{M}_n(k)$ satisfying the following conditions

$$L = k(Z), \text{ and } \partial Z = AZ,$$

∂Z being the matrix $(\partial z_{ij}) \in \mathrm{M}_n(L)$. We prove the differential analogue of Theorem 3.20 by applying the proof of 3.1. Namely,

THEOREM 3.23 (Theorem 3.20). *Let $L = k(z_{ij})_{1\le i,j\le n}/k$ be a differential Picard-Vessiot field. Then we have an isomorphism*

$$\mathrm{Lie}\,(\text{Inf-gal}\,(L/k)) \simeq \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^\natural$$

of $L^\natural$-Lie algebras.

So we examine the procedure of proof of Theorem 3.20 in 3.1 in the differential context. To this end, we need the list of replacements below.

Difference theory	*Differential theory*
$L^\natural$	$L^\natural$
Universal Euler morphism	Universal Taylor morphism
$F(\mathbb{N}, L^\sharp)$	$L^\sharp[[X]]$
Shift operator Σ	Derivation d/dX
$L^\sharp$	$L^\sharp$
$D_1, D_2, \ldots, D_d$	$D_1, D_2, \ldots, D_d$
$F(\mathbb{N}, L^\natural[[W]])$	$L^\natural[[W, X]]$
$A[[W]]$	$A[[W]]$
$F(\mathbb{N}, A[[W]])$	$A[[W, X]]$

For example, when we pass from difference theory to differential theory, we replace the universal Euler morphism ι in a difference assertion by the universal Taylor morphism (33) to get the differential analogue.

We mean by *Lemma xyz (Lemma abc)* that Lemma xyz is a differential analogue of Lemma abc in difference algebra.

Let us first study the differential analogue of Lemma 3.3.

LEMMA 3.24 (Lemma 3.3). *Set* $B := \iota(Z){Z^\sharp}^{-1} \in \mathrm{GL}_n(L^\sharp[[X]])$. *The matrix* $B \in \mathrm{GL}_n(L^\sharp[[X]])$ *is in* $\mathrm{GL}_n(k^\sharp[[X]])$.

Proof. The proof in difference algebra works also in differential algebra. ∎

LEMMA 3.25 (Lemma 3.5). *In the partial differential ring* $L^\sharp[[X]]$, *the sub-ring*

$$\iota(k[Z, (\det Z)^{-1}]).L^\sharp$$

coincides with the sub-ring $\iota(k)[B, (\det B)^{-1}].L^\sharp$.

Proof. The proof in difference algebra works also in differential algebra. ∎

LEMMA 3.26 (Lemma 3.6). *The sub-ring* $\iota(k[Z, (\det Z)^{-1}]).L^\sharp$ *is closed under* d/dX *and the derivations* D_i *for* $1 \le i \le d$. *So it is a differential sub-ring of* $L^\sharp[[X]]$.

Proof. The proof in difference algebra works also in differential algebra. ∎

LEMMA 3.27 (Lemma 3.8). *We have*

$$\mathcal{L} = \iota(k[Z, (\det Z)^{-1}]).L^\sharp = \iota(k)[B, (\det B)^{-1}].L^\sharp.$$

Proof. The proof in difference algebra works also in differential algebra. ∎

LEMMA 3.28 (Lemma 3.9). *Let* (R, ∂) *be a differential ring and* M *a differential sub-field. Then the field* M *and the sub-ring* C_R *of constants of* R *are linearly disjoint over* C_M.

This is a well known result. See for example, Chap. II, 1, Corollary 1 of Kolchin [9] as well as Lemma (1.1) in Umemura [17]. The proof in the difference case and the proof in the differential case are analogous.

LEMMA 3.29 (Lemma 3.11). *Let* $M[[X]], d/dX)$ *be the differential ring of power series with coefficients in a field* M. *Let* R *be a differential sub-ring of* $M[[X]]$ *containing the field* M *of constant power series. Then the ring* R *is a domain and the field* $Q(R)$ *of fractions of the differential domain* R *has a natural structure of differential field and we have*

$$C_{Q(R)} = M,$$

denoting by $C_{Q(R)}$ *the field of constants of the differential field* $Q(R)$.

Proof. The assertion is trivial. ∎

LEMMA 3.30 (Lemma 3.12). *The field* $C_{Q(\iota(k)[B,(\det B)^{-1}].k^\sharp)}$ *of differential constants of the differential field* $Q(\iota(k)[B, (\det B)^{-1}].k^\sharp)$ *of fractions of the differential domain*

$$\iota(k)[B, (\det B)^{-1}].k^\sharp$$

is $k^\sharp$.

Corollary 3.31 (Corollary 3.13). *The field of constants of the differential field*

$$(Q(\mathcal{L}), \Sigma)$$

is $L^\sharp$.

Proof. They are also trivial in the differential case. ∎

Lemma 3.32 (Lemma 3.14). *In the differential ring* $(\mathcal{L}, d/dX)$, *the differential sub-ring* $\iota(k)[B, (\det B)^{-1}].k^\sharp$ *and* $L^\sharp$ *are linearly disjoint over* $k^\sharp$. *So we have a* d/dX *differential isomorphism*

$$(\iota(k)[B, (\det B)^{-1}].k^\sharp) \otimes_{k^\sharp} L^\natural \simeq \mathcal{L}.$$

Proof. This holds for the same reason as in the difference case. ∎

Proposition 3.33 (Proposition 3.15). *We have a partial differential isomorphism*

$$(\iota(k)[B, (\det B)^{-1}].k^\sharp) \otimes_{k^\sharp} L^\natural \simeq k[Z, (\det Z)^{-1}] \otimes_C L^\natural$$

with respect to the derivations $\{d/dX, D_1, D_2, \ldots, D_d\}$.

Proof. The argument in difference algebra works also in differential algebra. ∎

We notice that the derivations $\{D_1, D_2, \cdots, D_d\}$ operate trivially.

Remark 3.17 in differential case is nothing but the linear disjointness theorem of Lemma 3.28.

Proposition 3.34 (Proposition 3.18). *In the partial differential algebra*

$$\left(L^\natural[[W, X]], \left\{\frac{d}{dX}, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_d}\right\}\right),$$

the partial differential sub-ring

$$\mathcal{L}.L^\natural = \iota(k[Z, (\det Z)^{-1}]).L^\natural.L^\sharp$$

generated by $\iota(k[Z, (\det Z)^{-1}]).L^\natural$ *and* $L^\sharp$ *is isomorphic to*

$$(\iota(k)[B, (\det B)^{-1}] \otimes_C L^\natural) \otimes_{k^\sharp} L^\sharp$$

as partial differential rings.

Proof. As we have shown all the differential analogues of lemmas, corollary and propositions necessary to prove the proposition, the proof in difference algebra works as well in differential algebra. ∎

So far we examined all the results that we need to prove Theorem 3.23, the differential version of Theorem 3.20. Therefore the proof of Theorem 3.23 is achieved as the proof of Theorem 3.20.

3.3. Hopf Galois theory. So far we treated difference equations and differential equations. Picard-Vessiot theory is a Galois theory of linear difference or differential equations. The idea of introducing Hopf algebra in Galois theory goes back to Sweedler [15]. Specialists in Hopf algebra succeeded in unifying Picard-Vessiot theories for difference equations and differential equations [1]. They further succeeded in generalizing the Picard-Vessiot theory for difference-differential equations, where the operators are not necessarily commutative. Heiderich [7] combined the idea of Picard-Vessiot theory via Hopf algebra with our general Galois theory for non-linear equations [18], [12]. His general theory includes a wide class of difference and differential algebras. It seems, however, that some algebras with operators are excluded from his theory. The sesquilinear difference algebra in André [2] is such an example.

There are two major advantages in his theory.

(1) Unified study of non-linear difference equations and differential equations.
(2) Generalization of universal Euler morphism and Taylor morphism.

Let C be a field. For C-vector spaces M, N, we denote by ${}_C\mathbf{M}(M,N)$ the set of C-morphisms from M to N.

EXAMPLE 3.35. Let $\mathcal{H} := C[\mathbb{G}_a] = C[t]$ be the C-Hopf algebra of the coordinate ring of the additive group scheme $\mathbb{G}_{aC}$. Let A be a C-algebra and

$$\Psi \in_C \mathbf{M}(A \otimes_C \mathcal{H}, A) =_C \mathbf{M}(A,_C \mathbf{M}(\mathcal{H}, A))$$

so that Ψ defines two C-linear maps

(1) $\Psi_1 : A \otimes_C \mathcal{H} \to A$,
(2) $\Psi_2 : A \to {}_C\mathbf{M}(\mathcal{H}, A)$.

DEFINITION 3.36. We say that (A, Ψ) is an $\mathcal{H}$-module algebra if the following conditions are satisfied.

(1) The C-linear map $\Psi_1 : A \otimes_C \mathcal{H} \to A$ defines an operation of the C-algebra $\mathcal{H}$ on the C-algebra A.
(2) The C-linear map
$$\Psi_2 : A \to {}_C\mathbf{M}(\mathcal{H}, A)$$
is a C-algebra morphism, the dual ${}_C\mathbf{M}(\mathcal{H}, A)$ of the co-algebra $\mathcal{H}$ being a C-algebra.

Concretely the dual algebra ${}_C\mathbf{M}(\mathcal{H}, A)$ is the formal power series ring $A[[X]]$.

It is a comforting exercise to examine that (A, Ψ) is an $\mathcal{H}$-module algebra if and only if A is a differential algebra with derivation δ such that $\delta(C) = 0$. When the equivalent conditions are satisfied, for every element a in the algebra A, $\Psi(a \otimes t) = \delta(a)$ and the C-algebra morphism

$$\Psi_2 : A \to {}_C\mathbf{M}(\mathcal{H}, A) = A[[X]]$$

is the universal Taylor morphism. See Heiderich [7], 2.3.4.

If we take an appropriate bialgebra for $\mathcal{H}$, we get difference structure and the universal Euler morphism. See [7], 2.3.1.

To illustrate what happens in the general situation, let us study Picard-Vessiot theory of iterative q-derivations by Hardouin [5], among these Picard-Vessiot theories.

We are going to see that

(1) a non-commutative algebra enters in the generalization of the universal Taylor or the Euler morphism but
(2) the Galois hull $\mathcal{L}$ which is a sub-algebra of the non-commutative algebra is commutative for a Picard-Vessiot extension.

Let C be a field, $q \neq 1$ an element of C. We use the standard notation of q-binomial coefficients. To this end, let Q be a variable over the field C.

We set $[n]_Q = \sum_{i=0}^{n-1} Q^i \in C[Q]$ for a positive integer n. We need also the Q-factorial

$$[n]_Q! := \prod_{i=1}^{n} [i]_Q \quad \text{for a positive integer } n \quad \text{and } [0]_Q! := 1.$$

So $[n]_Q! \in C[Q]$. The Q-binomial coefficient is defined for $m, n \in \mathbb{N}$ by

$$\binom{m}{n}_Q = \begin{cases} \frac{[m]_Q!}{[m-n]_Q![n]_Q!} & \text{if } m \geq n, \\ 0 & \text{if } m < n. \end{cases}$$

Then we can show that

$$\binom{m}{n}_Q \in C(Q)$$

is in fact a polynomial, i.e.

$$\binom{m}{n}_Q \in C[Q].$$

We have a ring morphism

$$C[Q] \to C[q], \quad Q \mapsto q$$

over C and we denote the image of

$$\binom{m}{n}_Q$$

under this morphism by

$$\binom{m}{n}_q.$$

3.3.1. *Iterative q-difference algebra.* Let t be a variable over the field C. We consider a C-automorphism

$$\sigma : C(t) \to C(t), \quad t \mapsto qt$$

of the field $C(t)$ so that $(C(t), \sigma)$ is a difference field.

DEFINITION 3.37. An *iterative q-difference algebra* $(R, \sigma, \delta^*) = (R, \sigma, \{\delta^{(i)}\}_{i\in\mathbb{N}})$ consists of a C-algebra R that is eventually non-commutative, a C-endomorphism $\sigma : R \to R$ of the algebra R such that $(R, \sigma)/(C(t), \sigma)$ is a difference algebra extension, and a set $\delta^* = \{\delta^{(i)}\}_{i\in\mathbb{N}}$ of C-linear maps, called q-difference operators,

$$\delta^{(i)} : R \to R \quad \text{for every } i \in \mathbb{N}$$

satisfying the following conditions.

(1) $\delta^{(0)} = \mathrm{Id}$, Id: $R \to R$ being the identity map.
(2) $(q-1)t\delta^{(1)} = \sigma - \mathrm{Id}$.
(3) $\delta^{(i)}(xy) = \sum_{l+m=i} \sigma^m(\delta^{(l)}(x))\delta^{(m)}(y)$, for every $x, y \in R$ and every $l \in \mathbb{N}$.
(4) $\delta^{(i)} \circ \delta^{(j)} = \binom{i+j}{i}_q \delta^{(i+j)}$, for every $i, j \in \mathbb{N}$, $\binom{i+j}{i}_q$ being the q-binomial coefficient.

Since as the second formula shows, the endomorphism σ is determined by $\delta^{(1)}$, we denote the iterative q-difference ring by (R, δ^*) or by $(R, \{\delta^{(i)}\}_{i\ \in\mathbb{N}})$.

Lemma 3.38 (Hardouin [5], Lemma 2.6). *Let (R, δ^*) be an iterative q-difference ring. We assume*

$$\delta^{(i)}(t) = 0 \text{ for } i \geq 2. \tag{38}$$

If R is commutative and if $\sigma : R \to R$ is a C-automorphism of the algebra R, we have the following commutation relation between σ and $\delta^{(i)}$ for every $i \in \mathbb{N}$:

$$q^i \sigma \delta^{(i)} = \delta^{(i)} \sigma.$$

Remark 3.39. Looking at the proof of [5, Lemma 2.6], maybe the author forgot to write the condition (38). From now on, when we speak of iterative q-difference algebras, we assume (38).

3.3.2. *q-skew iterative σ-differential algebra [6].* We need a slightly more general notion than iterative q-difference algebra.

Definition 3.40. Let C and q be as above. A *q-skew iterative σ-differential algebra* $(A, \sigma, \delta^*) = (A, \sigma, \{\theta^{(i)}\}_{i\in\mathbb{N}})$, a q-SI σ-differential algebra for short, consists of a C-algebra A that is eventually non-commutative, a C-endomorphism $\sigma : A \to A$ of the C-algebra A and a family

$$\theta^{(i)} : A \to A \quad \text{for } i \in \mathbb{N}$$

of C-linear maps satisfying the following conditions.

(1) $\theta^{(0)} = \mathrm{Id}$,
(2) $\theta^{(i)}\sigma = q^i \sigma \theta^{(i)}$ for every $i \in \mathbb{N}$,
(3) $\theta^{(i)}(ab) = \sum_{l+m=i} \sigma^m(\theta^{(l)}(a))\theta^{(m)}(b)$,
(4) $\theta^{(i)} \circ \theta^{(j)} = \binom{i+j}{i}_q \theta^{(i+j)}$.

We say that an element a of the q-SI σ-differential algebra A is a constant if $\sigma(a) = a$ and $\theta^{(i)} = 0$ for every $i \geq 1$.

A morphism of q-SI σ-differential algebras is a C-algebra morphism compatible with the endomorphisms σ and the derivations θ^*.

Lemma 3.41. *Let (R, δ^*) be an iterative q-difference algebra. If the algebra R is commutative and if the endomorphism $\sigma : R \to R$ is an automorphism, then the iterative q-difference algebra is a q-SI σ-differential algebra.*

Proof. This follows from Definitions 3.37, 3.40 and Lemma 3.38. ∎

3.3.3. *Example of q-SI σ-differential algebra.* An example of q-SI σ-differential algebra arises from a C-difference algebra (S, σ). Namely, the twisted power series ring $(S, \sigma)[[X]]$ over the difference ring (S, σ) has a natural q-SI σ-differential algebra structure.

Let (S, σ) be a C-difference ring so that $\sigma : S \to S$ is a C-endomorphism of the commutative ring S. We introduce the following twisted formal power series ring $(S, \sigma)[[X]]$ with coefficients in S that is the formal power series ring $S[[X]]$ as an additive group with the following commutation relation

$$aX = X\sigma(a) \quad \text{for every } a \in S.$$

So more generally

$$aX^n = X^n \sigma^n(a) \tag{39}$$

for every $n \in \mathbb{N}$. Therefore the twisted formal power series ring $(S,\sigma)[[X]]$ is non-commutative in general. By commutation relation (39), we can identify

$$(S,\sigma)[[X]] = \Big\{ \sum_{i=0}^{\infty} X^i a_i \,|\, a_i \in S \text{ for every } i \in \mathbb{N} \Big\}$$

as additive groups.

We are going to see that the twisted formal power series ring has a natural q-SI σ-differential structure. We define first a ring endomorphism $\hat{\sigma} : (S,\sigma)[[X]] \to (S,\sigma)[[X]]$ by setting

$$\hat{\sigma}\Big(\sum_{i=0}^{\infty} X^i a_i \Big) = \sum_{i=0}^{\infty} X^i q^i \sigma(a_i) \quad \text{for every } i \in \mathbb{N}, \tag{40}$$

for every element $\sum_{i=0}^{\infty} X^i a_i \in (S,\sigma)[[X]]$. The operators $\theta^* = \{\theta^{(l)}\}_{l\in\mathbb{N}}$ are defined by

$$\theta^{(l)}\Big(\sum_{i=0}^{\infty} X^i a_i \Big) = \sum_{i=0}^{\infty} X^i \binom{i+l}{l}_q a_{i+l}. \tag{41}$$

Hence the twisted formal power series ring $(S,\sigma)[[X]], \hat{\sigma}, \delta^*)$ is a non-commutative q-SI σ-differential ring. We denote this q-SI σ-differential ring simply by $(S,\sigma)[[X]]$. See [7], 2.3.

REMARK 3.42. If there exists a non-constant element a in the difference ring (S,σ) so that $\sigma(a) \neq a$, then the q-SI σ-differential algebra $(S,\sigma)[[X]]$ is not an iterative q-difference algebra.

In fact, then $0 = (q-1)t\delta^{(1)}(a)$ by (41). On the other hand, $\hat{\sigma}(a) - a = \sigma(a) - a \neq 0$ by (40) so that

$$(q-1)t\delta^{(1)}(a) \neq \hat{\sigma}(a) - a.$$

and the second condition of Definition 3.37 is not satisfied.

LEMMA 3.43. *The construction above is functorial for (S,σ). Namely let $\varphi : (S_1,\sigma_1) \to (S_2,\sigma_2)$ be a difference C-algebra morphism, Then the C-morphism φ induces a morphism*

$$(S_1,\sigma_1)[[X]] \to (S_2,\sigma_2)[[X]], \qquad \sum_{i=0}^{\infty} X^i a_i \mapsto \sum_{i=0}^{\infty} X^i \varphi(a_i)$$

of q-SI σ-differential algebras.

Proof. This follows from the definition of the q-SI σ-differential algebra structure on the twisted formal power series ring. ■

In particular, if we take as the coefficient difference ring S the difference ring

$$(F(\mathbb{N}, A), \Sigma)$$

of functions on $\mathbb{N}$ taking values in a ring A, where Σ is the shift operator, we obtain the q-SI σ-differential ring

$$(F(\mathbb{N}, A), \Sigma)[[X]].$$

PROPOSITION 3.44 (Heiderich [7], Proposition 2.3.17). *For a q-SI σ-differential algebra R, hence in particular for an iterative q-difference ring R, there exists a canonical* *morphism called the universal twisted Taylor morphism*

$$\iota : R \to (F(\mathbb{N}, R^\natural), \Sigma)[[X]], \qquad a \mapsto \sum_{i=0}^{\infty} X^i u[\delta^{(i)}(a)]$$

of q-SI σ-differential algebras. We denote here for an element $b \in R$ *by* $u[b]$ *a function on* $\mathbb{N}$ *taking values in the abstract ring* $R^\natural$ *such that*

$$u[b](n) = \sigma^n(b) \quad \textit{for every } n \in \mathbb{N}$$

so that $u[b] \in F(\mathbb{N}, R^\natural)$.

We can also characterize the twisted universal Taylor morphism as the solution of a universal mapping property.

Let us recall the following fact.

LEMMA 3.45. *Let* (R, σ, θ^*) *be a q-SI σ-differential domain. If the endomorphism* $\sigma : R \to R$ *is an automorphism, then the field* $Q(R)$ *of fractions of R has the unique structure of q-SI σ-differential field.*

If moreover R is an iterative q-difference algebra, then the field $Q(R)$ *of fractions of R is also an iterative q-difference field.*

Proof. See for example, Proposition 2.5 of [6]. ■

We can develop a general Galois theory for iterative q-difference field extensions analogous to our theories in [17] and [23]. Let L/k be an extension of q-iterative difference fields such that the abstract field $L^\natural$ is finitely generated over the abstract field $k^\natural$. Let us assume that we are in characteristic 0. The general theory in [7] and [8] works, however, also in characteristic $p \geq 0$. We have by Proposition 3.44 the universal twisted Taylor morphism

$$\iota : L \to (F(\mathbb{N}, L^\natural), \Sigma)[[X]] \tag{42}$$

so that the image $\iota(L)$ is a copy of the iterative q-difference field L. We have another copy of $L^\natural$. The set

$$\Big\{ f = \sum_{i=0}^{\infty} X^i a_i \in F(\mathbb{N}, L^\natural)[[X]] \mid a_i = 0 \text{ for every } i \geq 1 \text{ and } \Sigma(a_0) = a_0 \Big\}$$
$$= \{ f \in F(\mathbb{N}, L^\natural)[[X]] \mid \hat{\sigma}(f) = f,\ \theta^{(i)}(f) = 0 \text{ for every } i \geq 1 \} \tag{43}$$

forms the sub-ring of constants in the q-SI σ-differential algebra of the twisted power series

$$(F(\mathbb{N}, L^\natural), \Sigma)[[X]].$$

We may denote the sub-ring in (43) by $L^\natural$. In fact, as an abstract ring it is isomorphic to the abstract field $L^\natural$ and the endomorphism $\hat{\sigma}$ and the derivations $\theta^{(i)}$ $(i \geq 1)$ operate trivially on the sub-ring.

We choose a mutually commutative basis $\{D_1, D_2, \ldots, D_d\}$ of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/k^\natural)$ of k-derivations. So $L^\sharp := (L^\natural, \{D_1, D_2, \ldots, D_d\})$ is a differential field. We are now exactly in the same situation as in 2.2 of the difference case.

So we introduce derivations $D_1, D_2, \ldots, D_d$ operating on the coefficient ring $F(\mathbb{N}, L^\natural)$. In other words, we replace the target space $F(\mathbb{N}, L^\natural)[[X]]$ by $F(\mathbb{N}, L^\sharp)[[X]]$. Hence the twisted universal Taylor morphism in Proposition 3.44 becomes

$$\iota : L \to F(\mathbb{N}, L^\sharp)[[X]].$$

In the twisted formal power series ring $(F(\mathbb{N}, L^\sharp)[[X]], \hat{\sigma}, \delta^*)$, we add differential operators

$$D_1, D_2, \ldots, D_d.$$

So we have a set $\mathcal{D}$ of the following operators on the ring $(F(\mathbb{N}, L^\sharp), \Sigma)[[X]]$.

(1) The endomorphism $\hat{\sigma}$,

$$\hat{\sigma}\Big(\sum_{i=0}^{\infty} X^i a_i\Big) = \sum_{i=0}^{\infty} X^i q^i(\Sigma(a_i)),$$

$\Sigma : F(\mathbb{N}, L^\sharp) \to F(\mathbb{N}, L^\sharp)$ being the shift operator of the ring of functions on $\mathbb{N}$.

(2) The q-skew $\hat{\sigma}$-derivations $\theta^{(i)}$ in (41),

$$\theta^{(l)}\Big(\sum_{i=0}^{\infty} X^i a_i\Big) = \sum_{i=0}^{\infty} X^i \binom{l+i}{l} a_{i+l} \text{ for every } l \in \mathbb{N}.$$

(3) The derivations $D_1, D_2, \ldots, D_d$ operating through the coefficient ring $F(\mathbb{N}, L^\sharp)$.

Hence we may write $F(\mathbb{N}, L^\sharp), \mathcal{D})$, where

$$\mathcal{D} = \{\hat{\sigma}, D_1, D_2, \ldots, D_d, \theta^*\} \text{ and } \theta^* = \{\theta^{(i)}\}_{i\in\mathbb{N}}.$$

For an element $a \in L^\sharp$, we denote by $a^\sharp$ the constant function on $\mathbb{N}$ taking the value a so that

$$a^\sharp(n) = a \quad \text{for every } n \in \mathbb{N}.$$

Therefore $a^\sharp \in F(\mathbb{N}, L^\sharp)$. The latter is a sub-ring of the ring of twisted power series ring $F(\mathbb{N}, L^\sharp)[[X]]$. We can canonically identify the ring

$$F(\mathbb{N}, L^\sharp) = \{\sum_{i=0}^{\infty} X^i a_i \,|\, a_i = 0 \text{ for every } i \in \mathbb{N}^*\}.$$

Namely, we have canonical inclusions

$$L^\sharp \to F(\mathbb{N}, L^\sharp) \to F(\mathbb{N}, L^\sharp)[[X]].$$

We denote the image of an element $a \in L^\sharp$ by $a^\sharp$.

We are ready to define Galois hull as in Definition 2.1.

Definition 3.46. The Galois hull $\mathcal{L}/\mathcal{K}$ is a $\mathcal{D}$-invariant sub-algebra extension in $F(\mathbb{N}, L^\sharp)[[X]]$, where $\mathcal{L}$ is the $\mathcal{D}$-invariant sub-algebra generated by the image $\iota(L)$ and $L^\sharp$ and $\mathcal{K}$ is the $\mathcal{D}$-invariant sub-algebra generated by the image $\iota(k)$ and $L^\sharp$. So $\mathcal{L}/\mathcal{K}$ is a $\mathcal{D}$-algebra extension.

We pass to the task of defining the infinitesimal deformation functor $\mathcal{F}_{L/k}$ and the infinitesimal automorphism functor Inf-gal$\,(L/k)$. We have the universal Taylor morphism

$$\iota_{L^\sharp} : L^\sharp \to \left(L^\natural[[W_1, W_2, \ldots, W_d]], \left\{\frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_d}\right\}\right) \tag{44}$$

as in (5). So by (44), we have the canonical morphism

$$(F(\mathbb{N}, L^\sharp), \mathcal{D}) \to (F(\mathbb{N}, L^\natural[[W]]), \mathcal{D}), \tag{45}$$

where in the target space

$$\mathcal{D} = \left\{ \Sigma, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_d}, \theta^* \right\}$$

by abuse of notation.

For an $L^\natural$-algebra A, the structure morphism $L^\natural \to A$ induces the canonical morphism

$$(F(\mathbb{N}, L^\natural[[W]]), \mathcal{D}) \to (F(\mathbb{N}, A[[W]]), \mathcal{D}). \tag{46}$$

The composite of the $\mathcal{D}$-morphisms (45) and (46) gives us the canonical morphism

$$(F(\mathbb{N}, L^\sharp), \mathcal{D}) \to (F(\mathbb{N}, A[[W]]), \mathcal{D}). \tag{47}$$

The restriction of the morphism (47) to the $\mathcal{D}$-invariant sub-algebra $\mathcal{L}$ gives us the canonical morphism

$$\iota : (\mathcal{L}, \mathcal{D}) \to (F(\mathbb{N}, A[[W]]), \mathcal{D}). \tag{48}$$

We can define the functors exactly as in 2.5 and 2.6 in the difference case.

Definition 3.47. We define the functor

$$\mathcal{F}_{L/k} : (Alg/L^\natural) \to (Sets)$$

from the category $(Alg/L^\natural)$ of $L^\natural$-algebras to the category $(Sets)$ of sets, by associating to an $L^\natural$-algebra A, the set of infinitesimal deformations of the canonical morphism (47).

Hence

$$\begin{aligned}\mathcal{F}_{L/k}(A) = \{f : (\mathcal{L}, \mathcal{D}) \to (F(\mathbb{N}, A[[W_1, W_2, \ldots, W_d]])[[X]], \mathcal{D}) \mid f \text{ is a } \mathcal{D}\text{-morphism} \\ \text{congruent to the canonical morphism } \iota \text{ modulo nilpotent elements} \\ \text{such that } f = \iota \text{ when restricted to the sub-algebra } \mathcal{K}\}.\end{aligned}$$

The definition of the group functor Inf-gal (L/k) is similar.

Definition 3.48. The Galois group in our Galois theory is the group functor

$$\text{Inf-gal}\,(L/k) : (Alg/L^\natural) \to (Grp)$$

defined by

$$\begin{aligned}\text{Inf-gal}\,(L/k)(A) = \{\, f : \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \to \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \mid \\ f \text{ is a } \mathcal{K} \otimes_{L^\sharp} A[[W]]\text{-automorphism compatible with } \mathcal{D}, \\ \text{continuous with respect to the } W\text{-adic topology} \\ \text{and congruent to the identity modulo nilpotent elements}\}\end{aligned}$$

for an $L^\natural$-algebra A. See Definition 2.19 in [12].

Then the group functor Inf-gal (L/k) operates on the functor $\mathcal{F}_{L/k}$ in such a way that the operation (Inf-gal $(L/k), \mathcal{F}_{L/k}$) is a principal homogeneous space.

3.3.4. *Picard-Vessiot theory in iterative q-difference algebra.* From now on in this section, for an iterative q-difference algebra (R, σ, δ^*), we assume that the endomorphism $\sigma : R \to R$ is an automorphism. We denote by C_R the ring

$$\{a \in R \mid \delta^{(i)}(a) = 0 \text{ for every } i \in \mathbb{N}^*\}$$

of constants of R. Let L be a Picard-Vessiot extension of iterative q-difference field k as in Hardouin [5]. Hence we have

(1) $L = k(z_{ij})_{1 \le i,\, j \le n}$,
(2)
$$\delta^{(i)}(Z) = A_i Z \quad \text{for every } i \in \mathbb{N}^* = \{1, 2, 3, \dots\} \tag{49}$$
with $A_i \in \mathrm{M}_n(k)$, $Z = (z_{ij}) \in \mathrm{GL}_n(L)$, and such that
(3) $C := C_L = C_k$.
(4) $R = k[Z, (\det Z)^{-1}]$ is a Picard-Vessiot ring in the sense of Definition 4.3 in [5].

Theorem 3.49 (Theorem 4.12 in [5]). *The automorphism group* $\mathrm{Aut}(L/k)$ *has a natural structure of affine group scheme of finite type over the field* $C := C_k$.

In fact, to be more precise, Hardouin [5] proved that the group functor $\mathrm{Aut}(L/k)$ on the category (Alg/C_R) of C_R-algebra is representable. We denote the affine group scheme $\mathrm{Aut}(L/k)$ by $\mathrm{Gal}(L/k)$.

Theorem 3.50. *Let L/k be a iterative q-difference Picard-Vessiot extension. Then there exists a canonical isomorphism*
$$\mathrm{Lie}\,(\text{Inf-gal}\,(L/k)) \simeq \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^\natural$$
of Lie algebras over $L^\natural$.

This theorem is a particular case of the following general conjecture.

Conjecture 3.51. We can expect an extreme generalization of Theorem 3.50. Namely,

(1) We have a Picard-Vessiot theory with respect to a bialgebra D.
(2) One can develop a further generalization of our general Galois theory that we have seen in Section 2 for the difference case and in Sub-section 3.2 for the differential case.
(3) For a Picard-Vessiot extension L/k with respect to a bialgebra, we have a canonical isomorphism
$$\mathrm{Lie}\,(\text{Inf-gal}\,(L/k)) \simeq \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^\natural$$
of Lie algebras over $L^\natural$.

Heiderich [7], [8] constructed a general Galois theory and he answered affirmatively Conjecture 3.51 for a wide class of Picard-Vessiot extensions. In the last part of [7], however, he assumes that the bialgebra is a cocommutative Hopf algebra. So Picard-Vessiot theory of iterative q-difference equations is excluded from Corollary 4.2.8, [7]. Recent preprint of Masuoka [11] offers us a way to reduce iterative q-difference Picard-Vessiot theory to a cocommutative Hopf algebra.

Another uncomfortable point of Corollary 4.2.8 is that the isomorphism of Lie algebras is not defined over the field $L^\natural$ but over a finite separable extension L' of $L^\natural$.

We expect that the proof of Theorem 3.20 in 3.1 would yield a proof of Conjecture 3.51.

3.3.5. *Proof of Theorem 3.50.* In the difference case, we started from the universal Euler morphism
$$L \to (F(\mathbb{N}, L^\sharp), \Sigma)$$

and then we added the derivations $\{D_1, D_2, \dots, D_d\}$ to the difference ring $(F(\mathbb{N}, L^\natural), \Sigma)$. We worked in the difference-differential ring

$$(F(\mathbb{N}, L^\sharp), \Sigma, \{D_1, D_2, \dots, D_d\})$$

for construction of the Galois hull $\mathcal{L}$.

Similarly in the differential case, we use the universal Taylor morphism

$$L \to (L^\natural[[X]], d/dX)$$

to start with and then add the derivations $\{D_1, D_2, \dots, D_d\}$ to the differential ring

$$(L^\natural[[X]], d/dX).$$

We worked in the partial differential ring

$$(L^\natural[[X]], d/dX, \{D_1, D_2, \dots, D_d\})$$

to construct the Galois hull $\mathcal{L}$ in the partial differential ring.

The proof of Theorem 3.23 which is a differential analogue of Theorem 3.20 is done by using the differential counterparts according to the list in 3.2.

We are in the same situation for the iterative q-difference case. We start from the twisted universal Taylor morphism

$$L \to ((F(\mathbb{N}, L^\natural)[[X]], \hat{\sigma}, \theta^*)$$

and then we add the derivations $\{D_1, D_2, \dots, D_d\}$ to the q-SI σ-differential ring to get

$$((F(\mathbb{N}, L^\natural)[[X]], \hat{\sigma}, \theta^*, \{D_1, D_2, \dots, D_d\}.) \tag{50}$$

The Galois hull $\mathcal{L}$ is constructed in the ring (50). The proof of the theorem is more or less automatic if we replace, according to the list below, what is in difference algebra by its counterpart in iterative q-difference algebra.

Proposition 3.52. *We use the notation above so that L/k is a iterative q-difference Picard-Vessiot extension. Then*

$$\mathcal{L} = \iota(L).L^\sharp$$

that is a $\mathcal{D}$-sub-algebra of the non-commutative algebra $F(\mathbb{N}, L^\sharp)[[X]]$ of twisted formal power series. Moreover the Galois hull $\mathcal{L}$ is commutative.

Proof. Using the notation above, we set $B := \iota(Z){Z^\sharp}^{-1} \in \mathrm{GL}_n(F(\mathbb{N}, L^\sharp)[[X]])$.

Then iterative q-difference analogues of lemmas in the difference case hold without any change according to the list of replacements below. Then all the arguments in the proof of Lemmas 3.3, 3.5, 3.6 and 3.8 work showing

$$\mathcal{L} = \iota(L).L^\sharp \tag{51}$$

in the twisted formal power series ring $F(\mathbb{N}, L^\sharp)[[X]]$. Since $L^\sharp$ is in the center of the non-commutative ring $F(\mathbb{N}, L^\sharp)[[X]]$, the algebra $\mathcal{L}$ is commutative by (51). ∎

We give below the list of replacements.

Difference theory	*Iterative q-difference theory*
$L^\natural$	$L^\natural$
Universal Euler morphism	Universal twisted Taylor morphism
$(F(\mathbb{N}, L^\natural), \Sigma)$	$(F(\mathbb{N}, L^\natural)[[X]], \hat{\sigma}, \theta^*)$
Shift operator Σ	Difference operators $\hat{\sigma}, \theta^*$
$L^\sharp$	$L^\sharp$
$D_1, D_2, \ldots, D_d$	$D_1, D_2, \ldots, D_d$
$F(\mathbb{N}, L^\natural[[W]])$	$F(\mathbb{N}, L^\natural[[W]])[[X]]$
$A[[W]]$	$A[[W]]$
$F(\mathbb{N}, A[[W]])$	$F(\mathbb{N}, A[[W]])[[X]]$

Now using the list, we examine the procedure of proof of Theorem 3.20 in 3.1 in iterative q-difference context.

Once the Galois hull $\mathcal{L}$ is determined in Proposition 3.52, the next goal is to show an iterative q-difference analogue of Proposition 3.18 which implies Theorem 3.50. In the proof of Proposition 3.18, we use only Lemma 3.11 and linear disjointness theorems over constants of which Lemma 3.9 is a typical example.

As for the linear disjointness theorems, the following lemma generalizes Lemma 3.9 as well as Lemma 3.28.

Lemma 3.53. *Let (R, σ, θ^*) be a q-SI σ-differential ring and M a q-SI σ-differential sub-field. Then the field M and the ring C_R of constants of R are linearly disjoint over C_M.*

Proof. The proof of Lemma 3.9 and the proof of Lemma 3.28 depend on the same principle. In a q-SI σ-differential ring we have both an endomorphism σ and a family θ^* of C-linear maps. The combination of the proofs of Lemmas 3.9, 3.28 gives the proof of Lemma 3.53. ∎

We also need a linear disjointness theorem in a mixed version as in Remark 3.17. The proof of the mixed version poses no problem and Theorem 3.50 is proved. We have to discuss q-SI σ-differential analogue of Lemma 3.11. We can prove an analogue of Lemma 3.11 if q is not a root of 1. When q is a root of 1, however, a naive analogue of Lemma 3.11 looks false. We used Lemma 3.11, in the proof of Proposition 3.18, to show that $\iota(k)[B, (\det B)^{-1}]k^\natural$ and $L^\natural$ are linearly disjoint over $k^\natural = k^\sharp$.

To remedy this we prove Lemma 3.55 below so that we can prove Proposition 3.18 without Lemma 3.11. To this end, we first need

Lemma 3.54. *The field*

$$C_{Q(\iota(k)[B,(\det B)^{-1}]L^\sharp.L^\natural)}$$

of constants of the q-SI σ-differential field

$$Q(\iota(k)[B, (\det B)^{-1}]L^\sharp.L^\natural)$$

is $Q(L^\sharp.L^\natural)$.

Proof. We have $\iota(k)[B,(\det B)^{-1}]L^\sharp.L^\natural = \iota(k)[Z,(\det Z)^{-1}]L^\sharp$. So we have to show that

$$C_{Q(\iota(k)[Z,(\det Z)^{-1}]L^\sharp L^\natural)} = Q(L^\sharp L^\natural).$$

We have

$$\iota(k[Z,(\det Z)^{-1}]).L^\sharp.L^\natural \simeq (k[Z,(\det Z)^{-1}] \otimes_C L^\sharp) \otimes_{k^\sharp} L^\natural \tag{52}$$

$$\simeq k[Z,(\det Z)^{-1}] \otimes_C (L^\sharp \otimes_{k^\sharp} L^\natural) \tag{53}$$

as q-SI σ-differential rings. So the field $Q(\iota(k[Z,(\det Z)^{-1}]).L^\sharp.L^\natural)$ is obtained from the field $\iota(k[Z,(\det Z)^{-1}])$ by extending the constant field. Therefore the lemma is proved. ■

LEMMA 3.55. *The field*

$$C_{Q(\iota(k)[B,(\det B)^{-1}]k^\sharp)}$$

of constants of of the q-SI σ-differential field $Q(\iota(k)[B,(\det B)^{-1}]k^\sharp)$ is $k^\sharp$.

Proof. Let w be a constant of the q-SI σ-differential field $Q(\iota(k)[B,(\det B)^{-1}]k^\sharp)$. So there exist elements $f, g \in \iota(k)[B,(\det B)^{-1}]k^\sharp$ with $g \neq 0$ such that

$$w = \frac{f}{g}. \tag{54}$$

The elements f and g belong to $F(\mathbb{N}, k^\sharp)[[X]]$,

$$f = \sum_{i=0}^{\infty} X^i a_i(\underline{n}), \qquad g = \sum_{i=0}^{\infty} X^i b_i(\underline{n}). \tag{55}$$

where $a_i(\underline{n}), b_i(\underline{n}) \in F(\mathbb{N}, M)$ are functions on $\mathbb{N}$. We mean by $\underline{n}$ a variable varying in $\mathbb{N}$. If we work in the over-ring $F(\mathbb{N}, L^\natural((W)))[[X]]$ of $F(\mathbb{N}, L^\natural[[W))]][[X]]$, we have by Lemma 3.54, w is a constant function $w(\underline{n})$ taking a constant value $c \in Q(L^\sharp.L^\natural) \subset L^\natural((W))$. It follows from (54) that we have

$$f = cg,$$

which is an equality in $F(\mathbb{N}, L^\natural((W))[[X]]$ so that by (55),

$$\sum_{i=0}^{\infty} X^i a_i(\underline{n}) = c \sum_{i=0}^{\infty} X^i b_i(\underline{n}). \tag{56}$$

Therefore

$$a_i(\underline{n}) = b_i(\underline{n}) \text{ for every } i \in \mathbb{N}.$$

We choose an integer $l \in \mathbb{N}$ such that $b_l(\underline{n}) \neq 0$ so that

$$a_l(\underline{n}) = c b_l(\underline{n}). \tag{57}$$

Since $b_l(\underline{n}) \neq 0$, there exists an integer $n \in \mathbb{N}$ such that

$$b_l(n) \neq 0.$$

By (57)

$$c = a_l(n)/b_l(n) \in k^\sharp.$$

So we have proved the inclusion

$$C_{Q(\iota(k)[B,(\det B)^{-1}]k^\sharp)} \subset k^\sharp.$$

The opposite inclusion being trivial, the lemma is proved. ■

4. Beyond Picard-Vessiot theory

4.1. Strongly normal extensions. We can apply the argument of the proof of Theorem 3.20 not only to differential Picard-Vessiot extensions but also to strongly normal extensions in differential algebra or in Kolchin's Galois theory.

THEOREM 4.1 (Theorem (5.15) in [18]). *Let L/k be a strongly normal extension with Galois group $G = \mathrm{Gal}(L/k)$. Then we have an isomorphism*

$$\mathrm{Lie}\,(\mathrm{Inf\text{-}gal}\,(L/k)) \simeq \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^{\natural}$$

of $L^{\natural}$-Lie algebras.

Wibmer [24] defined difference strongly normal extensions.

QUESTION 4.2. *Let L/k be a strongly normal extension of difference fields. Do the arguments in the proof of Theorem 3.20 apply to this case so that we have an isomorphism*

$$\mathrm{Lie}\,(\mathrm{Inf\text{-}gal}\,(L/k)) \simeq \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^{\natural}$$

of $L^{\natural}$-Lie algebras?

4.2. Quantification of the Galois groupoid? In unified general Galois theory with Hopf algebra $\mathcal{H}$, the universal Euler morphism or the universal Taylor morphism is nicely generalized as the algebra morphism

$$\iota : L \to_C \mathbf{M}(\mathcal{H}, L).$$

As we have seen for an iterative q-difference algebra, the algebra ${}_C\mathbf{M}(\mathcal{H}, L)$, which is the dual of the coalgebra $\mathcal{H}$, is not commutative if the coalgebra $\mathcal{H}$ is not cocommutative. The Galois hull $\mathcal{L}$ is a sub-algebra of the eventually non-commutative algebra ${}_C\mathbf{M}(\mathcal{H}, L)$. We have, however, the following result generalizing Proposition 3.52.

LEMMA 4.3 (Heiderich [7]). *For a Picard-Vessiot extension L/k in Hopf Galois theory, the Galois hull $\mathcal{L} = \iota(L).L^{\sharp}$ and $\mathcal{L}$ is a commutative algebra.*

Proof. The proof of Proposition 3.52 works in a general setting. See Heiderich [7], Lemma 4.2.1. ∎

A similar assertion is expected for G-primitive extensions.

QUESTION 4.4. *Can we define G-primitive extension in the framework of Heiderich [7] so that we can prove an analogue of Lemma 4.3?*

This does not seem difficult.

We explained in [13], [21] and [22] that the Galois hull $\mathcal{L}/\mathcal{K}$ is an algebraic counterpart of the Galois groupoid. Hence Lemma 4.3 says that so far as we deal with linear difference, difference-differential equations, no matter how twisted or how non-commutative the operators are, the Galois hull $\mathcal{L}$ is commutative. In other words, we do not encounter the quantum groupoid.

The following natural and fundamental question arises.

QUESTION 4.5. *When the Galois hull $\mathcal{L}$ is non-commutative, what does it describe? Does it give a quantification of the algebraic groupoid?*

Since for Picard-Vessiot extensions or more generally G-primitive extensions if we assume an affirmative answer to Question 4.4, the Galois hull $\mathcal{L}$ is commutative, to explore Question 4.5 we must study an extension L/k far different from G-primitive extensions, a Picard-Vessiot extension being a particular G-primitive extension. We know in the differential case that Painlevé equations offer us examples of such extensions L/k.

To illustrate this, we recall, as a particular example, the following theorem that the first Painlevé equation is not reducible to the classical functions.

THEOREM 4.6 (Umemura [16]). *Let k be the differential field $(\mathbb{C}(x), d/dx)$ of the rational function field of 1-variable x. Let y be an element of a differential over-field M of $\mathbb{C}(x)$ satisfying the first Painlevé equation*

$$y'' = 6y^2 + x.$$

Then the differential field extension $L := k\langle y\rangle = \mathbb{C}(x, y, y')$ over the base field $k = \mathbb{C}(X)$ is not classical. Namely, there does not exist a tower

$$k = L_0 \subset L_1 \subset \cdots \subset L_n$$

of differential field extensions such that

$$L \subset L_n$$

and such that the extension L_{i+1}/L_i is of one of the following types:

(1) adjunction of constants,

(2) G-primitive extension, of which a Picard-Vessiot extension is a special case,

for $0 \le i \le n-1$.

The observation above leads Heiderich to the following proposal.

PROPOSAL 4.7 (Heiderich). *Shall we start our expedition with well-twisted Painlevé equations?*

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