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A SIMPLE PROOF OF THE NON-INTEGRABILITY OF THE FIRST AND THE SECOND PAINLEVÉ EQUATIONS

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Abstract. The first and the second Painlevé equations are explicitly Hamiltonian with time dependent Hamilton function. By a natural extension of the phase space one gets corresponding autonomous Hamiltonian systems in $\mathbb{C}^4$. We prove that the latter systems do not have any additional algebraic first integral. In the proof equations in variations with respect to a parameter are used.

1. Introduction. The equations

$$\frac{d^2x}{dt^2} = 6x^2 + t \tag{1}$$

and

$$\frac{d^2x}{dt^2} = 2x^3 + tx + \alpha \tag{2}$$

(α a parameter) are known as the *first* P1 and the *second* P2 *Painlevé equations* respectively (see [GLS]). They are Hamiltonian equations with time dependent Hamilton functions: $\frac{1}{2}\dot{x}^2 - 2x^3 - tx$ and $\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^4 - \frac{1}{2}tx^2 - \alpha x$.

Denoting $\dot{x} = y$, $t = q$ and introducing an additional variable p we get the systems

$$\dot{x} = y, \quad \dot{y} = 6x^2 + q, \quad \dot{q} = 1, \quad \dot{p} = x \tag{3}$$

and

$$\dot{x} = y, \quad \dot{y} = 2x^3 + xq + \alpha, \quad \dot{q} = 1, \quad \dot{p} = x^2/2, \tag{4}$$

where the dot denotes differentiation with respect to a new time (say τ). The latter

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systems are Hamiltonian with the (time independent) Hamilton functions

$$H = \frac{1}{2}y^2 - 2x^3 - xq + p \tag{5}$$

and

$$H = \frac{1}{2}y^2 - \frac{1}{2}x^4 - \frac{1}{2}x^2 q - \alpha q + p. \tag{6}$$

We obtain Hamiltonian systems with two degrees of freedom such that the functions (5) and (6) are their respective first integrals.

Also other Painlevé equations, i.e. P3, P4, P5 and P6, can be rewritten in the Hamiltonian form with time dependent Hamilton functions and admit extensions to autonomous Hamiltonian systems with two degrees of freedom. (The corresponding changes of coordinates and the Hamilton functions are rather complicated, so we do not present them here.) Recently E. Horozov and Ts. Stoyanova [HS, Sto] considered the question of integrability in the sense of Liouville and Arnold of the Hamiltonian system in $\mathbb{C}^4$ related to the sixth Painlevé equation. They proved that for some special (but not discrete) values of parameters $\alpha, \beta, \gamma, \delta$ (which appear in P6) this system is not integrable in the Liouville–Arnold sense.

Recall that an autonomous Hamiltonian vector field X_H on a (real or complex) symplectic manifold M (of dimension $2n$ and equipped with a symplectic 2-form ω) is *completely integrable* (or *integrable in the Liouville–Arnold sense*[1]) if there exist functionally independent first integrals $H_0 = H, H_1, \ldots, H_{n-1}$ which have zero pairwise Poisson brackets (see [Arn]). Here the *Hamiltonian vector field* X_H is defined by the condition

$$\omega(X_H, Z) = \langle dH, Z \rangle = \frac{\partial H}{\partial Z}$$

(for any vector field Z) and the *Poisson bracket* of two functions F and G equals

$$\{F, G\} = \omega(X_G, X_F) = \partial G / \partial X_F.$$

In our situation $M = \mathbb{C}^4$ and $\omega = dx \wedge dy + dq \wedge dp$. If H_j are algebraic functions on $\mathbb{C}^4$ then we say that the vector field X_H is *algebraically integrable in the Liouville–Arnold sense.*

The main result of the paper is the following

MAIN THEOREM 1.1. *Each of systems (3) and (4) does not admit any first integral which is an algebraic function of x, y, p, q and is independent of H.*

Horozov and Stoyanova applied a version of the Ziglin method [Zig], developed by J.-P. Ramis with J. Morales-Ruiz [M-R]. It uses the monodromy group (or the differential Galois group) of the normal variation equation for a particular algebraic solution of the corresponding Hamiltonian system. In the case of complete integrability with rational first integrals the identity component of this differential Galois group should be abelian. In the case of the P6 equation suitable algebraic solutions exist for special values of the parameters. By direct computation of the monodromy group Horozov and Stoyanova

[1]Horozov and Stoyanova say that the system is integrable in the Liouvillian sense. But in Differential Galois Theory there exists a notion of Liouvillian function (which is a function expressed in quadratures), so it is safer to speak about Liouville–Arnold integrability.

show that the identity component of the differential Galois group of the normal variation equation is not abelian.

Another method to prove the non-integrability is to exhibit heteroclinic orbits after perturbation of a system with some separatrix connection.

Our method of proof of the Main Theorem is different and probably new. By a suitable normalization of the variables we arrive at a perturbation of a completely integrable system with two algebraic first integrals. Then we consider the equation in variations with respect to a parameter (denoted by ε) around a particular solution which is a rather general elliptic curve. Then analysis of few initial terms in powers of ε of a possible first integral of the perturbed system leads to some properties of elliptic integrals which cannot be true.

Finally note that in [GLS] V. Gromak, I. Laine and S. Shimomura proved the non-existence of any first integral of the equation P1 which is an algebraic function of x, $\dot{x}$ and t. The latter result follows from the Main Theorem. Indeed, any such integral $F(x, \dot{x}, t)$, treated as a function of x, y, q, p, would be a first integral of the Hamiltonian system (3); moreover, independent of H.

2. Proof of the Main Theorem. In the proof we focus on system (3). System (4) is analyzed along the same lines and we briefly discuss it at the end of this section.

After applying the change $x \mapsto \mu^{-2}x$, $t \mapsto \mu t$ equation (1) becomes

$$\ddot{x} = 6x^2 + \varepsilon t, \quad \varepsilon = \mu^5.$$

Putting $\varepsilon t = q$, $\dot{x} = y$ and introducing an additional momentum p, we arrive at the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = 6x^2 + q, \quad \dot{q} = \varepsilon, \quad \dot{p} = x \tag{7}$$

with the Hamilton function

$$H_\varepsilon = \frac{1}{2}y^2 - 2x^3 - qx + \varepsilon p. \tag{8}$$

The change between the old and the new variables is the following:

$$x \mapsto \mu^{-2}x, \quad y \mapsto \mu^{-3}y, \quad q \mapsto \mu^{-4}q, \quad p \mapsto \mu^{-1}p. \tag{9}$$

Of course, if system (3) has an additional first integral F then also system (7) has an integral F_ε independent of H_ε. Moreover F_ε should depend algebraically on ε.

The following result is rather obvious.

LEMMA 2.1. *For $\varepsilon = 0$ system (7) is completely integrable with the functions $H_0 = H_\varepsilon|_{\varepsilon=0}$ and $H_1 = q$ playing the role of first integrals in involution.*

The common level sets

$$H_0 = h_0, \quad q = q_0$$

are of the form $\Gamma \times \mathbb{C}^1$ where Γ is the elliptic curve

$$\Gamma = \Gamma(q_0, h_0) = \left\{y^2 = 4x^3 + 2q_0x + 2h_0\right\} \tag{10}$$

and $\mathbb{C}^1$ is the line $\{(p,q) : q = q_0\}$.[2] The solutions to equation (7) for $\varepsilon = 0$ are

$$x = \mathcal{P}(t-t_0), \quad y = \mathcal{P}'(t-t_0), \quad p = p_0 + \mathcal{Q}(t-t_0), \quad q = q_0. \tag{11}$$

Here $\mathcal{P}(t)$ (the Weierstrass P-function), $\mathcal{P}'(t)$ and $\mathcal{Q}(t)$ are elliptic functions defined by the following formulas:

$$\int_{(x_0,y_0)}^{(\mathcal{P},\mathcal{P}')} \frac{dx}{y} = t, \quad \mathcal{Q}(t) = \int_0^t \mathcal{P}(s)ds = \int_{(x_0,y_0)}^{(\mathcal{P},\mathcal{P}')} \frac{xdx}{y}, \tag{12}$$

where the integral $\int_{(x_0,y_0)}^{(\mathcal{P},\mathcal{P}')}$ runs along a path in the complex curve Γ from some initial point (x_0, y_0) to the point $(x,y) = (\mathcal{P}(t), \mathcal{P}'(t))$. Below we fix the initial conditions by putting

$$y_0 = 0, \quad t_0 = 0 \tag{13}$$

and x_0 as some root of the equation $4x^3 + 2q_0x + 2h_0 = 0$.

Suppose that system (7) has an algebraic first integral $F_\varepsilon(x,y,p,q) = F(x,y,p,q;\varepsilon)$, which depends algebraically on ε and is independent of H_ε. Let us expand F_ε in (rational) powers of ε:

$$F_\varepsilon = F_0(x,y,p,q) + \varepsilon^{\alpha_1} F_1(x,y,p,q) + \dots, \tag{14}$$

where $0 < \alpha_1 < \alpha_2 < \dots$. We can assume the above form, otherwise we multiply the first integral by a power of ε.

Of course, F_0 is a first integral of system (7) for $\varepsilon = 0$. Therefore it is an algebraic function of H_0 and q,

$$F_0 = G_0(H_0, q).$$

We have two possibilities:

(i) G_0 depends on q;

(ii) G_0 does not depend on q, $G_0 = G_0(H_0)$.

In the second case we replace F_ε with $F_\varepsilon - G_0(H_\varepsilon)$ and divide it by a power of ε. After finitely many such operations we arrive to the form (14) with $F_0 = G_0(H_0, q)$ and $\partial G_0/\partial q \not\equiv 0$.

Take q_0 and h_0 in equation (2.4) such that

$$\frac{\partial G_0}{\partial q}(h_0, q_0) \neq 0. \tag{15}$$

Take also the solution (11) with the initial condition (13): $x = \mathcal{P}(t)$, $y = \mathcal{P}'(t)$, $p = p_0 + \mathcal{Q}(t)$, $q = q_0$.

We consider the *equation in variations with respect to the parameter along this solution.* Therefore we substitute

$$x = \mathcal{P}(t) + \varepsilon x_1(t),\ y = \mathcal{P}'(t) + \varepsilon y_1(t),\ p = p_0 + \mathcal{Q}(t) + \varepsilon p_1(t),\ q = q_0 + \varepsilon q_1(t), \tag{16}$$

[2]In the real case the elliptic curve has (typically) either one unbounded component (and $\Gamma \times \mathbb{R}^1$ is diffeomorphic to $\mathbb{R}^2$) or two components with one compact oval (the corresponding component of $\Gamma \times \mathbb{R}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^1$). This agrees with the Liouville–Arnold theorem [Arn] about completely integrable Hamiltonian systems: the connected components of common level surfaces of the first integrals H_j are of the form $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

$x_1(0) = y_1(0) = p_1(0) = q_1(0) = 0$, into system (7) and solve it modulo $O(\varepsilon^2)$. But for our purposes we do not need to solve the whole system, we need only the solution for q. It takes the form

$$q(t) = q_0 + \varepsilon t. \tag{17}$$

We have also

$$H_\varepsilon \equiv h_0 + \varepsilon h_1 + O(\varepsilon^2), \quad F_\varepsilon \equiv f_0 + \varepsilon^{\alpha_1} f_1 + \dots \tag{18}$$

on this solution; here $f_0 = G_0(h_0, q_0)$ and h_j and f_j depend on h_0, q_0 and p_0.

From equations (8), (17) and (18) we obtain

$$\begin{aligned} F_0 &= G_0(H_0, q) \approx G_0(H_\varepsilon - \varepsilon p, q_0 + \varepsilon t) \approx G_0(h_0 + \varepsilon h_1 - \varepsilon p, q_0 + \varepsilon t) \\ &= f_0 + \varepsilon \left\{ \frac{\partial G_0}{\partial H_0}(h_0, q_0) \cdot (h_1 - p_0 - \mathcal{Q}(t)) + \frac{\partial G_0}{\partial q}(h_0, q_0) \cdot t \right\} + O(\varepsilon^2) \\ &= f_0 + \varepsilon \{A + B\mathcal{Q} + Ct\} + O(\varepsilon^2), \end{aligned}$$

where A, B, $C \neq 0$ are constants not depending on t. We substitute it to the equation $F_0 + \varepsilon^{\alpha_1} F_1 + \dots \equiv f_0 + \varepsilon^{\alpha_1} f_1 + \dots$ and we arrive at an equation of the form

$$\varepsilon^{\alpha_1} \cdot \{F_1(\mathcal{P}, \mathcal{P}', p_0 + \mathcal{Q}, q_0) - f_1\} + \dots + \varepsilon \cdot \{A + B\mathcal{Q} + Ct\} + \dots \equiv 0, \tag{19}$$

Suppose $\alpha_1 < 1$. Then the term $\varepsilon^{\alpha_1} \cdot \{F_1 - f_1\}$ is dominating and hence it vanishes, it defines some relation between elliptic functions. The same statement holds for other terms in (19) with ε^{α_j}, $\alpha_1 < \alpha_j < 1$.

But for $\alpha_k = 1$ equation (19) implies a relation of the form

$$t \equiv \Phi(\mathcal{P}(t), \mathcal{P}'(t), \mathcal{Q}(t)),$$

where Φ is an algebraic function of its arguments. In other words, t is an algebraic function of the functions $\mathcal{P}(t)$, $\mathcal{P}'(t)$ and $\mathcal{Q}(t)$. Let us rewrite the corresponding algebraic equation in the following form:

$$\sum_{m,n} a_{m,n}(\mathcal{P}, \mathcal{P}') t^m \mathcal{Q}^n = 0, \tag{20}$$

where $a_{m,n}$ are polynomials of $\mathcal{P}$ and $\mathcal{P}'$.

Recall now that the Weierstrass function is doubly periodic:

$$\mathcal{P}(t + \omega_1) = \mathcal{P}(t), \quad \mathcal{P}(t + \omega_2) = \mathcal{P}(t),$$

where the periods ω_j are defined as complete elliptic integrals

$$\omega_1 = \oint_{\gamma_1} \frac{dx}{y}, \quad \omega_2 = \oint_{\gamma_2} \frac{dx}{y}$$

along two curves $\gamma_{1,2} \subset \Gamma$ which generate the first homology group of the Riemann surface Γ. If x_1, x_2, x_3 are zeroes of the polynomial $4x^3 + 2q_0 x + 2h_0$, then γ_1 (respectively γ_2) is a lift to the Riemann surface of the function $\sqrt{4x^3 + 2q_0 x + 2h_0}$ of a loop which surrounds the points x_1, x_2 (respectively x_1, x_3) in the x-plane.

The function $\mathcal{Q}(t)$ is not periodic, but it satisfies the following relations:

$$\mathcal{Q}(t + \omega_1) = \mathcal{Q}(t) + \eta_1, \quad \mathcal{Q}(t + \omega_2) = \mathcal{Q}(t) + \eta_2,$$

where

$$\eta_j = \int_0^{\omega_j} \mathcal{P}(t)dt = \oint_{\gamma_j} \frac{xdx}{y}, \quad j = 1, 2,$$

are also complete elliptic integrals. These statements reflect the fact that the integration path in equations (12) is not unique.

We need the following result whose proof we postpone to the end of the section.

Lemma 2.2. *We have*

$$\begin{vmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{vmatrix} \neq 0$$

for typical parameters q_0 *and* h_0.

Let us replace the function t with

$$\mathcal{R}(t) = t - \frac{\omega_1}{\eta_1}\mathcal{Q}(t).$$

It has the following properties:

$$\mathcal{R}(t + \omega_1) = \mathcal{R}(t), \quad \mathcal{R}(t + \omega_2) = \mathcal{R}(t) + \kappa, \quad \kappa = \omega_2 - \omega_1(\eta_2/\eta_1) \neq 0. \tag{21}$$

Equation (20) takes the form

$$\sum_{m,n} b_{m,n}(\mathcal{P}, \mathcal{P}')\mathcal{R}^m \mathcal{Q}^n \equiv 0. \tag{22}$$

Because only the function $\mathcal{Q}$ is not invariant with respect to translation by ω_1, we must have $n = 0$ in the above formula. But then also $m = 0$, because otherwise the left hand side is not invariant with respect to the translation by ω_2.

On the other hand, the degree with respect to $\mathcal{R}$ of the polynomial in equation (22) must be ≥ 1 since equation (2.14) defines t as an algebraic function of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{P}'$.

The latter contradiction proves the Main Theorem for the first Painlevé equation.

In the case of equation P2 an analogous normalization leads to the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = 2x^3 + qx + \beta, \quad \dot{q} = \varepsilon, \quad \dot{p} = x^2/2$$

with the Hamilton function

$$H_\varepsilon = \frac{1}{2}y^2 - \frac{1}{2}x^4 - \frac{1}{2}qx^2 - \beta x + \varepsilon p.$$

(Here β may depend on ε, but it does not change our argument). As before, for $\varepsilon = 0$ the system is completely integrable with H_0 and $H_1 = q$ as independent first integrals in involution. The common level surfaces of these first integrals are of the form $\Gamma \times \mathbb{C}^1$, where

$$\Gamma = \left\{y^2 = x^4 + q_0x^2 + 2\beta x + 2h_0\right\}$$

is also an elliptic curve. The solutions to the unperturbed system are as in (11), with $\mathcal{P}(t)$ an elliptic integral (but not just the Weierstrass P-function) and

$$\mathcal{Q}(t) = \frac{1}{2}\int_0^t \mathcal{P}^2(s)ds = \frac{1}{2}\int_{(x_0,y_0)}^{(\mathcal{P},\mathcal{P}')} \frac{x^2dx}{y}.$$

The remaining part of the proof is the same. The complete elliptic integrals ω_j are defined as above, with the loops γ_j being lifts to the Riemann surface of the function

$\sqrt{x^4+q_0x^2+2\beta x+2h_0}$ of loops (in the x-plane) which surround the pairs $\{x_1,x_2\}$ and $\{x_1,x_3\}$ respectively of zeroes of the polynomial $x^4+q_0x^2+2\beta x+2h_0$. The periods η_j are $\oint_{\gamma_j} x^2dx/y$. An analogue of Lemma 2.2 is proved in the same way as below.

The proof of the Main Theorem is complete. ■

Proof of Lemma 2.2. This result is well known in the theory of integrals of holomorphic forms along cycles in Riemann surfaces, see [Zol] for example. For the sake of completeness we present its quite standard proof.

Fix q_0 and assume that $h_0=-2\lambda^6<0$ is large. Then the normalization $x=\lambda^2X$, $y=2\lambda^3Y$ replaces the curve (10) with a curve close to $Y^2=X^3-1$. The elliptic integrals are

$$\omega_j\approx\frac{1}{2\lambda}\oint_{\gamma_j}\frac{dX}{Y},\quad \eta_j\approx\frac{\lambda}{2}\oint_{\gamma_j}\frac{XdX}{Y}.$$

We replace the cycles γ_1 and γ_2 by cycles along ridges of the cut of the x-plane along segments $[0,1]\cup[0,e^{2\pi i/3}]$ and $[0,1]\cup[0,e^{-2\pi i/3}]$ respectively. We also make the change $X=\zeta z^{1/3}$, $z\in[0,1]$ and $\zeta=1$ or $=e^{\pm 2\pi i/3}$, along corresponding segments. Each segment gives the contribution

$$2\zeta\int_0^1\frac{d\left(z^{1/3}\right)}{\sqrt{z-1}}=\frac{2\zeta}{3i}\int_0^1 z^{-2/3}(1-z)^{-1/2}dz$$

(with a proper sign) to the integral $\int dX/Y$, and an analogous contribution to $\int XdX/Y$. This gives

$$\omega_1\approx\frac{e^{2\pi i/3}-1}{3i\lambda}B(1/2,1/3),\qquad \omega_2\approx\frac{e^{-2\pi i/3}-1}{3i\lambda}B(1/2,1/3),$$
$$\eta_1\approx\frac{e^{-2\pi i/3}-1}{3i}\lambda B(1/2,2/3),\qquad \eta_2\approx\frac{e^{2\pi i/3}-1}{3i}\lambda B(1/2,2/3),$$

where $B(\cdot,\cdot)$ is the Euler beta function.

Now it is clear that the determinant from Lemma 2.2 is nonzero. (In fact, this determinant is constant as a function of h_0.) ■

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