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ρ-ORTHOGONALITY AND ITS PRESERVATION — REVISITED

JACEK CHMIELIŃSKI and PAWEŁ WÓJCIK

Institute of Mathematics, Pedagogical University
Podchorążych 2, 30-084 Kraków, Poland
E-mail: jacek@up.krakow.pl, pwojcik@up.krakow.pl

Abstract. The aim of the paper is to present results concerning the ρ-orthogonality and its preservation (both accurate and approximate) by linear operators. We survey on the results presented in [11] and [23], as well as give some new and more general ones.

1. Introduction. There is no unique way how to transfer the notion of orthogonality from inner product spaces to normed spaces. Perhaps the most useful is the notion of Birkhoff orthogonality; however many other can be used. One can also consider an axiomatic definition of the orthogonality relation and the orthogonality space. Following Rätz [21], a real linear space X with a binary relation $\perp$ in X is called an *orthogonality space* whenever

(i) $x\perp 0$ and $0\perp x$, $x \in X$;
(ii) if $x, y \in X \setminus \{0\}$ and $x\perp y$, then x and y are linearly independent;
(iii) if $x, y \in X$ and $x\perp y$, then $\alpha x\perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(iv) for any two-dimensional subspace P of X and for arbitrary $x \in P$, $\lambda \in [0, \infty)$, there exists $y \in P$ such that $x\perp y$ and $x + y\perp \lambda x - y$.

One of the possible notions of orthogonality is connected with so called norm's derivatives. In papers [11] and [23] authors considered ρ and $\rho_{\pm}$-orthogonalities as well as their approximate counterparts. Some properties of those relations have been proved. The classes of linear mappings which precisely or approximately preserve this kind of orthogonality were also considered in the above mentioned publications. Investigation on such a class of *linear mappings preserving orthogonality* can be considered as a part of

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the theory of *linear preservers*. The present paper has an expository character in part; however, many of the surveyed results will be essentially generalized.

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. If the norm is generated by an inner product $\langle\cdot|\cdot\rangle$, we consider the standard orthogonality relation: $x\perp y \Leftrightarrow \langle x|y\rangle = 0$. In general case, we may consider the definition introduced by Birkhoff [4] (cf. also James [16]):

$$x\perp_{\mathrm{B}} y \quad\Longleftrightarrow\quad \forall\,\lambda\in\mathbb{K}\ \ \|x+\lambda y\|\geq\|x\|.$$

The pair $(X, \perp_{\mathrm{B}})$ is an orthogonality space (cf. [1]).

Even if the norm in X does not come from an inner product there always exists (as noticed by G. Lumer [18] and J. R. Giles [14], cf. also [13]) a mapping $[\,\cdot\,|\,\cdot\,] : X\times X\to\mathbb{K}$ satisfying the properties:

- $[\lambda x+\mu y|z] = \lambda\,[x|z]+\mu\,[y|z]\,,\quad x,y,z\in X,\ \ \lambda,\mu\in\mathbb{K};$
- $[x|\lambda y] = \overline{\lambda}\,[x|y]\,,\quad x,y\in X,\ \ \lambda\in\mathbb{K};$
- $|\,[x|y]\,|\leq\|x\|\,\|y\|,\quad x,y\in X;$
- $[x|x]=\|x\|^2,\quad x\in X.$

Such a mapping is called a *semi-inner product* in X (generating the given norm $\|\cdot\|$ in X). There may exist infinitely many different semi-inner products in X. Its uniqueness is equivalent to the smoothness of the norm (which means that there is a unique supporting hyperplane at each point of the unit sphere S or, equivalently, the norm is Gâteaux differentiable on S—cf. [12, 13]). If X is an inner product space, the only semi-inner product on X is the inner product itself.

The *semi-orthogonality* of vectors x and y in X (with respect to a given semi-inner product) is defined as follows:

$$x\perp_{\mathrm{s}} y \quad\Longleftrightarrow\quad [y|x]=0.$$

Of course, in an inner product space we have $\perp_{\mathrm{B}} = \perp_{\mathrm{s}} = \perp$.

2. ρ-orthogonality. From now on we assume that the considered normed spaces are real and their dimensions are not less than 2. We define *norm derivatives*:

$$\rho'_{\pm}(x,y) := \lim_{t\to 0^{\pm}}\frac{\|x+ty\|^2-\|x\|^2}{2t} = \|x\|\lim_{t\to 0^{\pm}}\frac{\|x+ty\|-\|x\|}{t}\,,\qquad x,y\in X.$$

Convexity of the norm yields that the above definition is meaningful. The mappings ρ'_{+} and ρ'_{-} are sometimes called the *superior* and *inferior semi-inner products* and their following properties can be found, e.g., in [2, 3, 13].

- There is always $\rho'_{-}\leq\rho'_{+}$;
- $\rho'_{+}=\rho'_{-}$ if and only if X is smooth;
- for all $x,y\in X$ one has $\rho'_{\pm}(\alpha x,y)=\rho'_{\pm}(x,\alpha y)=\begin{cases}\alpha\rho'_{\pm}(x,y), & \alpha\geq 0,\\ \alpha\rho'_{\mp}(x,y), & \alpha\leq 0;\end{cases}$
- in particular, for all $x,y\in X$ one has $\rho'_{\pm}(-x,y)=\rho'_{\pm}(x,-y)=-\rho'_{\mp}(x,y)$ and $\rho'_{\pm}(x,0)=\rho'_{\pm}(0,y)=0$;
- the mappings $\rho'_{\pm}$ are continuous with respect to the second variable;

- for all $x, y \in X$, $\alpha \in \mathbb{R}$:
$$\rho'_{\pm}(x, \alpha x + y) = \alpha\|x\|^2 + \rho'_{\pm}(x, y); \tag{1}$$
- in particular, for all $x \in X$,
$$\rho'_{\pm}(x, x) = \|x\|^2;$$
- for all $x, y \in X$
$$|\rho'_{\pm}(x, y)| \leq \|x\|\,\|y\|;$$
- if $[\,\cdot\,|\,\cdot\,]$ is a given semi-inner product in X, then
$$\rho'_{\pm}(x, y) = \lim_{t\to 0^{\pm}} [y|x + ty]\,, \qquad x, y \in X,$$
and moreover, if X is smooth, then
$$\rho'_{\pm}(x, y) = [y|x]\,, \qquad x, y \in X;$$
- if $[\,\cdot\,|\,\cdot\,]$ is a given semi-inner product in X, then
$$\rho'_{-}(x, y) \leq [y|x] \leq \rho'_{+}(x, y), \qquad x, y \in X.$$

Let us define
$$\rho'(x, y) := \frac{1}{2}\left(\rho'_{+}(x, y) + \rho'_{-}(x, y)\right), \qquad x, y \in X.$$
(The above functional ρ' is also denoted by $\langle y|x\rangle_g$ and called an *M-semi-inner product*—cf. Miličić [19] and also [13].) As a consequence of (1) we have
$$\rho'(x, \alpha x + y) = \alpha\|x\|^2 + \rho'(x, y), \qquad x, y \in X,$$
and, in particular,
$$\rho'(x, x) = \|x\|^2, \qquad x \in X.$$
We have also
$$|\rho'(x, y)| \leq \|x\|\,\|y\|, \qquad x, y \in X.$$
Moreover, ρ' is continuous with respect to the second variable. A normed space X is called *semi-smooth* if ρ' is additive with respect to the second variable, i.e.,
$$\rho'(z, x + y) = \rho'(z, x) + \rho'(z, y), \qquad x, y, z \in X.$$
Each smooth space is semi-smooth in the above sense but not conversely (l^1 can serve as an example). If X is a real semi-smooth space, then $\langle\,\cdot\,|\diamond\rangle_g = \rho'(\diamond, \,\cdot\,)$ is a semi-inner product in the sense of Lumer–Giles.

Now we define orthogonality relations related to $\rho'_{\pm}$ (cf. [1, 2, 19]):
$$\begin{aligned} x\perp_{\rho_+} y &\iff \rho'_{+}(x, y) = 0;\\ x\perp_{\rho_-} y &\iff \rho'_{-}(x, y) = 0;\\ x\perp_{\rho} y &\iff \rho'(x, y) = 0. \end{aligned}$$
It is obvious that for a real inner product space all the above relations coincide with the standard orthogonality given by the inner product, i.e., $\perp_\rho = \perp_{\rho_+} = \perp_{\rho_-} = \perp$. It is proved in [1] that $(X, \perp_\rho)$ is an orthogonality space in the sense of Rätz.

The relations $\perp_{\rho_+}$, $\perp_{\rho_-}$ and $\perp_\rho$ are generally (unless X is smooth) incomparable.

THEOREM 2.1. *In a real normed space X the following conditions are equivalent*:

(a) $\perp_{\rho_+} \subset \perp_{\rho_-}$ (b) $\perp_{\rho_+} \supset \perp_{\rho_-}$ (c) $\perp_{\rho_+} = \perp_{\rho_-}$
(d) $\perp_{\rho_+} \subset \perp_{\rho}$ (e) $\perp_{\rho_+} \supset \perp_{\rho}$ (f) $\perp_{\rho_+} = \perp_{\rho}$
(g) $\perp_{\rho_-} \subset \perp_{\rho}$ (h) $\perp_{\rho_-} \supset \perp_{\rho}$ (i) $\perp_{\rho_-} = \perp_{\rho}$
(j) $\rho'_+ = \rho'_-$ (k) *X is smooth.*

This result was proved in [11, Theorem 1] and its generalization will be given in the subsequent section (Theorem 3.3).

In a real normed space X we have for arbitrary $x, y \in X$ and $\alpha \in \mathbb{R}$ (cf. [16, 3, 13]):

$$x \perp_{\mathrm{B}} (y - \alpha x) \iff \rho'_-(x, y) \le \alpha \|x\|^2 \le \rho'_+(x, y). \tag{2}$$

In particular, we have for arbitrary $x, y \in X$:

$$x \perp_{\mathrm{B}} y \iff \rho'_-(x, y) \le 0 \le \rho'_+(x, y). \tag{3}$$

In the next section we will generalize the above result (Theorem 3.1). It yields $\perp_\rho \subset \perp_{\mathrm{B}}$ and, if X is smooth, then $\perp_\rho = \perp_{\mathrm{B}}$. The next result (cf. [11, Theorem 2]) establishes the connection between $\perp_\rho$ and $\perp_{\mathrm{s}}$.

THEOREM 2.2. *Let X be a real normed space and let $[\cdot|\cdot]$ be a given semi-inner product in X. Then the following conditions are equivalent*:

(a) $\perp_\rho \subset \perp_{\mathrm{s}}$ (b) $\perp_\rho \supset \perp_{\mathrm{s}}$ (c) $\perp_\rho = \perp_{\mathrm{s}}$
(d) $\rho'(x, y) = [y|x], \quad x, y \in X.$

We will extend also this result in the next section (Theorem 3.5). Condition (d) yields that ρ' is additive with respect to the second variable, thus each of the (equivalent) conditions (a)–(d) implies semi-smoothness of X. For a non-semi-smooth norm, the orthogonalities $\perp_\rho$ and $\perp_{\mathrm{s}}$ are incomparable.

3. Approximate ρ-orthogonality.

In an inner product space an *approximate orthogonality* (ε-orthogonality, with $\varepsilon \in [0, 1)$) of vectors x and y is naturally defined by:

$$x \perp^\varepsilon y \iff |\langle x|y\rangle| \le \varepsilon \|x\| \, \|y\|.$$

In a normed space with a given semi-inner product $[\cdot|\cdot]$ one can define analogously:

$$x \perp^\varepsilon_{\mathrm{s}} y \iff |[y|x]| \le \varepsilon \|x\| \, \|y\|.$$

For an approximate Birkhoff orthogonality, we will follow the definition from [6]:

$$x \perp^\varepsilon_{\mathrm{B}} y \iff \forall\, \lambda \in \mathbb{K} \;\; \|x + \lambda y\|^2 \ge \|x\|^2 - 2\varepsilon \|x\| \|\lambda y\|.$$

The notions of an approximate $\rho_\pm$- and ρ-orthogonality were defined in [11] as follows.

$$x \perp^\varepsilon_{\rho_+} y \iff |\rho'_+(x, y)| \le \varepsilon \|x\| \, \|y\|;$$
$$x \perp^\varepsilon_{\rho_-} y \iff |\rho'_-(x, y)| \le \varepsilon \|x\| \, \|y\|;$$
$$x \perp^\varepsilon_{\rho} y \iff |\rho'(x, y)| \le \varepsilon \|x\| \, \|y\|.$$

It is easy to see that $x \perp^\varepsilon_{\rho_+} y \Rightarrow -x \perp^\varepsilon_{\rho_-} y$ and $x \perp^\varepsilon_{\rho_-} y \Rightarrow -x \perp^\varepsilon_{\rho_+} y$. Moreover, if $x \perp^\varepsilon_{\rho_+} y$ and $x \perp^\varepsilon_{\rho_-} y$, then $x \perp^\varepsilon_{\rho} y$. Obviously, if the norm in X comes from an inner product, then $\perp^\varepsilon_{\rho_+} = \perp^\varepsilon_{\rho_-} = \perp^\varepsilon_{\rho} = \perp^\varepsilon$ and for $\varepsilon = 0$ all the above approximate orthogonalities coincide with the related exact orthogonalities.

Now, we generalize (2) and (3).

THEOREM 3.1. *Let X be a real normed space and let $\varepsilon \in [0,1)$. Then, for arbitrary $x, y \in X$ and $\alpha \in \mathbb{R}$ we have*

$$x \perp_{\mathrm{B}}^{\varepsilon} (y - \alpha x) \iff \rho'_{-}(x,y) - \varepsilon\|x\|\,\|y-\alpha x\| \le \alpha\|x\|^2 \le \rho'_{+}(x,y) + \varepsilon\|x\|\,\|y-\alpha x\|.$$

In particular, we have for $x, y \in X$:

$$x \perp_{\mathrm{B}}^{\varepsilon} y \iff \rho'_{-}(x,y) - \varepsilon\|x\|\,\|y\| \le 0 \le \rho'_{+}(x,y) + \varepsilon\|x\|\,\|y\|. \tag{4}$$

Obviously, with $\varepsilon = 0$ one gets (2) and (3).

Proof. Assume $x \perp_{\mathrm{B}}^{\varepsilon} (y - \alpha x)$. Then, from the definition of $\perp_{\mathrm{B}}^{\varepsilon}$, for $\lambda > 0$ we have

$$-\varepsilon\|x\|\,\|y-\alpha x\| \le \frac{\|x+\lambda(y-\alpha x)\|^2 - \|x\|^2}{2\lambda}.$$

Letting $\lambda \to 0^+$ one gets $-\varepsilon\|x\|\,\|y-\alpha x\| \le \rho'_{+}(x, y-\alpha x) = \rho'_{+}(x,y) - \alpha\|x\|^2$. Similarly, for $\lambda < 0$ one gets $\rho'_{-}(x,y) - \alpha\|x\|^2 \le \varepsilon\|x\|\,\|y-\alpha x\|$.

Now we prove the converse. For $x = 0$ or $y = 0$ it is obvious, so assume that $x \ne 0 \ne y$. From the inequality $\rho'_{-}(x,y) - \alpha\|x\|^2 \le \varepsilon\|x\|\,\|y-\alpha x\|$ we get $\rho'_{-}(x, y-\alpha x) \le \varepsilon\|x\|\,\|y-\alpha x\|$ and

$$2\lim_{t\to 0^-} \frac{\|x+t(y-\alpha x)\|^2 - \|x\|^2}{2t} \le 2\varepsilon\|x\|\|y-\alpha x\|.$$

Fix $\gamma \in (0,1)$ to obtain

$$\lim_{t\to 0^-} \frac{\|x+t(y-\alpha x)\|^2 - \|x\|^2}{t} < 2(\varepsilon+\gamma)\|x\|\|y-\alpha x\|.$$

There exists $\delta_1 < 0$ such that

$$\forall\, t \in [\delta_1, 0) \quad \frac{\|x+t(y-\alpha x)\|^2 - \|x\|^2}{t} < 2(\varepsilon+\gamma)\|x\|\|y-\alpha x\|,$$

whence

$$\forall\, t \in [\delta_1, 0) \quad \|x\|^2 < \|x+t(y-\alpha x)\|^2 + 2(\varepsilon+\gamma)\|x\|\|t(y-\alpha x)\|. \tag{5}$$

From the inequality $-\varepsilon\|x\|\,\|y-\alpha x\| \le \rho'_{+}(x,y) - \alpha\|x\|^2 = \rho'_{+}(x, y-\alpha x)$ we get

$$-2\varepsilon\|x\|\|y-\alpha x\| \le 2\lim_{t\to 0^+} \frac{\|x+t(y-\alpha x)\|^2 - \|x\|^2}{2t}.$$

For the same γ as above we have

$$-2(\varepsilon+\gamma)\|x\|\|y-\alpha x\| < \lim_{t\to 0^+} \frac{\|x+t(y-\alpha x)\|^2 - \|x\|^2}{t},$$

whence there exists $\delta_2 > 0$ such that

$$\forall\, t \in (0, \delta_2] \quad -2(\varepsilon+\gamma)\|x\|\|y-\alpha x\| < \frac{\|x+t(y-\alpha x)\|^2 - \|x\|^2}{t}$$

and equivalently

$$\forall\, t \in (0, \delta_2] \quad \|x\|^2 < \|x+t(y-\alpha x)\|^2 + 2(\varepsilon+\gamma)\|x\|\|t(y-\alpha x)\|. \tag{6}$$

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) := \|x+t(y-\alpha x)\|^2 + 2(\varepsilon+\gamma)\|x\|\|t(y-\alpha x)\|$. The mapping φ is convex. To see this define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := \|x+t(y-\alpha x)\|$ and $g : [0,+\infty) \to \mathbb{R}$ by $g(t) := t^2$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) := 2(\varepsilon+\gamma)\|x\|\|t(y-\alpha x)\|$. Each of the functions f, g, h is convex and g is also nondecreasing. Thus $g \circ f$ is convex. Since $\varphi(t) = (g \circ f)(t) + h(t)$, and the sum of two convex mappings is convex, we get that φ is convex.

Inequalities (5) and (6) yield $\varphi(0) = \min\{\varphi(t) : t \in [\delta_1, \delta_2]\}$. Since φ is convex on $\mathbb{R}$, we have $\varphi(0) = \min\{\varphi(t) : t \in \mathbb{R}\}$. Thus

$$\|x\|^2 < \|x + t(y - \alpha x)\|^2 + 2(\varepsilon + \gamma)\|x\|\|t(y - \alpha x)\|, \qquad t \in \mathbb{R} \setminus \{0\}. \tag{7}$$

Fix an arbitrary $\lambda \neq 0$. From (7) we get

$$\|x\|^2 < \|x + \lambda(y - \alpha x)\|^2 + 2(\varepsilon + \gamma)\|x\|\|\lambda(y - \alpha x)\|. \tag{8}$$

Since γ was arbitrarily chosen from the interval $(0, 1)$, letting $\gamma \to 0^+$ in (8) we obtain

$$\|x\|^2 \leq \|x + \lambda(y - \alpha x)\|^2 + 2\varepsilon\|x\|\|\lambda(y - \alpha x)\|. \tag{9}$$

Obviously, (9) holds also for $\lambda = 0$, thus finally we get $x \perp^{\varepsilon}_{\mathrm{B}} (y - \alpha x)$. ∎

THEOREM 3.2. *For an arbitrary real normed space X and $\varepsilon \in [0, 1)$ we have*

$$\perp^{\varepsilon}_{\rho_+} \subset \perp^{\varepsilon}_{\mathrm{B}}; \tag{10}$$

$$\perp^{\varepsilon}_{\rho_-} \subset \perp^{\varepsilon}_{\mathrm{B}}; \tag{11}$$

$$\perp^{\varepsilon}_{\rho} \subset \perp^{\varepsilon}_{\mathrm{B}}; \tag{12}$$

$$\perp^{\varepsilon}_{\mathrm{s}} \subset \perp^{\varepsilon}_{\mathrm{B}}. \tag{13}$$

Moreover, if the norm is smooth, then

$$\perp^{\varepsilon}_{\rho} = \perp^{\varepsilon}_{\mathrm{s}} = \perp^{\varepsilon}_{\mathrm{B}}. \tag{14}$$

Proof. To prove (10) suppose $x \perp^{\varepsilon}_{\rho_+} y$. Using inequalities $-\varepsilon\|x\|\,\|y\| \leq \rho'_+(x, y) \leq \varepsilon\|x\|\,\|y\|$ and $\rho'_- \leq \rho'_+$, we get $\rho'_-(x, y) \leq \varepsilon\|x\|\,\|y\|$ and $-\varepsilon\|x\|\,\|y\| \leq \rho'_+(x, y)$. Then (4) yields $x \perp^{\varepsilon}_{\mathrm{B}} y$. Similarly one proves (11).

Inclusion (12) has been proved in [11, Theorem 3]; however, Theorem 3.1 enables us to present much simpler proof. Let $x \perp^{\varepsilon}_{\rho} y$. From the inequalities

$$-\varepsilon\|x\|\,\|y\| \leq \frac{1}{2}\left(\rho'_-(x, y) + \rho'_+(x, y)\right) \leq \varepsilon\|x\|\,\|y\| \quad \text{and} \quad \rho'_- \leq \rho'_+$$

we get

$$\frac{1}{2}\left(\rho'_-(x, y) + \rho'_-(x, y)\right) \leq \varepsilon\|x\|\,\|y\| \quad \text{and} \quad -\varepsilon\|x\|\,\|y\| \leq \frac{1}{2}\left(\rho'_+(x, y) + \rho'_+(x, y)\right).$$

An appeal to (4) verifies $x \perp^{\varepsilon}_{\mathrm{B}} y$.

Inclusion (13) has been proved in [6] whereas (14) in [6] and [11]. ∎

Generally, equalities (14) need not to hold—examples are provided in [6, Example 3.1] and [11, Example 5].

It is known (cf. [2, 13]) that the equality $\perp_{\rho} = \perp_{\mathrm{B}}$ yields the smoothness of the norm. In [11] the authors asked whether the smoothness of X resulted also from $\perp^{\varepsilon}_{\rho} = \perp^{\varepsilon}_{\mathrm{B}}$ with some $\varepsilon \in (0, 1)$. We will answer this question in Theorem 3.4. But first we need to prove the following generalization of Theorem 2.1.

THEOREM 3.3. *Let X be a real normed space and let $\varepsilon \in [0, 1)$. Then the following conditions are equivalent*:

(a) $\perp^{\varepsilon}_{\rho_+} \subset \perp^{\varepsilon}_{\rho_-}$	(b) $\perp^{\varepsilon}_{\rho_+} \supset \perp^{\varepsilon}_{\rho_-}$	(c) $\perp^{\varepsilon}_{\rho_+} = \perp^{\varepsilon}_{\rho_-}$
(d) $\perp^{\varepsilon}_{\rho_+} \subset \perp^{\varepsilon}_{\rho}$	(e) $\perp^{\varepsilon}_{\rho_+} \supset \perp^{\varepsilon}_{\rho}$	(f) $\perp^{\varepsilon}_{\rho_+} = \perp^{\varepsilon}_{\rho}$
(g) $\perp^{\varepsilon}_{\rho_-} \subset \perp^{\varepsilon}_{\rho}$	(h) $\perp^{\varepsilon}_{\rho_-} \supset \perp^{\varepsilon}_{\rho}$	(i) $\perp^{\varepsilon}_{\rho_-} = \perp^{\varepsilon}_{\rho}$
(j) $\rho'_+ = \rho'_-$	(k) *X is smooth.*	

Obviously, taking $\varepsilon = 0$ we get Theorem 2.1 as a particular case.

Proof. We start with proving (a)$\Rightarrow$(k). Let $[\cdot|\cdot]_1, [\cdot|\cdot]_2 : X \times X \to \mathbb{R}$ be two semi-inner products in X. Fix $x \in X$ such that $\|x\| = 1$ and choose $B \subset X$ such that $\{x\} \cup B$ forms a Hamel basis of X. Without loss of generality, we assume that $\|b\| = 1$ for all $b \in B$.

Now, fix $b \in B$. Suppose that $\rho'_+(x, b) > -\varepsilon$. Then we define a mapping $\varphi : [0,1] \to \mathbb{R}$ by $\varphi(t) := \rho'_+\big(x, \frac{t(-x)+(1-t)b}{\|t(-x)+(1-t)b\|}\big)$. Since φ is continuous, $\varphi(0) = \rho'_+(x,b) > -\varepsilon$ and $\varphi(1) = -1$, it follows that there exists $t_0 \in (0,1)$ such that $\varphi(t_0) = -\varepsilon$.

Since $\varphi(t_0) = -\varepsilon$, defining $\alpha := \frac{t_0}{\|t_0(-x)+(1-t_0)b\|}$, $\beta := \frac{(1-t_0)}{\|t_0(-x)+(1-t_0)b\|}$ and $y := \alpha(-x) + \beta b$ we get $\|y\| = 1$ and $\rho'_+(x,y) = -\varepsilon$, thus $x \perp^{\varepsilon}_{\rho_+} y$.

(In case $\rho'_+(x,b) < -\varepsilon$, we define $\psi : [0,1] \to \mathbb{R}$ by $\psi(t) := \rho'_+\big(x, \frac{tx+(1-t)b}{\|tx+(1-t)b\|}\big)$ and show that with some $t_0 \in (0,1)$, $\psi(t_0) = -\varepsilon$. Then we define $\alpha := \frac{t_0}{\|t_0x+(1-t_0)b\|}$, $\beta := \frac{(1-t_0)}{\|t_0x+(1-t_0)b\|}$ and $y := \alpha x + \beta b$ to obtain $x \perp^{\varepsilon}_{\rho_+} y$. In case $\rho'_+(x,b) = -\varepsilon$ we may take $t_0 := 0$ and $y := b$.)

Using (a), we get $x \perp^{\varepsilon}_{\rho_-} y$, which means $-\varepsilon \le \rho'_-(x,y) \le \varepsilon$. These inequalities, together with $\rho'_- \le \rho'_+$, yield $\rho'_-(x,y) = -\varepsilon$.

Define now $x_1^*, x_2^* \in X^*$, by $x_1^*(\cdot) := [\cdot|x]_1$, $x_2^*(\cdot) := [\cdot|x]_2$. For an arbitrary semi-inner product we have $\rho'_-(\cdot,\diamond) \le [\diamond|\cdot] \le \rho'_+(\cdot,\diamond)$, thus $[y|x]_1 = [y|x]_2$, because of $\rho'_-(x,y) = \rho'_+(x,y)$. Hence $x_1^*(y) = x_2^*(y)$, i.e., $x_1^*(\alpha(-x)+\beta b) = x_2^*(\alpha(-x)+\beta b)$. Since $\alpha, \beta > 0$ and $x_1^*(-x) = -1 = x_2^*(-x)$, there is $x_1^*(b) = x_2^*(b)$. We have shown that x_1^* and x_2^* coincides on the basis, thus they are equal: $x_1^* = x_2^*$. That means $[\cdot|x]_1 = [\cdot|x]_2$. Since x was arbitrarily chosen from the unit sphere, $[u|v]_1 = \big[u\big|\frac{v}{\|v\|}\big]_1\|v\| = \big[u\big|\frac{v}{\|v\|}\big]_2\|v\| = [u|v]_2$, i.e., $[\cdot|\cdot]_1 = [\cdot|\cdot]_2$. We have proved that each two semi-inner products are equal, i.e., there is only one semi-inner product on X. Thus X is smooth.

The proof of (b)$\Rightarrow$(k) runs similarly and we may consider (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(j)$\Leftrightarrow$(k) as shown (some of the implications are trivial).

Implication (d)$\Rightarrow$(k) can be shown similarly. In the first part of the above proof one should consider ρ' instead of ρ'_-. Then one applies inequality $\rho' \le \rho'_+$ instead of $\rho'_- \le \rho'_+$. On showing $\rho'(x,y) = -\varepsilon = \rho'_+(x,y)$, one derives $\rho'_-(x,y) = -\varepsilon = \rho'_+(x,y)$, since $\rho'(x,y) = \frac{1}{2}\big(\rho'_-(x,y) + \rho'_+(x,y)\big)$. The second part is the same. Thus we get (d)$\Leftrightarrow$(e)$\Leftrightarrow$(f)$\Leftrightarrow$(j)$\Leftrightarrow$(k) and similarly (g)$\Leftrightarrow$(h)$\Leftrightarrow$(i)$\Leftrightarrow$(j)$\Leftrightarrow$(k). ∎

Now, we can answer the question from [11] and give a sufficient condition for the smoothness of the space X.

THEOREM 3.4. *Let X be a real normed space and let $\varepsilon \in [0,1)$. If $\perp^{\varepsilon}_{\mathrm{B}} \subset \perp^{\varepsilon}_{\rho_+}$ or $\perp^{\varepsilon}_{\mathrm{B}} \subset \perp^{\varepsilon}_{\rho_-}$ or $\perp^{\varepsilon}_{\mathrm{B}} \subset \perp^{\varepsilon}_{\rho}$, then X is smooth.*

Proof. Since the proofs are similar we present only one. Assume that $\perp^{\varepsilon}_{\mathrm{B}} \subset \perp^{\varepsilon}_{\rho}$. From Theorem 3.2 we have $\perp^{\varepsilon}_{\rho_+} \subset \perp^{\varepsilon}_{\mathrm{B}}$. Thus $\perp^{\varepsilon}_{\rho_+} \subset \perp^{\varepsilon}_{\rho}$ and it follows from Theorem 3.3 ((d)$\Leftrightarrow$(k)) that X is smooth. ∎

Now, we are ready to present a generalization of Theorem 2.2.

Theorem 3.5. *Let X be a real normed space and let $[\,\cdot\,|\,\cdot\,]$ be a fixed semi-inner product in X. Then the following conditions are equivalent*:

(a) $\perp^{\varepsilon}_{\rho} \subset \perp^{\varepsilon}_{s}$ (b) $\perp^{\varepsilon}_{\rho} \supset \perp^{\varepsilon}_{s}$ (c) $\perp^{\varepsilon}_{\rho} = \perp^{\varepsilon}_{s}$

(d) $\rho'(x,y) = [y|x]\,, \quad x,y \in X.$

Moreover, each of the above conditions yields semi-smoothness of X.

Proof. For the proof of (a)⇒(d) fix arbitrarily two linearly independent vectors $x, y \in X$ such that $\|x\| = \|y\| = 1$. In a similar way as in the proof of Theorem 3.3 we obtain a vector $z \in \operatorname{lin}\{x,y\}$ such that $\|z\| = 1$ and $\rho'(x,z) = -\varepsilon$ (hence x and z are linearly independent). Now, we define mappings $\mu, \gamma : [0,1] \to \mathbb{R}$ as follows:

$$\mu(t) := \rho'\Big(x, \frac{(1-t)z+tx}{\|(1-t)z+tx\|}\Big); \qquad \gamma(t) := \left[\frac{(1-t)z+tx}{\|(1-t)z+tx\|}\,\middle|\, x\right].$$

Both mappings are continuous. Moreover, $\mu(0) = -\varepsilon$ and $\mu(1) = 1$, whence there exists $t_1 \in (0,1)$ such that $\mu(t_1) = \varepsilon$. Since we assume $\perp^{\varepsilon}_{\rho} \subset \perp^{\varepsilon}_{s}$, we have

$$\mu(0) \le \gamma(0) \quad \text{and} \quad \mu(t_1) \ge \gamma(t_1).$$

Using the Darboux property again we get for some $t_0 \in [0,t_1]$: $\mu(t_0) = \gamma(t_0)$. Thus for the vector

$$w := \frac{(1-t_0)z+t_0x}{\|(1-t_0)z+t_0x\|}$$

we have $\rho'(x,w) = [w|x]$. Notice that $w \in \operatorname{lin}\{x,y\}$, $w \ne x$, $w \ne -x$ and $\|w\| = 1$. Thus x, w are linearly independent, whence for some $\alpha, \beta \in \mathbb{R}$ we have $y = \alpha x + \beta w$. Applying properties of functionals ρ' and $[\,\cdot\,|\,\cdot\,]$ we get

$$\begin{aligned}\rho'(x,y) = \rho'(x,\alpha x+\beta w) = \alpha\rho'(x,x) + \beta\rho'(x,w) &= \alpha\,[x|x] + \beta\,[w|x] \\ &= [\alpha x+\beta w|x] = [y|x]\,.\end{aligned}$$

We have shown that functionals ρ' and $[\,\cdot\,|\,\cdot\,]$ are equal on unit, linearly independent vectors x, y. Now, let x, y be arbitrary vectors. If $y = \alpha x$, then $\rho'(x,y) = \alpha\|x\|^2 = [y|x]$. Suppose that x, y are linearly independent. Then we have

$$\rho'(x,y) = \|x\|\,\|y\|\rho'\Big(\frac{x}{\|x\|}, \frac{y}{\|y\|}\Big) = \|x\|\,\|y\|\left[\frac{y}{\|y\|}\,\middle|\,\frac{x}{\|x\|}\right] = [y|x]\,.$$

Hence $\rho' = [\,\cdot\,|\,\cdot\,]$. In a similar way one can prove (b)⇒(d). The rest is clear. ∎

Remark 3.6. The semi-smoothness implies none of the conditions (a)–(d). To see this consider a semi-smooth but not smooth space X. Then one can find two different semi-inner products. Thus at least one of them differs from ρ'.

4. Orthogonality preserving mappings. It is not difficult to prove that a linear mapping $f : X \to Y$ between inner product spaces which preserves orthogonality ($x \perp y \Rightarrow fx \perp fy$ for all $x, y \in X$) has to be a similarity (scalar multiple of an isometry); cf. e.g. [7]. It is much harder to see that the same is true for linear mappings between normed spaces, preserving the Birkhoff orthogonality, i.e., satisfying

$$x \perp_{\mathrm{B}} y \implies fx \perp_{\mathrm{B}} fy, \qquad x,y \in X.$$

For real spaces it has been proved by Koldobsky [17]; a proof including both real and complex spaces has been given by Blanco & Turnšek [5]. The same assertion can be also derived for linear mappings preserving a semi-orthogonality, i.e., satisfying

$$x \perp_s y \implies fx \perp_s fy, \qquad x, y \in X$$

with respect to some semi-inner product in X (cf. [5, Remark 3.2]).

4.1. ρ-orthogonality preserving mappings. Let us consider now linear mappings $f : X \to Y$ (between normed spaces X and Y) that preserve the $\rho(\rho_\pm)$-orthogonalities:

$$\begin{aligned} x \perp_{\rho_+} y &\implies fx \perp_{\rho_+} fy, & x, y \in X; \\ x \perp_{\rho_-} y &\implies fx \perp_{\rho_-} fy, & x, y \in X; \\ x \perp_{\rho} y &\implies fx \perp_{\rho} fy, & x, y \in X. \end{aligned}$$

The following characterization of $\rho_\pm$-orthogonality preserving mappings is in our disposal.

THEOREM 4.1. *Let X, Y be real normed spaces, $f : X \to Y$ a nonzero, linear mapping. Then, the following conditions are equivalent*:

(a) *f preserves ρ_+-orthogonality*;
(b) *f preserves ρ_--orthogonality*;
(c) *f preserves ρ-orthogonality*;
(d) $\|fx\| = \|f\| \, \|x\|, \quad x \in X$;
(e) $\rho'_+(fx, fy) = \|f\|^2 \rho'_+(x, y), \quad x, y \in X$;
(f) $\rho'_-(fx, fy) = \|f\|^2 \rho'_-(x, y), \quad x, y \in X$;
(g) $\rho'(fx, fy) = \|f\|^2 \rho'(x, y), \quad x, y \in X.$

The proof of this result was given in [11] and [23]. In [11, Theorem 5] the authors proved that (a), (b), (d), (e), (f), (g) are mutually equivalent and each of them implies (c). The lacking link was given in [23, Theorems 4.2, 4.3]. Thus in particular, linear mappings preserving ρ_+, ρ_-, ρ-orthogonality are similarities.

4.2. Equivalent norms and ρ-orthogonal sets. Now, we are going to present some new applications of the above theorem, which generalize some results from [2].

For the whole present subsection we assume that X is a real linear space endowed with two norms $\|\cdot\|_1$, $\|\cdot\|_2$, which generate respective functionals $(\rho'_\pm)_{1,2}$. We will consider only functionals $(\rho'_+)_{1,2}$ and to shorten the notation we will write $\rho_1 := (\rho'_+)_1$ and $\rho_2 := (\rho'_+)_2$. Following [2, Definition 2.4.1], we say that functionals ρ_1 and ρ_2 are *equivalent* if there exist constants $0 < A \le B$ such that

$$A\,|\rho_1(x, y)| \le |\rho_2(x, y)| \le B\,|\rho_1(x, y)|, \qquad x, y \in X. \tag{15}$$

The above defined equivalency can be characterized as follows (cf. [2, Theorem 2.4.2]).

THEOREM 4.2. *The functionals ρ_1 and ρ_2 are equivalent if and only if the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent and*

$$\frac{\rho_1(x, y)}{\|x\|_1^2} = \frac{\rho_2(x, y)}{\|x\|_2^2}, \qquad x, y \in X, \ x \ne 0.$$

In two theorems below, we generalize the above result. The "only if" part of Theorem 4.2 can be strengthen as follows.

THEOREM 4.3. *If there exists a constant $B > 0$ such that*

$$|\rho_2(x, y)| \le B\,|\rho_1(x, y)|, \qquad x, y \in X, \tag{16}$$

then there exists $\gamma > 0$ *such that*

$$\|x\|_2 = \gamma\|x\|_1 \quad \text{and} \quad \rho_2(x,y) = \gamma^2\rho_1(x,y), \qquad x,y \in X.$$

Proof. Define a linear mapping $h : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ by $h(x) := x$. It follows from (16) that h preserves ρ_+-orthogonality. According to Theorem 4.1, h is a similarity. Now it is enough to take $\gamma := \|h\|$ to obtain the assertion. ■

The "if" part of Theorem 4.2 can also be generalized.

THEOREM 4.4. *If*

$$\frac{\rho_1(x,y)}{\|x\|_1^2} = \frac{\rho_2(x,y)}{\|x\|_2^2}, \qquad x,y \in X,\ x \neq 0, \tag{17}$$

then the norms $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent, the functionals* ρ_1, ρ_2 *are equivalent and there exists* $\gamma > 0$ *such that*

$$\|x\|_2 = \gamma\|x\|_1 \quad \text{and} \quad \rho_2(x,y) = \gamma^2\rho_1(x,y), \qquad x,y \in X.$$

Proof. Define linear mapping $h : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ by $h(x) := x$. It follows from (17) that h preserves ρ_+-orthogonality, whence (applying again Theorem 4.1) h is a similarity. Now the assertion follows with $\gamma := \|h\|$. ■

Bearing in mind the above proofs, we notice that the inequalities in (15) cannot be sharp. Indeed, repeating the above reasonings and applying Theorem 4.1, one gets:

THEOREM 4.5. *The following conditions are equivalent*:

(i) $\frac{\rho_1(x,y)}{\|x\|_1^2} = \frac{\rho_2(x,y)}{\|x\|_2^2}$, $x,y \in X$, $x \neq 0$;
(ii) *there exist* $A, B > 0$ *such that* $A\,|\rho_1(x,y)| \leq |\rho_2(x,y)| \leq B\,|\rho_1(x,y)|$, $x,y \in X$;
(iii) *there exists* $B > 0$ *such that* $|\rho_2(x,y)| \leq B\,|\rho_1(x,y)|$, $x,y \in X$;
(iv) *there exists* $A > 0$ *such that* $A\,|\rho_1(x,y)| \leq |\rho_2(x,y)|$, $x,y \in X$;
(v) *there exist* $A, B \in \mathbb{R}$ *such that* $A\,\rho_1(x,y) \leq \rho_2(x,y) \leq B\,\rho_1(x,y)$, $x,y \in X$;
(vi) *there exists* $B > 0$ *such that* $\rho_2(x,y) = B\,\rho_1(x,y)$, $x,y \in X$;
(vii) *there exists* $B > 0$ *such that* $|\rho_2(x,y)| = B\,|\rho_1(x,y)|$, $x,y \in X$;
(viii) *there exists* $\gamma > 0$ *such that* $\|x\|_2 = \gamma\,\|x\|_1$, $x,y \in X$.

In particular, from implication (ii)⇒(vi) we derive the following result.

COROLLARY 4.6. *If there exist* $0 < A \leq B$ *such that*

$$A\,|\rho_1(x,y)| \leq |\rho_2(x,y)| \leq B\,|\rho_1(x,y)|, \qquad x,y \in X,$$

then there exists $\eta \in [A,B]$ *such that*

$$\rho_2(x,y) = \eta\,\rho_1(x,y), \qquad x,y \in X.$$

Equivalency (ii)⇔(viii) can be interpreted as follows.

COROLLARY 4.7. *Functionals* ρ_1 *and* ρ_2 *are equivalent if and only if the spaces* $(X, \|\cdot\|_1)$ *and* $(X, \|\cdot\|_2)$ *are isometrically isomorphic.*

Let us adopt the notion of B and ρ_+-orthogonal sets from [2, pp. 39–40]:

$$[x]^B_{\|\cdot\|} := \{y \in X : x \perp_B y\}, \quad [x]^{\rho_+}_{\|\cdot\|} := \left\{y \in X : x \perp_{\rho_+} y\right\}.$$

Theorems 2.5.1 and 2.5.2 from [2] can be extended.

THEOREM 4.8. *Each of the conditions* (i)–(viii) *from Theorem* 4.5 *is equivalent to each of the two below*:

(A) $[x]^B_{\|\cdot\|_1} = [x]^B_{\|\cdot\|_2}$ *for all* $x \in X$;
(B) $[x]^{\rho+}_{\|\cdot\|_1} = [x]^{\rho+}_{\|\cdot\|_2}$ *for all* $x \in X$.

Proof. It is visible that (viii) implies both (A) and (B). Define $h : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ by $h(x) = x$. Assuming (A) we get that h preserves Birkhoff orthogonality, thus it is a similarity (theorems of Koldobsky [17] and Blanco–Turnšek [5]), i.e., (viii) is true. Assuming (B) in turn, h preserves ρ_+-orthogonality and (viii) follows from Theorem 4.1. ∎

4.3. Another characterization of smooth spaces. A mapping $f : X \to Y$ between two normed spaces X and Y *changes orthogonality*, if it satisfies one of the following conditions (for all $x, y \in X$):

$$\begin{aligned} x\perp_{\rho_+} y &\implies fx\perp_{\rho_-} fy;\\ x\perp_{\rho_-} y &\implies fx\perp_{\rho_+} fy;\\ x\perp_{\rho_+} y &\implies fx\perp_{\rho} fy;\\ x\perp_{\rho} y &\implies fx\perp_{\rho_+} fy;\\ x\perp_{\rho_-} y &\implies fx\perp_{\rho} fy;\\ x\perp_{\rho} y &\implies fx\perp_{\rho_-} fy. \end{aligned}$$

Let us quote a result from [23, Theorem 5.1].

THEOREM 4.9. *A real normed space X is smooth if and only if there exists a normed space Y and a nonvanishing linear mapping $f : X \to Y$, such that f changes orthogonality.*

It follows that there is no mapping which essentially changes the orthogonality. If f changes orthogonality, then X is smooth, so $\perp_\rho = \perp_{\rho_+} = \perp_{\rho_-}$. Thus, in fact, f preserves ρ-orthogonality.

5. Approximately orthogonality preserving mappings. The class of linear mappings preserving orthogonality can be enlarged by admitting those mappings which only approximately preserve this relation. For inner product spaces X, Y one can consider linear mappings $f : X \to Y$ satisfying, for all $x, y \in X$,

$$x\perp y \implies fx\perp^{\varepsilon} fy.$$

For X, Y being normed spaces one can consider the condition

$$x\perp_{\mathrm{B}} y \implies fx\perp^{\varepsilon}_{\mathrm{B}} fy. \tag{18}$$

Similar classes of mappings can be considered for other orthogonality relations. The natural problems are: to describe such a class of approximately orthogonality preserving mappings (how far are they from similarities?) and to answer the stability question. By the latter we mean whether each approximately orthogonality preserving mapping has to be approximated by an orthogonality preserving one.

Investigations in this topic were carried out in inner product spaces [7, 8, 22], Hilbert C^*-modules [15] and in normed spaces with isosceles orthogonality [10] and Birkhoff orthogonality [20]. The respective results are extensively surveyed in [9].

As for ρ-orthogonality, one can consider the classes of linear mappings $f : X \to Y$ satisfying one of the following conditions (for all $x, y \in X$):

$$\begin{aligned} x\perp_\rho y &\implies fx\perp^\varepsilon_\rho fy; \\ x\perp_{\rho_+} y &\implies fx\perp^\varepsilon_{\rho_+} fy; \qquad (19)\\ x\perp_{\rho_-} y &\implies fx\perp^\varepsilon_{\rho_-} fy. \qquad (20)\end{aligned}$$

The latter two are equivalent. Indeed, suppose that f approximately preserves ρ_+-orthogonality and let $x\perp_{\rho_-} y$. Thus $-x\perp_{\rho_+} y$, hence

$$-fx\perp^\varepsilon_{\rho_+} fy$$

and finally

$$fx\perp^\varepsilon_{\rho_-} fy,$$

i.e., f approximately preserves ρ_--orthogonality. The proof of the reverse is the same.

We will prove that any linear mapping which approximately preserves ρ_+ (or ρ_-) orthogonality, also approximately preserves Birkhoff orthogonality.

Theorem 5.1. *Let X, Y be real normed spaces and let $f : X \to Y$ be linear and satisfy* (19) *or* (20)*. Then f satisfies* (18)*.*

Proof. Assuming (19) or (20) we have both of them. Fix $x, y \in X \setminus \{0\}$ and notice that, due to (1), we have

$$x\perp_{\rho_\pm}\Big(-\frac{\rho'_\pm(x,y)}{\|x\|^2}\,x+y\Big).$$

Therefore

$$fx\perp^\varepsilon_{\rho_\pm}\Big(-\frac{\rho'_\pm(x,y)}{\|x\|^2}\,fx+fy\Big),$$

and hence

$$\begin{aligned}\left|\rho'_\pm\Big(fx,\,-\frac{\rho'_\pm(x,y)}{\|x\|^2}\,fx+fy\Big)\right| &\le \varepsilon\,\|fx\|\,\Big\|-\frac{\rho'_\pm(x,y)}{\|x\|^2}\,fx+fy\Big\| \\ &\le \varepsilon\,\|fx\|\,\Big(\frac{|\rho'_\pm(x,y)|}{\|x\|^2}\,\|fx\|+\|fy\|\Big) \\ &= \varepsilon\,\|fx\|\,\|fy\|+\varepsilon\,\frac{\|fx\|^2}{\|x\|^2}\,|\rho'_\pm(x,y)|.\end{aligned}$$

Thus we have

$$\left|\rho'_\pm(fx,fy)-\frac{\|fx\|^2}{\|x\|^2}\,\rho'_\pm(x,y)\right|\le \varepsilon\,\|fx\|\,\|fy\|+\varepsilon\,\frac{\|fx\|^2}{\|x\|^2}\,|\rho'_\pm(x,y)|.$$

It follows from the above inequality that

$$\frac{\|fx\|^2}{\|x\|^2}\,\rho'_+(x,y)-\varepsilon\,\frac{\|fx\|^2}{\|x\|^2}\,|\rho'_+(x,y)|-\varepsilon\,\|fx\|\,\|fy\|\le\rho'_+(fx,fy) \qquad (21)$$

and

$$\rho'_-(fx,fy)\le\varepsilon\,\|fx\|\,\|fy\|+\varepsilon\,\frac{\|fx\|^2}{\|x\|^2}\,|\rho'_-(x,y)|+\frac{\|fx\|^2}{\|x\|^2}\,\rho'_-(x,y), \qquad (22)$$

for all $x, y \in X \setminus \{0\}$.

Now, fix $u, w \in X \setminus \{0\}$ and assume that $u \perp_{\mathrm{B}} w$. By (3) we have

$$\rho'_-(u,w) \leq 0 \leq \rho'_+(u,w)$$

and consequently $|\rho'_-(u,w)| = -\rho'_-(u,w)$ and $|\rho'_+(u,w)| = \rho'_+(u,w)$. From (21) we have

$$(1-\varepsilon)\,\frac{\|fu\|^2}{\|u\|^2}\,\rho'_+(u,w) - \varepsilon\|fu\|\,\|fw\| \leq \rho'_+(fu,fw).$$

Similarly, using (22) one gets

$$\rho'_-(fu,fw) \leq \varepsilon\|fu\|\,\|fw\| + (1-\varepsilon)\,\frac{\|fu\|^2}{\|u\|^2}\,\rho'_-(u,w).$$

Finally, we obtain

$$\rho'_-(fu,fw) \leq \varepsilon\|fu\|\,\|fw\| \quad \text{and} \quad -\varepsilon\|fu\|\,\|fw\| \leq \rho'_+(fu,fw)$$

and by (4) (Theorem 3.1),

$$fu \perp^{\varepsilon}_{\mathrm{B}} fw.$$

Thus f approximately preserves Birkhoff orthogonality. ∎

Turnšek & Mojškerc [20, Theorem 3.5, Remark 3.1] proved the following result.

THEOREM 5.2. *Let X, Y be real normed spaces, $\varepsilon \in [0, \frac{1}{8})$ and $T : X \to Y$ a linear mapping satisfying $x \perp_{\mathrm{B}} y \Rightarrow Tx \perp^{\varepsilon}_{\mathrm{B}} Ty$. Then*

$$(1-8\varepsilon)\,\|T\|\,\|x\| \leq \|Tx\| \leq \|T\|\,\|x\|, \qquad x \in X.$$

Combining this result with our Theorem 5.1 we obtain the following corollary.

THEOREM 5.3. *Let X, Y be real normed spaces and let $f : X \to Y$ be a nonzero linear mapping satisfying* (19) *or* (20) *with $\varepsilon \in [0, \frac{1}{8})$. Then f is injective, continuous and*

$$(1-8\varepsilon)\,\|f\|\,\|x\| \leq \|fx\| \leq \|f\|\,\|x\|, \qquad x \in X.$$

COROLLARY 5.4. *Let $\varepsilon \in [0, \frac{1}{8})$. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms in a linear space X. By $\perp_a$ and $\perp_b$ we denote the $\rho_\pm$-orthogonality with respect to one of the two norms. If $\perp_a \subset \perp^{\varepsilon}_b$, then both norms are equivalent and, with some $\gamma > 0$, we have*

$$(1-8\varepsilon)\gamma\|x\|_a \leq \|x\|_b \leq \gamma\|x\|_a, \qquad x \in X.$$

Proof. Define a linear mapping $h : (X, \|\cdot\|_a) \to (X, \|\cdot\|_b)$ by $hx := x$. Then h approximately preserves $\rho_\pm$-orthogonality and the assertion follows from Theorem 5.3. ∎

It is known from [20] that for the Birkhoff orthogonality the stability problem has an affirmative answer for some class of spaces (including finite-dimensional ones). Thus Theorem 5.3 extends these results to ρ_+ and ρ_- orthogonality. If we assume that the space X is smooth then we have also

$$\perp_{\rho_+} = \perp_{\rho_-} = \perp_{\rho} = \perp_{\mathrm{B}}$$

and

$$\perp^{\varepsilon}_{\rho_+} = \perp^{\varepsilon}_{\rho_-} = \perp^{\varepsilon}_{\rho} = \perp^{\varepsilon}_{\mathrm{B}}$$

for all $\varepsilon \in [0, 1)$. Thus, in this case, the stability problems for ρ, ρ_+, ρ_- and Birkhoff orthogonality preserving mappings are equivalent.

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