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ON AN ELEMENTARY INCLUSION PROBLEM AND GENERALIZED WEIGHTED QUASI-ARITHMETIC MEANS

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Dedicated to the 60th birthday of Professor László Székelyhidi

Abstract. The aim of this note is to characterize the real coefficients $p_1,\dots,p_n$ and $q_1,\dots,q_k$ so that

$$\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \in \operatorname{conv}\{x_1,\dots,x_n\}$$

be valid whenever the vectors $x_1,\dots,x_n$, $y_1,\dots,y_k$ satisfy

$$\{y_1,\dots,y_k\} \subseteq \operatorname{conv}\{x_1,\dots,x_n\}.$$

Using this characterization, a class of generalized weighted quasi-arithmetic means is introduced and several open problems are formulated.

1. Introduction. Given a nonempty convex subset D of a linear space X and $n,k\in\mathbb{N}$, define the set $D_{n,k}\subset D^{n+k}$ by

$$D_{n,k} := \big\{(x_1,\dots,x_n,y_1,\dots,y_k)\in D^{n+k} \mid \{y_1,\dots,y_k\}\subseteq \operatorname{conv}\{x_1,\dots,x_n\}\big\}.$$

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The aim of this paper is to characterize those real coefficients $p_1, \dots, p_n$ and $q_1, \dots, q_k$ for which the inclusion

$$\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \in \operatorname{conv}\{x_1, \dots, x_n\}$$

holds for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$.

After answering the above question, we introduce the notion of generalized weighted quasi-arithmetic means and we give the results for some its subclasses. We also formulate several open problems.

2. An elementary inclusion problem. We recall the notation for the positive and negative parts of real numbers defined by

$$q^+ := \begin{cases} q & \text{if } q \geq 0 \\ 0 & \text{if } q < 0, \end{cases} \qquad q^- := \begin{cases} 0 & \text{if } q > 0 \\ -q & \text{if } q \leq 0 \end{cases} \qquad (q \in \mathbb{R}).$$

It is obvious that $q = q^+ - q^-$ and $|q| = q^+ + q^-$ for all $q \in \mathbb{R}$.

For $n = 1$, the description of the set $D_{n,k}$ is trivial:

$$D_{1,k} = \{(x, y_1, \dots, y_k) \in D^{1+k} \mid y_1 = \ldots = y_k = x\} = \{(x, x, \dots, x) \mid x \in D\},$$

therefore, in the following theorem, which contains our main result, we consider only the case $n \geq 2$.

THEOREM 2.1. *Let X be a linear space, let $D \subseteq X$ be a convex set containing at least two distinct elements. Let $n, k \in \mathbb{N}$, $n \geq 2$ and $p_1, \dots, p_n, q_1, \dots, q_k \in \mathbb{R}$. Then the following statements are equivalent*:

(i) *The inclusion*

$$\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \in \operatorname{conv}\{x_1, \dots, x_n\} \tag{1}$$

holds for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$.

(ii) *The coefficients $p_1, \dots, p_n, q_1, \dots, q_k$ satisfy the conditions*

$$\sum_{i=1}^{n} p_i + \sum_{j=1}^{k} q_j = 1 \qquad \textit{and} \qquad \min\{p_1, \dots, p_n\} \geq \sum_{j=1}^{k} q_j^-. \tag{2}$$

(iii) *For all convex functions $f : X \to \mathbb{R}$, the inequality*

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\right) \leq \sum_{i=1}^{n} p_i f(x_i) + \sum_{j=1}^{k} q_j f(y_j) \tag{3}$$

holds for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$.

Proof. (i)$\Rightarrow$(ii). Assume that (1) is satisfied for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$. By the assumption, D contains two distinct elements, say $a, b \in D$ with $a \neq b$. We may also assume that $a \neq 0$. Then, taking $x_1 = \ldots = x_n = y_1 = \ldots = y_k = a$, we can see that $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$ is valid, hence, from (1), we obtain

$$\left(\sum_{i=1}^{n} p_i + \sum_{j=1}^{k} q_j\right) a = \sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \in \operatorname{conv}\{x_1, \dots, x_n\} = \{a\},$$

which yields the first equality in (2).

To prove the second inequality in (2), let $\ell \in \{1, \dots, n\}$ be a fixed index, let $x_\ell := a$ and $x_i := b$ for $i \in \{1, \dots, n\} \setminus \{\ell\}$ and define, for $j \in \{1, \dots, k\}$, $y_j := a$ if $q_j \le 0$ and $y_j := b$ if $q_j > 0$. Then

$$\{y_1, \dots, y_k\} \subseteq \{a, b\} \subseteq \operatorname{conv}\{a, b\} = \operatorname{conv}\{x_1, \dots, x_n\},$$

i.e., $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$ is valid and hence (1) holds. Observe that $q_j y_j = q_j^+ b - q_j^- a$, thus (1) and the first equality in (2) yield

$$\Big(p_\ell - \sum_{j=1}^k q_j^-\Big)a + \Big(1 - p_\ell + \sum_{j=1}^k q_j^-\Big)b = \Big(p_\ell - \sum_{j=1}^k q_j^-\Big)a + \Big(\sum_{i=1}^n p_i - p_\ell + \sum_{j=1}^k q_j^+\Big)b$$
$$= \sum_{i=1}^n p_i x_i + \sum_{j=1}^k q_j y_j \in \operatorname{conv}\{x_1, \dots, x_n\} = \operatorname{conv}\{a, b\} = \{ta + (1-t)b \mid t \in [0,1]\}.$$

Therefore, using $a \neq b$, for $\ell \in \{1, \dots, n\}$, we get

$$0 \le p_\ell - \sum_{j=1}^k q_j^- \le 1.$$

From the left hand side inequality here the second inequality of (2) follows.

(ii)$\Rightarrow$(i). Assume that the conditions in (2) are satisfied. Let $(x_1, \dots, x_n, y_1, \dots, y_k)$ be an element of $D_{n,k}$. Then, for $(i, j) \in \{1, \dots, n\} \times \{1, \dots, k\}$, there exist $\lambda_{ij} \in [0, 1]$ such that, for $j \in \{1, \dots, k\}$, we have

$$\sum_{i=1}^n \lambda_{ij} = 1 \qquad \text{and} \qquad \sum_{i=1}^n \lambda_{ij} x_i = y_j. \tag{4}$$

Thus,

$$\sum_{i=1}^n p_i x_i + \sum_{j=1}^k q_j y_j = \sum_{i=1}^n p_i x_i + \sum_{j=1}^k q_j \sum_{i=1}^n \lambda_{ij} x_i = \sum_{i=1}^n \Big(p_i + \sum_{j=1}^k q_j \lambda_{ij}\Big) x_i. \tag{5}$$

To prove the validity of (1), it suffices to show that the right hand side expression in (5) is a convex combination of $x_1, \dots, x_n$, i.e.,

$$\sum_{i=1}^n \Big(p_i + \sum_{j=1}^k q_j \lambda_{ij}\Big) = 1 \qquad \text{and} \qquad p_i + \sum_{j=1}^k q_j \lambda_{ij} \ge 0 \quad (i \in \{1, \dots, n\}). \tag{6}$$

Using the first equalities in (2) and (4), we get

$$\sum_{i=1}^n \Big(p_i + \sum_{j=1}^k q_j \lambda_{ij}\Big) = \sum_{i=1}^n p_i + \sum_{j=1}^k q_j \sum_{i=1}^n \lambda_{ij} = \sum_{i=1}^n p_i + \sum_{j=1}^k q_j = 1,$$

which proves the first equality in (6). On the other hand, by $\lambda_{ij} \in [0, 1]$ and the second inequality in (2), we get

$$p_i + \sum_{j=1}^k q_j \lambda_{ij} = p_i + \sum_{j=1}^k (q_j^+ - q_j^-)\lambda_{ij} \ge p_i - \sum_{j=1}^k q_j^- \ge 0.$$

(ii)$\Rightarrow$(iii). Assume that the conditions in (2) are satisfied. Let $f : X \to \mathbb{R}$ be a convex function and let $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$. Then, for $(i, j) \in \{1, \dots, n\} \times \{1, \dots, k\}$, there exist $\lambda_{ij} \in [0,1]$ such that, for $j \in \{1,\dots,k\}$, we have (4). By the convexity of f, we have

$$f(y_j) = f\Big(\sum_{i=1}^{n} \lambda_{ij} x_i\Big) \le \sum_{i=1}^{n} \lambda_{ij} f(x_i) \qquad (j \in \{1,\dots,k\}).$$

Thus

$$\sum_{j=1}^{k} q_j^- f(y_j) \le \sum_{j=1}^{k} q_j^- \sum_{i=1}^{n} \lambda_{ij} f(x_i) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{k} q_j^- \lambda_{ij}\Big) f(x_i). \tag{7}$$

On the other hand, (ii) and (4) yield that

$$\sum_{i=1}^{n} \Big(p_i - \sum_{j=1}^{k} q_j^- \lambda_{ij}\Big) + \sum_{j=1}^{k} q_j^+ = 1$$

$$p_i - \sum_{j=1}^{k} q_j^- \lambda_{ij} \ge 0 \quad (i \in \{1,\dots,n\}), \qquad \text{and} \qquad q_j^- \ge 0 \quad (j \in \{1,\dots,k\}),$$

i.e., these numbers form a system of convex combination coefficients. Hence, by the convexity of f,

$$\begin{aligned} f\Big(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\Big) &= f\Big(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j^+ y_j - \sum_{j=1}^{k} q_j^- y_j\Big) \\ = f\Big(\sum_{i=1}^{n}\Big(p_i - \sum_{j=1}^{k} q_j^- \lambda_{ij}\Big) x_i + \sum_{j=1}^{k} q_j^+ y_j\Big) &\le \sum_{i=1}^{n}\Big(p_i - \sum_{j=1}^{k} q_j^- \lambda_{ij}\Big) f(x_i) + \sum_{j=1}^{k} q_j^+ f(y_j). \end{aligned}$$

Adding inequality (7) to this inequality, we get

$$f\Big(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\Big) + \sum_{j=1}^{k} q_j^- f(y_j) \le \sum_{i=1}^{n} p_i f(x_i) + \sum_{j=1}^{k} q_j^+ f(y_j),$$

which is equivalent to (3).

(iii)⇒(i). Assume that there exists $(x_1,\dots,x_n,y_1,\dots,y_k) \in D_{n,k}$ such that (1) is not valid, i.e.,

$$\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \notin \operatorname{conv}\{x_1,\dots,x_n\}. \tag{8}$$

Then, by the standard separation theorem, there exists a linear function $\varphi : X \to \mathbb{R}$ such that

$$c := \sup\{\varphi(u) \mid u \in \operatorname{conv}\{x_1,\dots,x_n\}\} < \varphi\Big(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\Big).$$

Define $f : X \to \mathbb{R}$ by $f(x) := \max\{\varphi(x) - c, 0\} = (\varphi(x) - c)^+$. Then f is a convex function, $f(u) = 0$ for all $u \in \operatorname{conv}\{x_1,\dots,x_n\}$, and

$$f\Big(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\Big) = \varphi\Big(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\Big) - c > 0.$$

On the other hand, by (3),

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\right) \le \sum_{i=1}^{n} p_i f(x_i) + \sum_{j=1}^{k} q_j f(y_j) = 0.$$

The contradiction obtained proves that (8) cannot hold, i.e., (1) must be valid.

The proof of the theorem is complete. ∎

REMARK 2.2. The implication (ii)⇒(iii) is valid more generally: If (ii) holds then (3) is satisfied for all convex functions $f : D \to \mathbb{R}$ and for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in D_{n,k}$. Indeed, as it follows from the implication (ii)⇒(i), inclusion (1) is valid, hence, by the convexity of D,

$$\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \in D,$$

proving that the left hand side of (3) is well defined. Now, the argument followed in the proof of Theorem 2.1 yields that (3) is valid.

Based on Theorem 2.1, the set of those $(n + k)$-tuples $(p_1, \dots, p_n, q_1, \dots, q_k)$ that satisfy the two conditions of (2) will be denoted by $K_{n,k}$.

COROLLARY 2.3. *Let $I \subseteq \mathbb{R}$ be an interval containing at least two distinct elements. Let $n, k \in \mathbb{N}$, $n \ge 2$ and $p_1, \dots, p_n, q_1, \dots, q_k \in \mathbb{R}$. Then the following statements are equivalent:*

(i) *The inequality*

$$\min\{x_1, \dots, x_n\} \le \sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j \le \max\{x_1, \dots, x_n\} \tag{9}$$

holds for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in I_{n,k}$.

(ii) *The coefficients $p_1, \dots, p_n, q_1, \dots, q_k$ satisfy (2), i.e., $(p_1, \dots, p_n, q_1, \dots, q_k) \in K_{n,k}$.*

(iii) *For all convex functions $f : \mathbb{R} \to \mathbb{R}$, the inequality*

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{k} q_j y_j\right) \le \sum_{i=1}^{n} p_i f(x_i) + \sum_{j=1}^{k} q_j f(y_j) \tag{10}$$

holds for all $(x_1, \dots, x_n, y_1, \dots, y_k) \in I_{n,k}$.

Proof. Apply Theorem 2.1 in the particular case $X = \mathbb{R}$ and $D = I$, and observe that, for $x_1, \dots, x_n, y \in \mathbb{R}$, the inclusion $y \in \operatorname{conv}\{x_1, \dots, x_n\}$ holds if and only if the inequalities

$$\min\{x_1, \dots, x_n\} \le y \le \max\{x_1, \dots, x_n\}$$

are satisfied. ∎

Some of the consequences of the statement of Theorem 2.1 are important of their own. We consider the case when $k = 1$. Then the equality condition in (2) implies that $q_1 = 1 - p_1 - \ldots - p_n$.

COROLLARY 2.4. *Let X be a linear space, let $D \subseteq X$ be a convex set containing at least two distinct elements. Let $n \ge 2$ and $p_1, \dots, p_n \in \mathbb{R}$. Then the following statements are equivalent:*

(i) *The inclusion*

$$\sum_{i=1}^{n} p_i x_i + \Big(1 - \sum_{i=1}^{n} p_i\Big) y \in \operatorname{conv}\{x_1, \dots, x_n\}$$

holds for all $(x_1, \dots, x_n, y) \in D_{n,1}$.

(ii) *The coefficients* $p_1, \dots, p_n$ *satisfy the conditions*

$$\min\{p_1, \dots, p_n\} \geq 0 \qquad \text{and} \qquad \min\{p_1, \dots, p_n\} + 1 \geq \sum_{i=1}^{n} p_i. \tag{11}$$

(iii) *For all convex functions* $f : X \to \mathbb{R}$, *the inequality*

$$f\Big(\sum_{i=1}^{n} p_i x_i + \Big(1 - \sum_{i=1}^{n} p_i\Big) y\Big) \leq \sum_{i=1}^{n} p_i f(x_i) + \Big(1 - \sum_{i=1}^{n} p_i\Big) f(y)$$

holds for all $(x_1, \dots, x_n, y) \in D_{n,1}$.

Proof. Apply Theorem 2.1 in the case $k = 1$ for the coefficients $p_1, \dots, p_n$ and $q_1 := 1 - p_1 - \dots - p_n$. Then (2) holds, i.e., $(p_1, \dots, p_n, 1 - p_1 - \dots - p_n) \in K_{n,1}$ is valid if and only if

$$\min\{p_1, \dots, p_n\} \geq \Big(1 - \sum_{i=1}^{n} p_i\Big)^{-} = \max\Big(0, \sum_{i=1}^{n} p_i - 1\Big).$$

This condition is trivially equivalent to (11). ∎

Remark 2.5. In the particular case $n = 2$ the conditions of (11) are easily seen to be equivalent to $p_1, p_2 \in [0,1]^2$. Thus, with $p_1 := p_2 := 1$, we have that, for all $x_1, x_2 \in D$, $y \in \operatorname{conv}\{x_1, x_2\}$ and for all convex functions $f : D \to \mathbb{R}$,

$$x_1 + x_2 - y \in \operatorname{conv}\{x_1, x_2\} \qquad \text{and} \qquad f(x_1 + x_2 - y) \leq f(x_1) + f(x_2) - f(y).$$

3. Generalized weighted quasi-arithmetic means. Given a nonvoid open interval $I \subset \mathbb{R}$, a function $M : I^n \to \mathbb{R}$ is said to be an n-variable mean on I if

$$\min\{x_1, \dots, x_n\} \leq M(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\}$$

holds for all $(x_1, \dots, x_n) \in I^n$.

To recall the notion of n-variable weighted quasi-arithmetic means, denote by $\mathcal{CM}(I)$ the class of continuous and strictly monotone real valued functions defined on the interval I. For $\varphi \in \mathcal{CM}(I)$ and $(p_1, \dots, p_n) \in [0,1]$ with $p_1 + \dots + p_n = 1$, define $\mathcal{A}_{\varphi}^{(p_1,\dots,p_n)} : I^n \to \mathbb{R}$ by

$$\mathcal{A}_{\varphi}^{(p_1,\dots,p_n)}(x_1, \dots, x_n) := \varphi^{-1}\big(p_1 \varphi(x_1) + \dots + p_n \varphi(x_n)\big). \tag{12}$$

The following result enables us to construct a more general class of means.

Theorem 3.1. *Let* $n \geq 2$ *and* $k \geq 1$ *be natural numbers,* $(p_1, \dots, p_n, q_1, \dots, q_k) \in K_{n,k}$, $\varphi \in \mathcal{CM}(I)$ *and* $M_1, \dots, M_k : I^n \to \mathbb{R}$ *be* n*-variable means. Then the function* $\mathcal{A}_{\varphi, M_1, \dots, M_k}^{(p_1,\dots,p_n,q_1,\dots,q_k)} : I^n \to \mathbb{R}$ *defined by*

$$\mathcal{A}_{\varphi, M_1, \dots, M_k}^{(p_1,\dots,p_n,q_1,\dots,q_k)}(x_1, \dots, x_n) := \varphi^{-1}\Big(\sum_{i=1}^{n} p_i \varphi(x_i) + \sum_{j=1}^{k} q_j \varphi(M_j(x_1, \dots, x_n))\Big) \tag{13}$$

is an n*-variable mean on* I.

Proof. By the monotonicity of φ and by the mean value property of the means $M_1, \dots, M_n$, for all $(x_1, \dots, x_n) \in I^n$ and $j \in \{1, \dots, k\}$, we have

$$\min\{\varphi(x_1), \dots, \varphi(x_n)\} \le \varphi(M_j(x_1, \dots, x_n)) \le \max\{\varphi(x_1), \dots, \varphi(x_n)\}. \tag{14}$$

In other words,

$$\big(\varphi(x_1), \dots, \varphi(x_n), \varphi(M_1(x_1, \dots, x_n)), \dots, \varphi(M_k(x_1, \dots, x_n))\big) \in (\varphi(I))_{n,k}. \tag{15}$$

Hence, by Corollary 2.3, we get

$$\min\{\varphi(x_1), \dots, \varphi(x_n)\} \le \sum_{i=1}^{n} p_i \varphi(x_i) + \sum_{l=1}^{k} q_j \varphi(M_j(x_1, x_2, \dots, x_n)) \\ \le \max\{\varphi(x_1), \dots, \varphi(x_n)\}.$$

Using the monotonicity of φ^{-1}, we obtain that

$$\varphi^{-1}\Big(\sum_{i=1}^{n} p_i \varphi(x_i) + \sum_{j=1}^{k} q_l \varphi(M_j(x_1, \dots, x_n))\Big)$$

is between $\min\{x_1, \dots, x_n\}$ and $\max\{x_1, \dots, x_n\}$, which proves that $\mathcal{A}_{\varphi, M_1, \dots, M_k}^{(p_1, \dots, p_n, q_1, \dots, q_k)}$ is an n-variable mean. ∎

We call this mean the *generalized weighted quasi-arithmetic* mean generated by the function φ, the means $M_1, \dots, M_k$, and the weights $(p_1, \dots, p_n, q_1, \dots, q_k) \in K_{n,k}$. In the particular case $q_1 = \ldots = q_k = 0$, the generalized weighted quasi-arithmetic mean in (5) reduces to the weighted quasi-arithmetic mean in (4).

Concerning these means, we may consider the following three basic problems:

(i) Equality problem;
(ii) Comparison problem;
(iii) Matkowski–Sutô type problem.

In the following subsections we briefly describe some basic results as well as some open problems related to these questions.

(i) Equality Problem. Given the n variable means $M_1, \dots, M_k : I^n \to \mathbb{R}$, characterize those weights $(p_1, \dots, p_n, q_1, \dots, q_k), (r_1, \dots, r_n, s_1, \dots, s_k) \in K_{n,k}$ and pairs of functions $\varphi, \psi \in \mathcal{CM}(I)$ such that, for all $x_1, \dots, x_n \in I$,

$$\mathcal{A}_{\varphi, M_1, \dots, M_k}^{(p_1, \dots, p_n, q_1, \dots, q_k)}(x_1, \dots, x_n) = \mathcal{A}_{\psi, M_1, \dots, M_k}^{(r_1, \dots, r_n, s_1, \dots, s_k)}(x_1, \dots, x_n). \tag{16}$$

For this problem, we have the following sufficient condition.

THEOREM 3.2. *Let $n \ge 2$ and $k \ge 1$ be natural numbers, $(p_1, \dots, p_n, q_1, \dots, q_k) \in K_{n,k}$, $\varphi, \psi \in \mathcal{CM}(I)$ and let $M_1, \dots, M_k : I^n \to I$ be n-variable means. Assume that there exist $a, b \in \mathbb{R}$, $a \ne 0$, such that*

$$\psi(x) = a\varphi(x) + b \qquad (x \in I). \tag{17}$$

Then, for all $x_1, \dots, x_n \in I$,

$$\mathcal{A}_{\varphi, M_1, \dots, M_k}^{(p_1, \dots, p_n, q_1, \dots, q_k)}(x_1, \dots, x_n) = \mathcal{A}_{\psi, M_1, \dots, M_k}^{(p_1, \dots, p_n, q_1, \dots, q_k)}(x_1, \dots, x_n). \tag{18}$$

Proof. It follows from (17) that $\psi^{-1}(t) = \varphi^{-1}\big(\frac{t-b}{a}\big)$ for $t \in \psi(I)$. This, and the definition (13) of the means, immediately results (18). ∎

Open Problem 3.3. Under the assumption of Theorem 3.2, is the equality (17) necessary for the validity of (18)?

The answer to this question is affirmative in several particular cases.

— In the case when $q_1 = \ldots = q_n = 0$, the equality problem (18) reduces to the equality problem of weighted quasi-arithmetic means when the necessity of (17) is known to be valid (cf. [14], [17], [22], [24]).

— In the case $n = 2$, $k = 1$, the necessity of (17) was obtained in [8], where the following result was proved: *Let $M : I^2 \to \mathbb{R}$ be a strict and continuous two variable mean, $p, q \in]0, 1]$, $\varphi, \psi \in \mathcal{CM}(I)$. Then, in order that $\mathcal{A}^{(p,q,1-p-q)}_{\varphi,M} = \mathcal{A}^{(p,q,1-p-q)}_{\psi,M}$, that is, the functional equation*

$$\varphi^{-1}\big(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi(M(x,y))\big) = \psi^{-1}\big(p\psi(x) + q\psi(y) + (1-p-q)\psi(M(x,y))\big)$$

be satisfied for all $x, y \in I$, it is necessary and sufficient that (17) *be valid for some $a, b \in \mathbb{R}$ with $a \neq 0$.*

The more general equality problem (16) has also been solved in other particular cases. In [5] the case of the equality problem $\mathcal{A}^{(p,q,1-p-q)}_{\varphi,M} = \mathcal{A}^{(r,1-r,0)}_{\psi,M}$, that is, the functional equation

$$\varphi^{-1}\big(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi(M(x,y))\big) = \psi^{-1}\big(r\psi(x) + (1-r)\psi(y)\big)$$

was investigated with the additional assumption that M is a (symmetric) quasi-arithmetic mean on I.

(ii) Comparison Problem. Given the n variable means $M_1, \ldots, M_k : I^n \to \mathbb{R}$, characterize those weights $(p_1, \ldots, p_n, q_1, \ldots, q_k), (r_1, \ldots, r_n, s_1, \ldots, s_k) \in K_{n,k}$ and pairs of functions $\varphi, \psi \in \mathcal{CM}(I)$ such that, for all $x_1, \ldots, x_n \in I$,

$$\mathcal{A}^{(p_1,\ldots,p_n,q_1,\ldots,q_k)}_{\varphi,M_1,\ldots,M_k}(x_1, \ldots, x_n) \leq \mathcal{A}^{(r_1,\ldots,r_n,s_1,\ldots,s_k)}_{\psi,M_1,\ldots,M_k}(x_1, \ldots, x_n).$$

For this problem, we have the following sufficient condition.

Theorem 3.4. *Let $n \geq 2$ and $k \geq 1$ be natural numbers, $(p_1, \ldots, p_n, q_1, \ldots, q_k) \in K_{n,k}$, $\varphi, \psi \in \mathcal{CM}(I)$ and let $M_1, \ldots, M_k : I^n \to I$ be n-variable means. Assume that ψ is increasing [decreasing] on I and $\psi \circ \varphi^{-1}$ is convex [concave] on $\varphi(I)$. Then, for all $x_1, \ldots, x_n \in I$,*

$$\mathcal{A}^{(p_1,\ldots,p_n,q_1,\ldots,q_k)}_{\varphi,M_1,\ldots,M_k}(x_1, \ldots, x_n) \leq \mathcal{A}^{(p_1,\ldots,p_n,q_1,\ldots,q_k)}_{\psi,M_1,\ldots,M_k}(x_1, \ldots, x_n). \tag{19}$$

Proof. Assume that ψ is increasing define $f := \psi \circ \varphi^{-1}$. To prove (19), let $x_1, \ldots, x_n \in I$. By the monotonicity of φ and by the mean value property of $M_1, \ldots, M_n$, for all $(x_1, \ldots, x_n) \in I^n$ and $j \in \{1, \ldots, k\}$, we deduce that (14) holds, and hence (15) is also valid. Thus, by Theorem 2.1(iii) and the convexity of f, it follows that

$$\psi\Big(\mathcal{A}^{(p_1,\ldots,p_n,q_1,\ldots,q_k)}_{\varphi,M_1,\ldots,M_k}(x_1, \ldots, x_n)\Big) = f\Bigg(\sum_{i=1}^{n} p_i\varphi(x_i) + \sum_{j=1}^{q} \varphi(M_j(x_1, \ldots, x_n))\Bigg)$$

$$\leq \sum_{i=1}^{n} p_i f(\varphi(x_i)) + \sum_{j=1}^{k} f\big(\varphi(M_j(x_1, \ldots, x_n))\big) = \sum_{i=1}^{n} p_i\psi(x_i) + \sum_{j=1}^{k} \psi(M_j(x_1, \ldots, x_n)).$$

By applying ψ^{-1} to both sides of this inequality, (19) results. ∎

Open Problem 3.5. Under the assumption of Theorem 3.4 and provided that ψ is increasing [decreasing] on I, is the convexity [concavity] of $\psi\circ\varphi^{-1}$ on $\varphi(I)$ necessary for the validity of inequality (19)?

The answer to this problem is also affirmative in several particular cases.

— In the case when $q_1 = \ldots = q_n = 0$, the comparison problem (18) reduces to the comparison problem of weighted quasi-arithmetic means when the necessity of the convexity [concavity] of $\psi\circ\varphi^{-1}$ on $\varphi(I)$ (provided that ψ is increasing [decreasing] on I) is known to be valid (cf. [17], [22], [24], [14]).

— In the case $n = 2$, $k = 1$, the necessity of the convexity [concavity] of $\psi\circ\varphi^{-1}$ on $\varphi(I)$ was obtained in [8], where the following result was proved: *Let $M : I^2 \to \mathbb{R}$ be a strict and continuous two variable mean, $p, q \in]0, 1]$, $\varphi, \psi \in \mathcal{CM}(I)$. Then, in order that*

$$\mathcal{A}^{(p,q,1-p-q)}_{\varphi,M} \le \mathcal{A}^{(p,q,1-p-q)}_{\psi,M},$$

that is, the functional inequality

$$\varphi^{-1}\big(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi(M(x,y))\big) \\ \le \psi^{-1}\big(p\psi(x) + q\psi(y) + (1-p-q)\psi(M(x,y))\big)$$

be satisfied for all $x, y \in I$, it is necessary and sufficient that $\psi\circ\varphi^{-1}$ be convex [concave] on $\varphi(I)$ provided that ψ is increasing [decreasing] on I. A more general result was obtained in [1].

(iii) Matkowski–Sutô Problem. Given the two variable means $M_1, \ldots, M_k : I^2 \to \mathbb{R}$, characterize those weights $(p_1, p_2, q_1, \ldots, q_k), (r_1, r_2, s_1, \ldots, s_k) \in K_{2,k}$, $t \in [0, 1]$, and pairs of functions $\varphi, \psi \in \mathcal{CM}(I)$ such that, for all $x, y \in I$,

$$\mathcal{A}^{(p_1,p_2,q_1,\ldots,q_k)}_{\varphi,M_1,\ldots,M_k}(x,y) + \mathcal{A}^{(r_1,r_2,s_1,\ldots,s_k)}_{\psi,M_1,\ldots,M_k}(x,y) = 2(tx + (1-t)y).$$

The particular case when $q_1 = \ldots = q_k = 0$, i.e., when the two means on the left hand side of this equation are ordinary weighted quasi-arithmetic means, was considered by several authors (see [2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 19, 20, 21, 23, 25, 26]).

The special case $k = 1$, $p_1 = p_2$, $t = \frac{1}{2}$ and M_1 is the arithmetic mean was investigated in [10].

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