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DETERMINANT CRITERIA OF SOLVABILITY

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Abstract. In the paper [3] the determinant criterion of solvability for the Kuczma equation [4] was given. This criterion appeared in the natural way as barycenter of some mass system. It turned out that determinants do appear in many different situations as solvability criteria. The present article is aimed to review the mostly classical results in the theory of functional equations from this point of view. We begin with classical results of the linear functional equations and the determinant equations solved by F. Neuman. Using natural hierarchy we select some class of well-known equations in order to catch a pattern in the solvability criteria. This approach gives some sort of hypothesis.

1. Introduction. In the review a hierarchy of functional equations of polynomial type on the function of two arguments is proposed. The main results in the theory of such equations were obtained in the seventies. The solution spaces of a few series of cyclic type equations were described, and some individual equations were investigated. The most famous equation was the one finally solved by F. Neuman, this equation is a determinant of values. This review was inspired by the complete solution of the Kuczma equation for the generalized mean, received recently by the author. The class of such means is the solution space of the third degree homogeneous equation, which is the equality of two determinants. Indeed, there is a natural reason why determinants occur in the equations possessing a nontrivial solution space. The main trick in solution of the functional equation is the reduction—equating the independent variables. With the reduction the expression either becomes tautological or gives a new equation with fewer variables. In the latter case, if at least one of the reduced equation is not solvable, then the original equation is not. If all reductions are solvable, then a system of functional equations appears, which is quite strong constraint on the unknown functions. That is, reducible equations are generally unsolvable. However, among polynomial equations there are those

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almost all reductions of which are tautological, those containing determinants. Each such equation might have a nontrivial solution space.

This review cannot claim to be complete, even citations are given not for all equations mentioned in the review.

2. Hierarchy of polynomial functional equations. Let us consider the system of independent variables $\{x_i\}_{i=1}^n$. An equation on a function of two variables $F : S\times S \to M$

$$p(F(x_i, x_j)) = 0,$$

where S is a specified set, M is a module over some ring K, $p(\nu_{ij})$ is a polynomial of matrix variables with coefficients from K, is called a *polynomial functional equation.*

For every polynomial of matrix variables we can calculate quantity of the left indexes in nonzero coefficients, say k, and analogously the right indexes, s, and indexes which are in both sides, d ($d \leqslant \min(k, s)$).

DEFINITION 2.1. The number $\dim p = k + s - d$ is called the *dimension* of the functional equation ($\dim p \geqslant \max(k, s)$). The equation with $n = \dim p$ is called *effective*, with $d = 0$ — *free*, with $d = n$ — *complete.*

REMARK. Any free equation could be considered for a function $F : S_1 \times S_2 \to M$ where the sets are different and independent variables are divided in two groups. We will apply our notation for such equations.

For example, for the Sincov equation

$$F(x, y) + F(y, z) = F(x, z)$$

we have $p(x_{11}, \dots, x_{33}) = x_{12}+x_{23}-x_{13}$, $k = 2$, $s = 2$, $d = 1$, $\dim p = 3$. It is clear that we can renumber variables and get an equivalent equation, but with the same characteristics $(k, s; d)$ which is called the *type of the equation.* Generally, we can assume that $k \leqslant s$. It is easy to calculate that the quantity $m(s)$ of types for the effective equations of dimension s is

$$m(s) = \Big[\frac{s^2 + 4s}{4}\Big].$$

DEFINITION 2.2. The polynomial is called *solvable* if there is at least one nontrivial solution in prescribed settings.

It means that the notion of solvability is relative.

DEFINITION 2.3. The *reduction* of the functional equation is the system

$$\begin{cases} p(F) = 0 \\ x_i = x_j, \quad i \neq j \end{cases}$$

We will denote such a reduction by (ij).

If any function satisfies the reduction (ij) it is called *tautological.* The reduction decreases the dimension of the equation. If some reduction of the equation is unsolvable then the equation also is unsolvable. So, if we investigate hierarchy of equations from lower to higher dimension (so called *pexiderization* process) we get rather wide class of equations with the solution space described in a new way.

3. Known series of equations

3.1. Linear equations of any dimension. The equation corresponding to the polynomial of degree 1 is called *linear*. The complete classification of solvable linear equations of dimension $\leqslant n$

$$\sum_{1\leqslant i,j\leqslant n} a_{ij}F(x_i,x_j)=a_0$$

was given by D. D. Adamović [1]. In his paper $S=M$ is a linear space over a field of characteristic not equal to 2.

THEOREM 3.1 ([1]). *For a homogeneous linear equation of dimension* 2

$$aF(x,y)+bF(y,x)+cF(x,x)+dF(y,y)=0$$

there are ten classes of equations with equal solution space. The non-homogeneous case is trivially reduced to the homogeneous one.

It is remarkable that for any higher dimension there are only finite set of classes with different solution spaces.

3.2. Free determinant equations of even dimension. In the paper of F. Neuman [9] the solution of the homogeneous determinant equation of dimension $2n$ (type $(n,n;0)$)

$$F^n(\mathbf{x},\mathbf{y})=\det\|F(x_i,y_j)\|_{i,j=1}^n=0$$

was given under some regularity condition.

THEOREM 3.2 ([9]). *For arbitrary sets X and Y (intervals, discrete sets, etc.) a function $F: X\times Y\to\mathbb{R}$ (or $\mathbb{C}$) is of the form*

$$F(x,y)=\sum_{i=1}^{n-1}\varphi_i(x)\psi_i(y) \tag{3.1}$$

with linearly independent sets $\{\varphi_i\}$ and $\{\psi_i\}$ if and only if the maximal rank of the matrices $\|F(x_i,y_j)\|_{i,j=1}^n$ is $n-1$ for all $\mathbf{x}\in X^n$ and $\mathbf{y}\in Y^n$.

The degree of polynomial is here n.

The equations described by J. Šimša [10] have the same solution space. The dimension of the free equation (3.2) is $4n$ and the degree is equal to $2n$ (type $(2n,2n;0)$).

THEOREM 3.3 ([10]). *If a function $F: X\times Y\to\mathbb{K}$ has the form* (3.1)*, then the equality*

$$F^n(\mathbf{x},\mathbf{y})F^n(\mathbf{u},\mathbf{v})=F^n(\mathbf{x},\mathbf{v})F^n(\mathbf{u},\mathbf{y}) \tag{3.2}$$

holds for all $(\mathbf{x},\mathbf{y})\in X^n\times Y^n$ and all $(\mathbf{u},\mathbf{v})\in X^n\times Y^n$. Conversely, if (3.2) *holds for all $(\mathbf{x},\mathbf{y})\in X^n\times Y^n$ and some $(\mathbf{u},\mathbf{v})\in X^n\times Y^n$ such that $F^n(\mathbf{u},\mathbf{v})\neq 0$, then F is of the form* (3.1).

REMARK. The functions of the form (3.1) satisfy all reductions of the determinant equation. In particular, if $X=Y$ (type $(n,n;n)$)

$$F^n(\mathbf{x},\mathbf{x})=\det\|F(x_i,x_j)\|_{i,j=1}^n=0.$$

But even in the case $n = 2$ the solution space is more rich. The general solution can be described by

$$F(x, y) = S(x, y) \pm \sqrt{S^2(x, y) - S(x, x)S(y, y)},$$

where S is any symmetrical function satisfying the inequality $S^2(x, y) \geqslant S(x, x)S(y, y)$ and the signs in the formula are taken different in the domains $x > y$ and $x < y$. Further we will denote this solution space by $\mathcal{D}$.

3.3. Determinant equation of type $(2n, 3n; 2n)$. In the same article [10] the $3n$-dimensional equation of type $(3n, 2n; 2n)$ was solved.

THEOREM 3.4 ([10]). *If a function $F : X \times X \to \mathbb{K}$ is of the form*

$$F(x, y) = \sum_{i,j=1}^{n} \alpha_{ij} f_i(x) f_j(y), \tag{3.3}$$

then the equality

$$F^n(\mathbf{x}, \mathbf{y})F^n(\mathbf{z}, \mathbf{z}) = F^n(\mathbf{x}, \mathbf{z})F^n(\mathbf{y}, \mathbf{z}) \tag{3.4}$$

holds for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^n$. Conversely, if $F^n(\mathbf{z}, \mathbf{z}) \neq 0$ for some $z = (z_1, z_2, \ldots, z_n) \in X^n$ and (3.4) holds for all $\mathbf{x}, \mathbf{y} \in X^n$, then F is of the form (3.3), with constants $\alpha_{ij} = F(z_i, z_j)$, $1 \leqslant i, j \leqslant n$.

3.4. Nonlinear cyclic equations. Some series of nonlinear functional equations of cyclic type were investigated in the articles of D. S. Mitrinović, S. B. Prešić, P. M. Vasić, and R. R. Janić.

The first series of type $(2n-1, 2n; 2n-1)$, investigated in [6],

$$\begin{aligned} &F(x_1, x_2)F(x_3, x_4) \ldots F(x_{2n-1}, x_{2n}) \\ &+ F(x_1, x_3)F(x_4, x_5) \ldots F(x_{2n}, x_2) \\ &\quad \ldots \\ &+ F(x_1, x_{2n})F(x_2, x_3) \ldots F(x_{2n-2}, x_{2n-1}) = 0 \end{aligned}$$

has only trivial solution for $n > 2$ and, as shown in [11] for $n = 2$, the general solution has the form $F(x, y) = \varphi(x)\psi(y) - \psi(x)\varphi(y)$. P. M. Vasić [12] noted that the equation of type $(3, 3; 2)$

$$F(x_1, x_2)F(x_3, x_4) - F(x_1, x_3)F(x_2, x_4) - F(x_1, x_4)F(x_3, x_2) = 0$$

is equivalent (has the same solution space) to the above one.

In order to describe the next series of cyclic type equations let us consider the functional equation

$$\psi(x_1, x_2, \ldots, x_{2n}) + \psi(x_1, x_3, \ldots, x_{2n}, x_2) + \ldots + \psi(x_1, x_{2n}, \ldots, x_{2n-1}) = 0,$$

where ([7])

$$\begin{aligned} \psi(x_1, \ldots, x_{2n}) = \big(&F(x_1, x_2) + F(x_3, x_4) + \ldots + F(x_{2k-1}, x_{2k})\big) \\ &\times \big(F(x_{2k+1}, x_{2n}) + F(x_{2k+2}, x_{2n-1}) + \ldots + F(x_{k+n}, x_{k+n+1})\big) \end{aligned}$$

or

$$\psi(x_1,\dots,x_{2n}) = F(x_1,x_2)\sum_{j=0}^{n-3} A_j(F(x_{j+3},x_{j+4}) + F(x_{2n-j-1},x_{2n-j})) \\ + A_{n-2}F(x_1,x_2)F(x_{n+1},x_{n+2}),$$

here $\sum_{j=0}^{n-2} A_j \neq 0$, or, more general ([8])

$$\psi(x_1,\dots,x_{2n}) = \big(F(x_1,u_0) + F(x_2,u_1) + \ldots + F(x_{2k-2},u_{k-1})\big) \\ \times \big(F(x_{2k},v_0) + F(x_{2k+2},v_1) + \ldots + F(x_{2n-2},v_{n-k-1})\big),$$

where $u_i \in \{x_{2l-1}\}_{l=1}^{k}$, $v_j \in \{x_{2l-1}\}_{l=k+1}^{n}$, and $u_\nu \neq u_\mu$, $v_\nu \neq v_\mu$ for $\nu \neq \mu$, and finally ([2])

$$\psi(x_1,\dots,x_{2n}) = \big(F(x_1,x_2) + F(x_3,x_4) + \ldots + F(x_{2k-1},x_{2k})\big) \\ \times \Big(\sum_{j=0}^{n+k-1} A_jF(x_{2k+j+1},x_{2n-j})\Big).$$

All these equations are solvable.

4. Determinant equations of the second degree. All functions are supposed to be real, continuous and with monotone sections for simplicity. We consider many equations of the second degree, but in contrast to those solved by D. S. Mitrinović, S. B. Prešić, P. M. Vasić, and R. R. Janić they are not cyclic. Let us consider the simple equation of type $(2,2;1)$

$$F(x_1,x_4)F(x_2,x_1) - F(x_1,x_1)F(x_2,x_4) = 0.$$

It is easy to see that it has the general solution $F(x,y) = \varphi(x)\psi(y)$. Let us denote the solution space of this equation by $\mathcal{F}$. Note that the only nontrivial reduction $x_4 = x_2$ leads to the determinant equation considered in Section 3.2 and has more rich solution space $\mathcal{D}$. Now apply the "pexiderization" replacing some variables by new ones in such a way that the general solution still satisfies it. This process gives some new equations with the same solution space.

Theorem 4.1. *A general solution of all equations listed below is* $F(x,y) = \varphi(x)\psi(y)$

$$\text{(i)}\quad \begin{vmatrix} F(x_1,x_5) & F(x_8,x_7) \\ F(x_2,x_6) & F(x_3,x_4) \end{vmatrix} = \begin{vmatrix} F(x_3,x_5) & F(x_8,x_6) \\ F(x_2,x_7) & F(x_1,x_4) \end{vmatrix} \qquad \text{(type } (4,4;0)\text{)},$$

$$\text{(ii)}\quad \begin{vmatrix} F(x_1,x_5) & F(x_3,x_7) \\ F(x_2,x_6) & F(x_3,x_4) \end{vmatrix} = \begin{vmatrix} F(x_3,x_5) & F(x_3,x_6) \\ F(x_2,x_7) & F(x_1,x_4) \end{vmatrix} \qquad \text{(type } (3,4;0)\text{))},$$

$$\text{(iii)}\quad \begin{vmatrix} F(x_1,x_5) & F(x_3,x_4) \\ F(x_2,x_6) & F(x_3,x_4) \end{vmatrix} = \begin{vmatrix} F(x_3,x_5) & F(x_3,x_6) \\ F(x_2,x_4) & F(x_1,x_4) \end{vmatrix} \qquad \text{(type } (3,3;0)\text{))},$$

$$\text{(iv)}\quad \begin{vmatrix} F(x_1,x_5) & F(x_3,x_4) \\ F(x_2,x_1) & F(x_3,x_4) \end{vmatrix} = \begin{vmatrix} F(x_3,x_5) & F(x_3,x_1) \\ F(x_2,x_4) & F(x_1,x_4) \end{vmatrix} \qquad \text{(type } (3,3;1)\text{))},$$

$$\text{(v)}\quad \begin{vmatrix} F(x_1,x_2) & F(x_3,x_4) \\ F(x_2,x_1) & F(x_3,x_4) \end{vmatrix} = \begin{vmatrix} F(x_3,x_2) & F(x_3,x_1) \\ F(x_2,x_4) & F(x_1,x_4) \end{vmatrix} \qquad \text{(type } (3,3;2)\text{))}.$$

Proof. All equations are just the sequence of reductions applied to the first one, namely $x_8 = x_3$, $x_7 = x_4$, $x_6 = x_1$, $x_5 = x_2$, $x_3 = x_1$. ■

Let us now describe the full hierarchy of the item (iii). We took this equation just to demonstrate the method and because the amount of work is not great for it. The reduction $x_1 = x_2$, $x_3 = x_4 = x_5 = x_6$ will be denoted by (12)(3456) and the type of such reduction by $2|4$.

I. Six reductions of type $1|5$ are tautological, 15 ones of type $2|4$ and 10 ones of type $3|3$ are tautological or $\mathcal{D}$.
II. 15 reductions $1|1|4$ are tautological or $\mathcal{D}$. 60 ones of type $1|2|3$ give three equations (if not tautological)

$$\mathcal{F}: \quad \begin{vmatrix} F(x_1,x_2) & F(x_1,x_4) \\ F(x_2,x_1) & F(x_1,x_4) \end{vmatrix} = \begin{vmatrix} F(x_1,x_2) & F(x_1,x_1) \\ F(x_2,x_4) & F(x_1,x_4) \end{vmatrix};$$

$$\mathcal{A}: \quad \begin{vmatrix} F(x_1,x_2) & F(x_2,x_3) \\ F(x_1,x_1) & F(x_2,x_3) \end{vmatrix} = \begin{vmatrix} F(x_2,x_2) & F(x_2,x_1) \\ F(x_1,x_3) & F(x_1,x_3) \end{vmatrix};$$

$$\mathcal{B}: \quad \begin{vmatrix} F(x_1,x_3) & F(x_2,x_2) \\ F(x_1,x_1) & F(x_2,x_2) \end{vmatrix} = \begin{vmatrix} F(x_2,x_3) & F(x_2,x_1) \\ F(x_1,x_2) & F(x_1,x_2) \end{vmatrix}.$$

The reductions of type $2|2|2$ (15 variants) give four exceptional equations besides $\mathcal{F}$- and $\mathcal{D}$-equations

$$\mathcal{H}: \quad \begin{vmatrix} F(x_1,x_2) & F(x_3,x_3) \\ F(x_2,x_1) & F(x_3,x_3) \end{vmatrix} = \begin{vmatrix} F(x_3,x_2) & F(x_3,x_1) \\ F(x_2,x_3) & F(x_1,x_3) \end{vmatrix},$$

this is the only equation whose any reduction is tautological;

$$\mathcal{A}' \subset \mathcal{D}: \quad \begin{vmatrix} F(x_1,x_3) & F(x_3,x_2) \\ F(x_2,x_1) & F(x_3,x_2) \end{vmatrix} = \begin{vmatrix} F(x_3,x_3) & F(x_3,x_1) \\ F(x_2,x_2) & F(x_1,x_2) \end{vmatrix};$$

$$\mathcal{B}' \subset \mathcal{D}: \quad \begin{vmatrix} F(x_1,x_1) & F(x_3,x_3) \\ F(x_2,x_2) & F(x_3,x_3) \end{vmatrix} = \begin{vmatrix} F(x_3,x_1) & F(x_3,x_2) \\ F(x_2,x_3) & F(x_1,x_3) \end{vmatrix};$$

$$\mathcal{C}' \subset \mathcal{D}: \quad \begin{vmatrix} F(x_1,x_1) & F(x_3,x_2) \\ F(x_2,x_3) & F(x_3,x_2) \end{vmatrix} = \begin{vmatrix} F(x_3,x_1) & F(x_3,x_3) \\ F(x_2,x_2) & F(x_1,x_2) \end{vmatrix}$$

So we have the complete list of reductions up to three variables. For every reduction except $\mathcal{F}$-equation we should write down all equations reducible to them. The quantity of such reductions is 260. It is easy to check that all of them except 8 tautological ones are reducible to $\mathcal{F}$-equation. The last step is to check these 8 reductions in the same way. All 27 equations over them are reducible to $\mathcal{F}$-equation.

Now let us solve the exceptional equations.

THEOREM 4.2. *The $\mathcal{H}$-equation*

$$\begin{vmatrix} F(x_1,x_2) & F(x_3,x_3) \\ F(x_2,x_1) & F(x_3,x_3) \end{vmatrix} = \begin{vmatrix} F(x_3,x_2) & F(x_3,x_1) \\ F(x_2,x_3) & F(x_1,x_3) \end{vmatrix} \tag{4.1}$$

is solvable in continuous real functions and the general solution has locally the form

$$F(x,y) = \begin{vmatrix} a & \varphi(y) & b \\ \psi(x) & 0 & \varphi(x) \\ c & \psi(y) & d \end{vmatrix},$$

where $a,b,c,d \in \mathbb{R}$ and $\varphi(x)$, $\psi(x)$ are arbitrary functions.

Proof. It is easy to check that symmetric and skew-symmetric functions satisfy the equation. Let us decompose the function into symmetric and skew-symmetric parts

$$F(x,y) = S(x,y) + \Omega(x,y). \tag{4.2}$$

The equation can be rewritten now in the form

$$S(x,x)\Omega(b,a) + S(a,x)\Omega(x,b) + S(x,b)\Omega(a,x) = 0. \tag{4.3}$$

Suppose now that in some domain, say $(\alpha,\beta) \times (\gamma,\mu)$, $S \neq 0$ and $\Omega \neq 0$, but $S(x,x) \equiv 0$ over (α,β). Fix $a \in (\alpha,\beta)$. Then in this domain $S(a,x) \neq 0$ and

$$\Omega(x,b) = \frac{S(x,b)\Omega(x,a)}{S(a,x)} = \lambda_a(x)S(x,b).$$

Therefore $\Omega(b,x) = \lambda_a(b)S(b,x)$, i.e. $(\lambda_a(x) - \lambda_a(b))S(x,b) \equiv 0$. This implies that $\lambda_a(x) \equiv 0$ and function is symmetric despite our assumption. Now we will solve the equation over some domain where three functions S, Ω, and $S(x,x)$ are bounded away from zero. Fix now two different points $a,b \in (\alpha,\beta)$ and consider five nonzero functions

$$\varphi(x) = \Omega(a,x);\ \psi(x) = \Omega(b,x);\ s_1(x) = S(x,x); s_a(x) = S(a,x);\ s_b(x) = S(b,x).$$

Our aim is to express successively s_a, s_b, s_1, S, and Ω via φ, ψ and some constants.

Change x and b in (4.3) and express s_b as

$$s_b(x) = \frac{s_1(b)\varphi(x) - s_b(a)\psi(x)}{\varphi(b)}. \tag{4.4}$$

Analogously after changing a and b in (4.3) we have

$$s_a(x) = \frac{s_b(a)\varphi(x) - s_1(x)\psi(x)}{\varphi(b)}. \tag{4.5}$$

Immediately from (4.3) we get

$$s_1(x) = \frac{s_b(x)\varphi(x) - s_a(x)\psi(x)}{\varphi(b)}.$$

Now we can substitute (4.4) and (4.5) here:

$$s_1(x) = \frac{1}{\varphi^2(x)}\big(s_1(b)\varphi^2(x) - 2s_b(a)\varphi(x)\psi(x) + s_1(a)\psi^2(x)\big). \tag{4.6}$$

Let us change now in the main identity (4.3) $b \to x$, $a \to y$, and $x \to b$, and express $\Omega(x,y)$ as

$$\Omega(x,y) = \frac{s_b(x)\psi(y) - s_b(y)\psi(x)}{s_1(b)}.$$

Substituting (4.4) here we obtain

$$\Omega(x,y) = \frac{\varphi(x)\psi(y) - \varphi(y)\psi(x)}{\varphi(b)}. \tag{4.7}$$

Finally after changing $b \to y$ in (4.3) we get

$$S(x,y) = \frac{s_1(x)\varphi(y) - s_a(x)\Omega(x,y)}{\varphi(x)}\,.$$

Substituting the known expressions here we get the formula

$$S(x,y) = \frac{1}{\varphi^2(b)}\big[s_1(b)\varphi(x)\varphi(y) + s_1(a)\psi(x)\psi(y) - s_b(a)(\varphi(y)\psi(x) + \varphi(x)\psi(y))\big]. \tag{4.8}$$

Summing (4.7) and (4.8) we get the final result. Any solution of (4.1) locally has the form

$$F(x,y) = a\varphi(x)\varphi(y) + b\psi(x)\psi(y) + c\varphi(x)\psi(y) + d\varphi(y)\psi(x). \tag{4.9}$$

The last formula can be written in the form of determinant. A straightforward calculation shows that any function of such form satisfies the equation (4.1). ■

It is easy to solve the rest of exceptional equations. $\mathcal{A}$-, $\mathcal{A}'$-, $\mathcal{B}$-, and $\mathcal{C}'$-equations have the solution space $\mathcal{F}$. $\mathcal{B}'$-equation is of type $\mathcal{D}$. Now we can formulate the final result.

THEOREM 4.3. *Every equation of* 4 *or* 5 *variables which can be reduced from the equation* (iii) (*Theorem* 4.1) *has the solution space $\mathcal{F}$. There is the only exceptional equation of three variables* ($\mathcal{H}$, *Theorem* 4.2) *reduced from* (iii), *whose solution space differs from $\mathcal{F}$ and $\mathcal{D}$.*

5. Some determinant equations of the third degree. In the paper [3] of the author the following theorem was proved.

THEOREM 5.1. *The equation of type* $(3,3;2)$

$$\begin{vmatrix} F(x_1,x_2) & F(x_1,x_2) & F(x_3,x_4) \\ F(x_3,x_1) & F(x_1,x_4) & F(x_3,x_4) \\ F(x_3,x_2) & F(x_2,x_4) & F(x_3,x_4) \end{vmatrix} = \begin{vmatrix} F(x_1,x_4) & F(x_2,x_4) & F(x_2,x_4) \\ F(x_1,x_4) & F(x_1,x_2) & F(x_3,x_2) \\ F(x_3,x_1) & F(x_3,x_1) & F(x_3,x_2) \end{vmatrix} \tag{5.1}$$

is solvable for a smooth monotone function $F(x,y)$ over square and the general form of the solution is

$$F(x,y) = \frac{\varphi\big((\varphi')^{-1}(x)\big) - \varphi\big((\varphi')^{-1}(y)\big)}{(\varphi')^{-1}(x) - (\varphi')^{-1}(y)}\,, \tag{5.2}$$

where φ is a smooth convex function over a segment.

We see that all types of equations described above have very general form of solutions or have no solution. The last example can be regarded in this context as the first equation among low degree ones with nontrivial solution space.

Historically Marek Kuczma offered in his last work [4] the functional equation (5.2) for special left side. The equation (5.2) in general form and its properties were discussed in the paper of J. Matkowski [5]. The complete solution was received by the author in [3]. Now we can consider equation (5.1) as determinant criteria of the Kuczma equation solvability. It turned out that the Kuczma equation is related to many areas of mathematics and even economics. The method of hierarchy offered in this paper gives the possibility to find exceptional equations among myriads of polynomial ones. It is natural to suppose that we can find among them absolutely new and rich ones, which can be regarded as determinant criteria of solvability. Moreover, the process of searching allows the program realization.

Let us consider now the reductions of the above equation. The reductions (12), (13), and (24) are tautological. It is easy to show that the reductions (14) and (23) have symmetrical functions as solution space. But the reduction (34) is slightly more difficult.

THEOREM 5.2. *The equation of type* $(3,3;3)$

$$\begin{vmatrix} F(x_1,x_2) & F(x_1,x_2) & F(x_3,x_3) \\ F(x_3,x_1) & F(x_1,x_3) & F(x_3,x_3) \\ F(x_3,x_2) & F(x_2,x_3) & F(x_3,x_3) \end{vmatrix} = \begin{vmatrix} F(x_1,x_3) & F(x_2,x_3) & F(x_2,x_3) \\ F(x_1,x_3) & F(x_1,x_2) & F(x_3,x_2) \\ F(x_3,x_1) & F(x_3,x_1) & F(x_3,x_2) \end{vmatrix}$$

is solvable for a function monotone in each of its arguments and a general solution is just a symmetrical function.

Proof. We see that symmetrical function satisfies the equation. It could be rewritten now in the form

$$F(x_1,x_2)\left(\begin{vmatrix} F(x_1,x_3) & F(x_3,x_3) \\ F(x_2,x_3) & F(x_3,x_3) \end{vmatrix} - \begin{vmatrix} F(x_3,x_1) & F(x_3,x_3) \\ F(x_3,x_2) & F(x_3,x_3) \end{vmatrix} + \begin{vmatrix} F(x_2,x_3) & F(x_1,x_3) \\ F(x_3,x_2) & F(x_3,x_1) \end{vmatrix}\right)$$
$$= \begin{vmatrix} F(x_1,x_3) & F(x_2,x_3) & F(x_2,x_3) \\ F(x_1,x_3) & F(x_3,x_3) & F(x_3,x_2) \\ F(x_3,x_1) & F(x_3,x_1) & F(x_3,x_2) \end{vmatrix}.$$

If the right multiplier in the left side is identically zero then the reduction $x_3 = x_2$ gives trivially either $F(x,y) = F(y,x)$ or $F(x,y) = -F(y,x)$. In the latter case, the symmetry of F follows immediately from the right side of equation which is zero automatically and we get the contradiction to monotonicity of F. If the considered multiplier is not equal to zero for some value x_3 then we can divide the right part of the equation and receive an explicit expression of $F(x_1,x_2)$. The numerator and the denominator change their signs under permutation of x_1 and x_2. It implies the symmetry of F. ■

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