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REFINEMENT TYPE EQUATIONS: SOURCES AND RESULTS

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Abstract. It has been proved recently that the two-direction refinement equation of the form

$$f(x)=\sum_{n\in\mathbb{Z}}c_{n,1}f(kx-n)+\sum_{n\in\mathbb{Z}}c_{n,-1}f(-kx-n)$$

can be used in wavelet theory for constructing two-direction wavelets, biorthogonal wavelets, wavelet packages, wavelet frames and others. The two-direction refinement equation generalizes the classical refinement equation $f(x)=\sum_{n\in\mathbb{Z}}c_nf(kx-n)$, which has been used in many areas of mathematics with important applications. The following continuous extension of the classical refinement equation $f(x)=\int_{\mathbb{R}}c(y)f(kx-y)\,dy$ has also various interesting applications. This equation is a special case of the continuous refinement type equation of the form

$$f(x)=\int_{\Omega}|K(\omega)|f(K(\omega)x-L(\omega))\,dP(\omega),$$

which has been studied recently in connection with probability theory. The purpose of this paper is to give a survey on the above refinement type equations. We begin with a brief introduction of types of refinement equations. In the first part we present several problems from different areas of mathematics which lead to the problem of the existence/nonexistence of integrable solutions of refinement type equations. In the second part we discuss and collect recent results on integrable solutions of refinement type equations, including some necessary and sufficient conditions for the existence/nonexistence of integrable solutions of the two-direction refinement equation. Finally, we say a few words on the existence of extremely non-measurable solutions of the two-direction refinement equation.

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The sections of this paper are:

1. Introduction. Throughout this paper $(\Omega, \mathcal{A}, P)$ denotes a complete probabilistic space.

We are interested in the following refinement type equation

$$f(x) = \int_\Omega |\det \varphi'_x(x,\omega)|\, f(\varphi(x,\omega))\, dP(\omega), \tag{R}$$

where $\varphi : \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is a map fulfilling the conditions:

(i) for every $\omega \in \Omega$ the map $\mathbb{R}^m \ni x \mapsto \varphi(x,\omega) \in \mathbb{R}^m$ is a diffeomorphism onto $\mathbb{R}^m$;
(ii) for every $x \in \mathbb{R}^m$ the map $\Omega \ni \omega \mapsto \varphi(x,\omega) \in \mathbb{R}^m$ is $\mathcal{A}$-measurable;
(iii) for every Borel set $B \in \mathbb{R}^m$ with $l_m(B) = 0$ we have $(l_m \otimes P)(\varphi^{-1}(B)) = 0$;

here and in the sequel m is a fixed positive integer. The symbol l_m denotes the m-dimensional Lebesgue measure and the symbol φ'_x denotes the derivative of φ with respect to the first variable.

We say that $f \in L^1(\mathbb{R}^m)$ is an L^1-*solution* of equation (R), if every representative of f satisfies (R) for almost all $x \in R^m$ with respect to l_m; this definition is well posed by the above assumptions (see [99] for details). It is clear that the set of all L^1-solutions of equation (R) forms a linear subspace of $L^1(\mathbb{R}^m)$; we denote this subspace by $Sol(\mathsf{R})$.

2. Kinds of refinement type equations. Fix $\mathcal{A}$-measurable functions $K : \Omega \to \mathbb{R}^{m\times m}$, $L : \Omega \to \mathbb{R}^m$ and define the map $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ by $\varphi(x,\omega) = K(\omega)x - L(\omega)$ with $\det K(\omega) \neq 0$ for every $\omega \in \Omega$. It is easy to verify that the map φ satisfies conditions (i)–(iii) and equation (R) takes the form

$$f(x) = \int_\Omega |\det K(\omega)|\, f(K(\omega)x - L(\omega))\, dP(\omega). \tag{R$_1$}$$

Note that if $P(\det K = 0) > 0$, then equation (R_1) has no nontrivial solution (see e.g. [103]; cf. [36]). It turns out that integrable solutions of equation (R_1) play an important role in many interesting problems in pure and applied mathematics (see Sections 3.2, 3.3, 3.4, 3.5 and 3.8 for details).

If $m = 1$, $K(\Omega) \subset \{-k, k\}$ with a fixed positive real number k and $L(\Omega) \subset \mathbb{Z}$, then putting $c_{n,\varepsilon} = kP(K = \varepsilon k, L = n)$ for any $n \in \mathbb{Z}$ and $\varepsilon \in \{-1, 1\}$ we have

$$\sum_{n\in\mathbb{Z}} (c_{n,-1} + c_{n,1}) = k, \tag{c}$$

and equation (R_1) reduces to the two-direction refinement equation

$$f(x) = \sum_{n\in\mathbb{Z}} c_{n,1} f(kx - n) + \sum_{n\in\mathbb{Z}} c_{n,-1} f(-kx - n). \tag{R_2}$$

It has been proved recently that equation (R_2) can be used in wavelet theory (see Section 3.7 for details) and in spline theory (see Section 3.6 for details). Furthermore, using equation (R_2) we can characterize some of the well known special functions (see Section 3.1 for details). Equation (R_2) generalizes the classical refinement equation

$$f(x) = \sum_{n\in\mathbb{Z}} c_n f(kx - n); \tag{R_3}$$

condition (c) takes now the form $\sum_{n\in\mathbb{Z}} c_n = k$. Equation (R_3), as well as its matrix version, has been used, among others, in such field as wavelet theory, approximation theory, theory of subdivision schemes, computer graphics, physics, combinatorial number theory, and others (see e.g. [10, 13, 22, 24, 26, 32, 35, 42, 43, 47, 48, 68, 69, 73, 78, 83, 90, 137, 138, 140, 164, 169, 174, 175, 177, 214]).

If $m = 1$ and $K(\omega) = \alpha \in \mathbb{R} \setminus \{0\}$ for every $\omega \in \Omega$ then equation (R_1) takes the form

$$f(x) = \int_\Omega |\alpha| f\big(\alpha x - L(\omega)\big)\, dP(\omega). \tag{R_4}$$

Put $\Omega = \mathbb{R}$, $L = \mathrm{id}_\mathbb{R}$ and define the measure P, on the σ-algebra of all Lebesgue measurable subsets on the real line, by $P(A) = \frac{1}{|\alpha|}\int_A c(y)\, dy$, where $c \in L^1(\mathbb{R})$ is a given nonnegative function such that $\int_\mathbb{R} c(y)\, dy = |\alpha|$. Then equation (R_4) reduces to the continuous refinement equation of the form

$$f(x) = \int_\mathbb{R} c(y) f(\alpha x - y)\, dy. \tag{R_5}$$

Equation (R_5) has many significant applications and has been studied by many authors (see e.g. [18, 31, 40, 41, 58, 89, 123]).

All the above equations will be called *refinement equations*. In this paper we are interested only in homogeneous refinement equations, however there are many papers concerning inhomogeneous versions of many particular cases of equation (R) (see e.g. [45, 46, 86, 87, 93, 126, 127, 128, 185, 186]).

3. Problems leading to refinement type equations. In this section we present problems from different areas of mathematics closely connected with integrable solutions of refinement equations. We restrict ourselves only to some of them because, in our opinion, the complete presentation all of them is impossible.

3.1. Special functions. Special functions are examined by many authors and in many cases graphs of such functions look like fractals (see [96] and the references given there). To see that many special functions are closely related to refinement equations we will follow an example from [37]. For every $a \in \mathbb{R}$ define the sequence $(f_{a,n})_{n\in\mathbb{N}_0}$ putting

$$f_{a,0}(x) = \begin{cases} 1+x, & \text{if } x \in [-1,0], \\ 1-x, & \text{if } x \in [0,1], \\ 0, & \text{if } x \notin [-1,1], \end{cases}$$

$$f_{a,n+1}(x) = f_{a,n}(3x) + \frac{1-a}{2}\Big[f_{a,n}(-3x-1) + f_{a,n}(3x+1)\Big] + \frac{1+a}{2}\Big[f_{a,n}(-3x-2) + f_{a,n}(3x+2)\Big].$$

We can now formulate our first proposition.

PROPOSITION 3.1. *For every $|a| < 1$ the sequence $(f_{a,n})_{n\in\mathbb{N}_0}$ converges pointwise to a continuous function f_a, which is the unique (up to a multiplicative constant) L^1-solution of the equation*

$$f_a(x) = f_a(3x) + \frac{1-a}{2}\big[f_a(-3x-1) + f_a(3x+1)\big] + \frac{1+a}{2}\big[f_a(-3x-2) + f_a(3x+2)\big]. \tag{3.1.1}$$

Equation (3.1.1) is a special case of (R_2). Moreover, since $1 + 2\,\frac{1-a}{2} + 2\,\frac{1+a}{2} = 3$, it follows that condition (c) holds. One can show that among all functions from Proposition 3.1 there are continuous and nowhere differentiable functions (e.g. the de Rham function $f_{1/3}$), singular functions (e.g. f_0 which coincides with the Cantor function on the interval $[-1,0]$ and on the interval $[0,1]$ is its mirror image) and more regular functions (e.g. the cardinal B-spline function of the first order $f_{-1/3}$).

Let us mention that an example of a continuous and nowhere differentiable function as a solution of a functional equation close to equation (3.1.1) is given in [171] (cf. [108, Chapter 10.5] where more details on continuous and nowhere differentiable functions can be found).

3.2. Iterated function systems and Foias operators. To give another reason showing that graphs of solutions of refinement equations are closely related with fractals we associate with (R_1) the iterated function system $\{S_\omega : \omega \in \Omega\}$ consisting of maps $S_\omega : \mathbb{R}^m \to \mathbb{R}^m$ given by $S_\omega(x) = K(\omega)^{-1}(x + L(\omega))$. Iterated function systems have been originally introduced and studied in [84] (see also [3]). It can be shown (see [148]) that if

$$\sup_{\omega\in\Omega} \|S_\omega(0)\| < +\infty \quad \text{and} \quad \sup_{\omega\in\Omega} \sup_{\|x\|=1} \|K(\omega)^{-1}x\| < 1, \tag{3.2.1}$$

then the iterated function system $\{S_\omega : \omega \in \Omega\}$ is asymptotically stable; i.e., there is a unique nonempty compact set $A_* \subset \mathbb{R}^m$ such that $H(A_*) = A_*$ and $\lim_{n\to\infty} H^n(A) = A_*$ for every nonempty and compact set $A \subset \mathbb{R}^m$, where $H : \mathbb{R}^m \to \mathbb{R}^m$ is the corresponding Hutchinson operator given by $H(A) = \mathrm{cl}\bigcup_{\omega\in\Omega} S_\omega(A)$ and the convergence holds in the Hausdorff metric. The set A_* is said to be an attractor.

Consider now the Foias operator $M : \mathcal{M}_1(\mathbb{R}^m) \to \mathcal{M}_1(\mathbb{R}^m)$ given by

$$M\mu(A) = \int_\Omega \int_{\mathbb{R}^m} \mathbf{1}_A(S_\omega(x))\, d\mu(x)\, dP(\omega) \quad \text{for Borel sets } A \subset \mathbb{R}^m, \tag{3.2.2}$$

where $\mathcal{M}_1(\mathbb{R}^m)$ denotes the family of all Borel probability measures on $\mathbb{R}^m$. The operator M corresponds to a regular stochastic dynamical system and it is a special case of Markov operator (see [113, Chapter 12.4]). If

$$\int_\Omega \|S_\omega(0)\|\, dP(\omega) < +\infty \quad \text{and} \quad \int_\Omega \sup_{\|x\|=1} \|K(\omega)^{-1}x\|\, dP(\omega) < 1, \tag{3.2.3}$$

then the operator M is asymptotically stable; i.e., there exists a unique measure $\mu_* \in \mathcal{M}_1(\mathbb{R}^m)$ such that $M\mu_* = \mu_*$ and $\lim_{n\to\infty} M^n\mu = \mu_*$ for every $\mu \in \mathcal{M}_1(\mathbb{R}^m)$, where the convergence holds in the Fortet–Mourier metric, or equivalently, in the weak sense. The unique measure μ_*, called an *attractor*, is of pure type and it is absolutely continuous with respect to l_m if and only if there exists a unique density $f \in L^1(\mathbb{R}^m)$ satisfying (R_1). Moreover, if (3.2.1) holds and μ_* has a density f, then this density is an L^1-solution of (R_1) with support contained in the attractor A_* of the iterated function system $\{S_\omega : \omega \in \Omega\}$ (see [148] for details); in the discrete case where $\Omega = \{\omega_1, \dots, \omega_N\}$ and $P(\omega_n) > 0$ for every $n \in \{1, \dots, N\}$ we have $\operatorname{supp} f = A_*$ (see e.g. [84]). Clearly, condition (3.2.1) implies condition (3.2.3).

We end this section by a proposition ensuring sufficient conditions for the existence of a nontrivial solution of equation (R_1) with deterministic matrix K; cf. [41, 89, 100, 148].

Proposition 3.2. *Assume that the matrix K does not depend on ω, $\sup_{\|x\|=1} \|K^{-1}x\| < 1$ and $\int_\Omega \|L(\omega)\|\, P(d\omega) < +\infty$. If an absolutely continuous part of the Jordan decomposition of a probability law of L is nonzero, then equation (R_1) has a nontrivial and nonnegative L^1-solution f. Moreover, if L is bounded, then the support of f is contained in the attractor of the iterated function system $\{S_\omega : \omega \in \Omega\}$.*

3.3. Self-similar measures. In the previous section we have looked for asymptotically stability of the operator M given by (3.2.2). Sometimes it is enough to know that there exists a measure $\mu \in \mathcal{M}_1(\mathbb{R}^m)$ such that $M\mu = \mu$; such a measure is called *stationary*. If the operator M has stationary measure μ, then

$$\mu = \int_\Omega \mu \circ S_\omega^{-1}\, dP(\omega) \tag{S}$$

It is known that in some cases there exists a unique measure $\mu \in \mathcal{M}_1(\mathbb{R}^m)$, called *self-similar*, satisfying (S) (see e.g. [49, 84, 187]). If such a measure exists it is of purely type; i.e., it is either singular or absolutely continuous with respect to l_m (see e.g. [85, 161]). The problem of deciding when a given self-similar measure is absolutely continuous is very difficult. Even in the simplest case of the Erdős problem it is still far from being solved (see e.g. [179, 180]). Certain connection between self-similar measures satisfying (S) and L^1-solutions of equation (R_1) can be formulated as follows.

Proposition 3.3. *If a Borel measure μ satisfying* (S) *has a density f, then f satisfies* (R_1). *On the other hand, if f is a nontrivial and nonnegative L^1-solution of equation* (R_1), *then the formula $\mu(B) = \frac{1}{\|f\|} \int_B f(t)\, dt$ defines a Borel measure satisfying* (S).

Proposition 3.3 has been used in very particular cases of equation (R_1) in many papers (see e.g. [80, 120, 147]) and it is strictly connected with a construction of Haar-type wavelets (see e.g. [66, 77, 110]).

If equation (R_1) has no nontrivial L^1-solution, then according to Proposition 3.3 we can expect that the corresponding self-similar measure μ is singular; i.e., $l_m(\operatorname{supp}\mu) = 0$. In such a case, to get more detailed information on the self-similar measure μ, one can try to calculate some of dimensions of μ studied in the literature (see e.g. [2, 16, 60, 67, 107, 114, 115, 117, 118, 119, 150, 151, 152, 153, 155, 158, 162, 181, 62] for details).

3.4. Perpetuities. Assume that $(\xi_n, \eta_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed random variables such that $P(\xi_1 = 0) = 0$, $-\infty < \int_\Omega \log|\xi_1(\omega)|\, P(d\omega) < 0$, $\int_\Omega \log\max\{|\eta_1(\omega)|, 1\}\, P(d\omega) < +\infty$ and $P(\eta_1 + c\xi_1 = c) < 1$ for every $c \in \mathbb{R}$. According to [74] (see also [105, 189]) we know that the series

$$\sum_{n=1}^{\infty} \eta_n \prod_{k=1}^{n-1} \xi_k \tag{G}$$

is almost surely convergent and its probability distribution F is of purely type; i.e., F is either absolutely continuous or continuous and singular. The limiting random variable (G) is the probabilistic formulation of the actuarial notion of a perpetuity; it represents the present value of a permanent commitment to make a payment at regular intervals, say annually, into the future forever (see e.g. [59]). We will see later that convergence of special perpetuities plays important role in determining the Fourier transform of L^1-solutions of refinement equations.

Put $K = \xi_1^{-1}$, $L = \xi_1^{-1}\eta_1$. From [74] it follows that the probability distribution F fulfils

$$F(x) = \int_{K>0} F(K(\omega)x - L(\omega))\, dP(\omega) + \int_{K<0} [1 - F(K(\omega)x - L(\omega))]\, dP(\omega). \tag{D}$$

Equation (D) has an extensive literature; we refer the reader to [4, 5] (see also [108]) for detailed and current review of linear iterative equations including many cases of equation (D). Equation (D) is closely related to the iterates of random-valued functions (see e.g. [6, 97, 98]); in this context it is not surprising that a probabilistic approach occurs often in study of equation (D) (see e.g. [40]). Let us also mention that a characterization of the Cantor function via a very special case of equation (D) can be found in [178] (see also [141, 142]).

What is the connection between equation (D) and refinement type equations? The answer to this question is given by the following proposition.

PROPOSITION 3.4 (see [100]; cf. [39, 99, 168, 195]). *If a probability distribution F is a solution of equation* (D) *and has a density $f \in L^1(\mathbb{R})$, then f satisfies* (R_1). *On the other hand, if $f \in L^1(\mathbb{R})$ is a nontrivial solution of equation* (R_1), *then the formula $F(x) = \frac{1}{\|f\|}\int_{-\infty}^{x} |f(t)|\, dt$ defines an absolutely continuous function satisfying* (D).

3.5. Distributional fixed points. As we have seen in the previous section, in the one-dimensional case the probability distribution of a density satisfying (R_1) fulfils equation (D). As we have seen, in the case where $m = 1$, equation (D) consists of two terms and it is written explicitly in a form useful for refinement equations. In the case where $m \geq 2$ a counterpart of equation (D) can be written explicitly in a useful form in some particular cases only (see [102]). Hence, instead of probability distributions we examine laws of solutions (which have to be densities up to normalization). This leads us to the notion of a distributional fixed point. To introduce this notion assume that $\xi : \Omega \to \mathbb{R}^{m\times m}$ and $\eta : \Omega \to \mathbb{R}^m$ are $\mathcal{A}$-measurable functions. Let Ψ be the random affine map given by

$$\Psi(t,\omega) = \eta(\omega) + \xi(\omega)t \quad \text{for all } t \in \mathbb{R}^m,\ \omega \in \Omega.$$

Following [189] consider a stochastic fixed-point equation

$$\Phi \stackrel{d}{=} \xi\Phi + \eta, \tag{F}$$

where the symbol $\stackrel{d}{=}$ means equality of probability laws. According to [71] we say that a distributional fixed point of a random affine map Ψ is a probability law of a random vector $\Phi : \Omega \to \mathbb{R}^m$ such that (F) holds and Φ is independent of (η, ξ).

Let $\Pi_{(0,1)}$ denote the projection map given by $\Pi_{(0,1)}(x,\omega) = x$ for $\omega \in \Omega$ and $x \in (0,1)$. Here the unit interval $(0,1)$ is endowed with the Lebesgue measure. The main result of this section characterizes the existence of nontrivial L^1-solutions of equation (R_1) by distributional fixed points of special random affine maps. It also gives a characterization of the dimension of the space $Sol(\mathsf{R}_1)$.

Proposition 3.5. *Assume that $\Phi : (0,1) \to \mathbb{R}^m$ is a random variable with a density f. Then f is an L^1-solution of equation (R_1) if and only if the probability law of $\Phi \circ \Pi_{(0,1)}$ is a distributional fixed point of the random affine map*

$$\Psi(t,\omega,x) = K^{-1}(\omega)L(\omega) + K^{-1}(\omega)\,t, \quad t \in \mathbb{R}^m,\ \omega \in \Omega,\ x \in (0,1). \tag{A}$$

Moreover, $\dim Sol(\mathsf{R}_1) = n$ *if and only if random affine map* (A) *has exactly n absolutely continuous distributional fixed points with linearly independent densities.*

Proposition 3.5 can be found in [104] in the case where $m = 1$. To prove its moreover part it is enough to do the following observation. If f is an L^1-solution of equation (R_1), then so is $|f|$. Consequently, any basis of the space $Sol(\mathsf{R}_1)$ can be replaced by a basis consisting of densities.

Let us notice that under some non-degeneracy and moment conditions the space $Sol(\mathsf{R}_1)$ is at most one-dimensional (see e.g. [103]). So, in such a case the existence of a nontrivial L^1-solution of equation (R_1) is equivalent to $\dim Sol(\mathsf{R}_1) = 1$. For example, if $m = 1$, then under weak assumptions the space $Sol(\mathsf{R}_1)$ is at most one-dimensional, and moreover, if K and L are independent and a probability law μ is a distributional fixed point of random affine map (A), then $\dim Sol(\mathsf{R}_1) = 1$ provided L is absolutely continuous or K is absolutely continuous and $P(L + y = 0) = 0$ for μ-a.e. $y \in \mathbb{R}$ (see [104]).

3.6. Refinable splines. Consider the cardinal B-spline functions defined as follows:

$$N_1 = \chi_{[0,1]} \quad \text{and} \quad N_{m+1} = N_m * N_1 \quad \text{for every } m \in \mathbb{N}.$$

It is known (see e.g. [17]) that the cardinal B-spline N_m is symmetric about $m/2$ and satisfies (R_3) with $k = 2$, $c_n = m!/(2^{m-1}n!(m-n)!)$ for $n \in \{0,1,\ldots,m\}$ and $c_n = 0$ for $n \in \mathbb{Z}\setminus\{0,1,\ldots,m\}$. Fix an arbitrary sequence $(c_{n,1})_{n\in\mathbb{Z}}$ of reals and put $c_{n,-1} = c_n - c_{n,1}$

for every $n \in \mathbb{Z}$. A short calculation shows that

$$N_m(x) = \sum_{n\in\mathbb{Z}} c_{n,1} N_m(2x-n) + \sum_{n\in\mathbb{Z}} c_{n,-1} N_m(-2x+n+m)$$

and (c) holds with $k=2$. Similar motivation for studying equation (R_2) with arbitrary k comes from [203] (see also [146, 154, 204]).

Any compactly supported piecewise polynomial function on the real line is called a *spline*. Among the most useful splines there are those that are also refinable; i.e., splines being solutions of equation (R_3) with a real number k such that $\sum_{n\in\mathbb{Z}} c_n = |k| > 1$ and $c_n \neq 0$ only for finitely many $n \in \mathbb{Z}$. Consequently, refinable splines are solutions of equation (R_2) with $c_{n,1} = 0$ for all $n \in \mathbb{Z}$ or with $c_{n,-1} = 0$ for all $n \in \mathbb{Z}$. Complete classification of refinable splines can be found in [33] (see also [32, 122]) and in [170], where not only refinable splines, but all refinable piecewise-smooth functions have been analysed.

Refinable splines, as well as cardinal B-splines, form the foundation for theory of compactly supported wavelets, theory of subdivision schemes, fractal geometry and self-affine tilings (see e.g. [13, 19, 30, 35, 61, 111]).

3.7. Orthonormal basis of wavelets. It is well known (see [133], cf. [34]) that having a multiresolution analysis one can construct a wavelet; i.e., a function $\psi \in L^2(\mathbb{R})$ such that the family $\{2^{j/2}\psi(2^j\cdot - l) : j,l \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. A *multiresolution analysis* is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ such that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$, there exists $\phi \in V_0$ such that $\{\phi(\cdot - l) : l \in \mathbb{Z}\}$ is an orthonormal basis of V_0, and moreover, $V_j \subset V_{j+1}$ and $f \in V_0 \Leftrightarrow f(2^{-j}\cdot) \in V_j$ for every $j \in \mathbb{Z}$. The function ϕ is called a *scaling function*. It is an easy matter to show that the scaling function ϕ satisfies equation (R_3) with $k=2$. Even more, it turns out that wavelets can be constructed via equation (R_3). Namely, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfies (R_3) and there are positive reals A, B such that $A \leq \sum_{l\in\mathbb{Z}} |\hat{f}(x+2l\pi)|^2 \leq B$ for almost all $x \in \mathbb{R}$, then a multiresolution analysis is the sequence $(V_j)_{j\in\mathbb{Z}}$ given by $V_j = \mathrm{cl}(\mathrm{lin}\{\mathrm{f}(2^{\mathrm{j}}\cdot - \mathrm{l}) : \mathrm{l} \in \mathbb{Z}\})$ (see e.g. [35]). A multidimensional multiresolution analysis has been treated, among others, in [23, 76, 77, 132, 136, 196].

It has been proved recently that in many constructions in wavelet theory equation (R_3) can be replaced by two-direction refinement equation (R_2). More precisely, equation (R_2) has been used for constructing two-direction wavelets, biorthogonal wavelets, wavelet packets, wavelet frames and multiwavelets (see e.g. [109, 124, 125, 129, 130, 131, 160, 197, 198, 199, 201, 203, 204, 205, 208]). All the constructions are based on the classical ones, but they are a little bit more complicated to be presented here.

3.8. Subdivision schemes. Computer Aided Geometric Design (shortly CAGD) is a branch of applied mathematics concerned with algorithms for the design of smooth curves, surfaces and many important geometrical quantities (see e.g. [63, 65, 194]); general methods in CAGD are explained in [11], whereas the history of curves and surfaces in CAGD can be found in [64].

Subdivision schemes have become important and efficient ways to generate smooth curves and surfaces (see e.g. [1, 51, 54, 163, 173, 213]). To explain the idea of subdivision schemes in the simplest binary case assume that we have a sequence $(c_n)_{n\in\mathbb{Z}}$, called the *mask* of the subdivision scheme, and an initial data $v^0 = (v_l^0)_{l\in\mathbb{Z}}$. Then we recursively define a new data sequence $(v^n)_{n\in\mathbb{N}} = \big((v_l^n)_{l\in\mathbb{Z}}\big)_{n\in\mathbb{N}}$ by the refinement rule

$$v_l^n = \sum_{m\in\mathbb{Z}} c_{l-2m} v_m^{n-1}$$

for all $l \in \mathbb{Z}$ and $n \in \mathbb{N}$. It turns out that under a suitable condition on the refinement rule, the data sequence $(v^n)_{n\in\mathbb{N}}$ starting with $v^0 = (\delta_{0,l})_{l\in\mathbb{Z}}$ converges to a smooth curve f (being a nontrivial and continuous function from the space $L^1(\mathbb{R})$) in the following sense: $\lim_{n\to+\infty} \sup_{l\in\mathbb{Z}} |f(2^{-l}n) - v_l^n| = 0$ and the limit curve f satisfies (R_3) with $k = 2$. Moreover, for any initial data $v^0 \in l^\infty(\mathbb{Z})$ the data sequence $(v^n)_{n\in\mathbb{N}}$ converges to a curve f_0 which can be expressed as $f_0(x) = \sum_{l\in\mathbb{Z}} v_l^0 f(x-l)$ for every $x \in \mathbb{R}$, where f is the smooth curve obtained above (see e.g. [13, 138, 139]).

It is well known that a necessary condition for the convergence of the subdivision scheme is the so called the *first sum rule condition*

$$\sum_{n\in\mathbb{Z}} c_{2n} = \sum_{n\in\mathbb{Z}} c_{2n+1} = 1. \tag{3.8.1}$$

The convergence analysis of subdivision schemes can be found in many papers (see e.g. [13, 43, 55]). In many problems arising from CAGD, the mask $(c_n)_{n\in\mathbb{Z}}$ consists of nonnegative reals (see e.g. [50, 138]). In such a case condition (3.8.1) is close to be sufficient for the convergence of the subdivision scheme (see e.g. [72, 91, 135, 193, 211]).

More details on subdivision schemes, some generalizations and applications can be found, among others, in [14, 15, 27, 28, 29, 38, 41, 52, 53, 56, 57, 79, 81, 159, 176].

4. Integrable solutions of refinement type equations. In this section we are interested in the space $Sol(\mathsf{R})$. In general, it is very difficult to describe the space $Sol(\mathsf{R})$, but in special cases it is possible to determine its dimension and obtain some properties of its elements. As already mentioned, we are interested in L^1-solutions, however we begin with a few words on distributional solutions of refinement equations.

4.1. Distributional solutions. Distributional solutions of refinement equations have been considered by many authors in the homogeneous case (see e.g. [12, 25, 83, 92, 94, 112, 165, 203, 204, 210]), as well as in the inhomogeneous case (see e.g. [45, 86, 87, 185]). We will not present the formal definition of distributional solutions of refinement equations. For a precise definition we refer the reader to aforementioned references.

Distributional solutions of refinement equations can be constructed via the Fourier transform. The key fact is that the Fourier transform converts a refinement equation to some equivalent form. In the case of equation (R_3) with $m = 1$, application of the Fourier transform and iteration leads to $\widehat{f}(x) = \prod_{n=1}^{\infty} h(x/k^n)\widehat{f}(0)$, where h is a characteristic function given by $h(x) = \frac{1}{k}\sum_{n\in\mathbb{Z}} c_n e^{inx}$. This is the well known representation of solutions of equation (R_3) (see e.g. [34]). Under some conditions this representation can be extended to equation (R_1) as follows

$$\widehat{f}(x) = \int_{\Omega^\infty} \exp\Big(ix \cdot \sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n} K(\omega_k)^{-1}\Big) L(\omega_n)\Big) \widehat{f}(0)\, dP^\infty(\omega_1, \omega_2, \dots). \tag{4.1.1}$$

Here the Fourier transform of $f \in L^1(\mathbb{R}^m)$ at a point x is defined by $\widehat{f}(x) = \int_{\mathbb{R}^m} e^{ix\cdot t} f(t)\,dt$ and $\cdot$ denotes the inner product in $\mathbb{R}^m$. Representation (4.1.1) of solutions of refinement equations can be generalized to the case of matrix refinement type equations (see e.g. [103]). However, in more general setting additional assumptions on asymptotic behavior of the product of random matrices like in [182] (see also [106, 149]) are needed.

In general it is very difficult to decide when the right hand side of (4.1.1) describes the Fourier transform of a function $f \in L^1(\mathbb{R}^m)$. But we may say that equation (R) has a unique (up to a multiplicative constant) solution being a distribution, provided the right hand side of (4.1.1) is a distribution. More precisely, in the case where $m = 1$, we have the following proposition.

Proposition 4.1. *Assume that*

$$\lim_{n\to\infty} \frac{L(\omega_n)}{K(\omega_1)\cdots K(\omega_n)} = 0 \qquad a.s.\ on\ \Omega^\infty \tag{4.1.2}$$

and

$$\lim_{n\to\infty} |K(\omega_1)\cdots K(\omega_n)| = +\infty \qquad a.s.\ on\ \Omega^\infty. \tag{4.1.3}$$

Then the formula $H(x) = \int_{\Omega^\infty} \exp\big(ix\sum_{n=1}^{\infty} \frac{L(\omega_n)}{K(\omega_1)\cdots K(\omega_n)}\big)\,dP^\infty(\omega_1,\omega_2,\dots)$ *is well defined for every* $x \in \mathbb{R}$. *Moreover*:

(i) *Equation* (R_1) *has a unique* (*up to a multiplicative constant*) *distributional solution, if* H *is a distribution.*

(ii) *If* H *is the Fourier transform of* $f_0 \in L^1(\mathbb{R})$, *then* $Sol(\mathsf{R}_1) = \{\lambda f_0 : \lambda \in \mathbb{R}\}$.

Condition (4.1.2) holds obviously in the case where $P(L = 0) = 1$. If $P(L = 0) < 1$, then condition (4.1.3) holds if and only if condition (4.1.2) is satisfied and

$$\int_1^\infty \frac{\log y}{\int_0^{\log y} P(\log|K| > x)\,dx}\, dP(|K^{-1}L| \le y) < +\infty$$

(see [71]). Consequently, one of assumptions (4.1.2), (4.1.3) of Proposition 4.1 is satisfied trivially.

If H from Proposition 4.1 is a well defined function, then equation (R_1) has a unique distributional solution and, by the Paley–Wiener theorem for distributions, this distributional solution is supported in $[-A, A]$ if and only if H can be extended to an entire analytic function $\widetilde{H}$ such that $\widetilde{H}(z) \le C(1 + |z|)^N e^{A|\mathrm{Im}\, z|}$ for $z \in \mathbb{C}$ with $N \in \mathbb{N}$ and $C > 0$.

It may happen that the function H defined in Proposition 4.1 is constant. Then $Sol(\mathsf{R}_1) = \{0\}$ and the Dirac delta distribution is the only distributional solution of (R_1). It concerns, e.g., the equation $f(x) = 2f(2x)$ (see [36]), although from Theorems 1 and 2 in [203] (see also [204]) it would result that this equation has more than one even compactly supported distributional solution. Similarly, equation $f(x) = f(2x) - f(-2x)$ has no nontrivial distributional solutions, whereas from Lemma 1 in [201] (see also Theorem 1 in [199]) it would result that it is not the case.

4.2. The space of integrable solutions. As we have seen in Proposition 3.5 the dimension of the space $Sol(\mathsf{R}_1)$ depends on the number of absolutely continuous distributional fixed points of the random affine map of form (A). These distributional fixed points are in fact perpetuities and in most cases we also have the uniqueness. Therefore in such a case the space $Sol(\mathsf{R}_1)$ is at most one-dimensional. The same conclusion can be drawn from the representation formula described in the previous section; for example, $\dim Sol(\mathsf{R}_1) \leq 1$ under assumptions of Proposition 4.1.

Equation (R) without additional assumptions on φ is not easy to examine, especially the problem of existence of a solution of equation (R) is difficult. In particular, it is rather impossible to get a representation of a solution of equation (R) in its full extent. However, we can give some assumptions which imply that the trivial function is the unique L^1-solution of equation (R) with $m = 1$. According to [99] (cf. also [102]) we have the following proposition.

PROPOSITION 4.2. *Assume that* $|\varphi(x,\omega) - \varphi(y,\omega)| \leq K(\omega)|x-y|$ *for all* $x, y \in \mathbb{R}$, $\omega \in \Omega$. *If* $K : \Omega \to (0,+\infty)$ *is measurable and* $-\infty < \int_\Omega \log K(\omega)\, dP(\omega) < 0$, *then* $\dim Sol(\mathsf{R}) = 0$.

In contrast to Proposition 4.2 we have the following result concerning the dimension of the space $Sol(\mathsf{R}_4)$.

THEOREM 4.3 (see [100]). *Assume that* $L : \Omega \to \mathbb{R}$ *is a continuous random variable such that* $\int_\Omega \log\max\{|L(\omega)|, 1\}\, dP(\omega) < +\infty$. *If* $|\alpha| > 1$ *then* $\dim Sol(\mathsf{R}_4) = 1$.

At the end of this section let us only mention that in the case of a matrix refinement type equation the space of all its L^1-solutions is at most p-dimensional in general, where p is the number of coordinates of the unknown L^1-solution $f = (f_1, \ldots, f_p) \in L^1(\mathbb{R}^m)^p$ (see e.g. [83, 103]).

4.3. Basic properties of integrable solutions. From now on, by $Sol(\mathsf{R}_2)$ we will mean the space of all L^1-solutions of equation (R_2) under condition (c).

Here and throughout we will examine equation (R_2) only in the case where $m = 1$ with $k > 1$ and $c_{n,\varepsilon} \geq 0$ for all $(n,\varepsilon) \in \mathbb{Z} \times \{-1, 1\}$, however there are papers concerning the case $m > 1$, as well as more general equation (see e.g. [200, 202, 206, 207, 209]). Moreover, to simplify the presented result we restrict ourselves to the case where $c_{n,\varepsilon} > 0$ only for finitely many $(n,\varepsilon) \in \mathbb{Z} \times \{-1, 1\}$; i.e., we consider the equation

$$f(x) = \sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon} f(\varepsilon k x - n), \tag{R_2'}$$

where the set $\mathbf{S} = \{(n,\varepsilon) \in \mathbb{Z} \times \{-1, 1\} : c_{n,\varepsilon} > 0\}$ is finite. Now, condition (c) takes the form $\sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon} = k$.

We already know that the space $Sol(\mathsf{R}_2')$ is at most one-dimensional. According to [203] (cf. [7, 36]) it is also known that every element $f \in Sol(\mathsf{R}_2')$ is compactly supported with $\operatorname{supp} f \subset \mathbf{J} = [-\frac{N}{k-1}, \frac{N}{k-1}]$, where $N = \max\{|n| : c_{n,\varepsilon} \in \mathbf{S}\}$. The next result gives precise information on the support of elements from $Sol(\mathsf{R}_2')$.

THEOREM 4.4 (see [147]). *Let* $\mathbf{J_S}$ *be the attractor of the iterated function system* $\{S_{n,\varepsilon} : (n,\varepsilon) \in \mathbf{S}\}$ *consisting of maps* $S_{n,\varepsilon} : \mathbb{R} \to \mathbb{R}$ *given by* $S_{n,\varepsilon}(x) = \frac{x+n}{\varepsilon k}$. *If* $f \in Sol(\mathsf{R}_2')$ *is nontrivial, then* $\operatorname{supp} f = \mathbf{J_S} \subset \mathbf{J}$.

One can ask when $\mathbf{J_S} = \mathbf{J}$. For example, it is the case where $2N+1 \geq k$ and $\mathbf{S} \cap \{(n,1),(-n,-1)\} \neq \emptyset$ for every $n \in \{-N,\dots,N\}$. On the other hand, if $2N+1 < k$ or if $\operatorname{card}\mathbf{S} < k$, then $\dim Sol(\mathsf{R}_2')= 0$.

As we have seen every element from the space $Sol(\mathsf{R}_2')$ must be compactly supported. The converse is false even in the case of equation (R_3) (see [184]). Furthermore, equation (R_3) with finitely many nonzero coefficients may also have a non-compactly supported L^2-solution (see [134]).

The next result generalizes a known property of elements from the space $Sol(\mathsf{R}_3)$ (see e.g. [39, 70, 80, 144, 168, 195]) correlative to an open problem proposed in [135].

THEOREM 4.5 (see [147]). *If $f \in Sol(\mathsf{R}_2')$, then f is of constant sign. Moreover, if $\mathbf{J_S}$ is an interval, then f is either essentially positive or essentially negative on $\mathbf{J_S}$.*

To see an example of a nonnegative function $f \in L^1(\mathbb{R})$ which is not essentially positive on its support take as f the characteristic function of an arbitrary Cantor set C of positive Lebesgue measure on the real.

We end this section with a result which allows us to approximate elements from the space $Sol(\mathsf{R}_2')$.

THEOREM 4.6 (see [147]; cf. [34, 70, 190]). *Assume that $f_0 \in Sol(\mathrm{R}_2')$, $\|f_0\|_1 = 1$ and $f_0 > 0$ on $\mathbf{J}$. If f is a density with support contained in $\mathbf{J}$, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} M^n f = f_0,$$

where M is the Foias operator connected with equation (R_2').

4.4. Necessary and sufficient conditions for the existence of nontrivial integrable solutions. The main purpose of this section is to give necessary and sufficient conditions for $\dim Sol(\mathsf{R}_2')= 1$. However, we begin with two results which give sufficient conditions for $\dim Sol(\mathsf{R}_2')= 0$ and for $\dim Sol(\mathsf{R}_2')= 1$. The first one is closely related to self-similar measures (see e.g. [157]).

THEOREM 4.7 (see [147, 148]).

(i) *If $\sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon}\log c_{n,\varepsilon} > 0$, then $\dim Sol(\mathsf{R}_2')= 0$.*
(ii) *If $\sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon}\log c_{n,\varepsilon} = 0$ and $c_{n,\varepsilon} \neq 1$ for some $(n,\varepsilon) \in \mathbf{S}$, then $\dim Sol(\mathsf{R}_2')= 0$.*
(iii) *If $c_{n,\varepsilon} = 1$ for all $(n,\varepsilon) \in \mathbf{S}$, then $\dim Sol(\mathsf{R}_2')= 1$ if and only if $l_1(\mathbf{J_S}) > 0$.*

Condition $l_1(\mathbf{J_C}) > 0$ in assertion (iii) of Theorem 4.7 can be replaced by the so-called open set condition, which reads as follows: There exists an open set $U \subset \mathbf{J}$ such that $T_{n,\varepsilon}(U) \subset U$ and $T_{n,\varepsilon}(U) \cap T_{m,\eta}(U) = \emptyset$ for $(n,\varepsilon),(m,\eta) \in \mathbf{S}$ with $(n,\varepsilon) \neq (m,\eta)$ (see [157]). From [156] it follows that if $l_1(\mathbf{J_S}) > 0$, then $\operatorname{int}\mathbf{J_S} \neq \emptyset$ and $\operatorname{cl}(\operatorname{int}\mathbf{J_S}) = \mathbf{J_S}$. In consequence, $l_1(\mathbf{J_S}) > 0$ if and only if the set $\operatorname{int}\mathbf{J_S}$ is nonempty and satisfies the open set condition. Although we can write $\mathbf{J_S} = \left\{\sum_{l\in\mathbb{N}} \frac{\varepsilon_1\dots\varepsilon_j}{k^l} n_l : (n_l,\varepsilon_l)_{l\in\mathbb{N}} \in \mathbf{S}^{\mathbb{N}}\right\}$ it is very difficult to determine the Lebesgue measure of the set $\mathbf{J_S}$ in the general situation. For example, from [147] it can be derived that the refinement equation

$$f(x) = f(2\varepsilon x - n) + f(2\eta x - m)$$

has a nontrivial L^1-solution if and only if $\varepsilon+\eta\neq 0$ or $m+3n\neq 0$.

To formulate the second result assume that $k\geq 2$ is an integer number.

THEOREM 4.8 (see [101, 147]).

(i) *If*

$$\sum_{n\in\mathbb{Z}} c_{kn+j,\varepsilon}=\sum_{n\in\mathbb{Z}} c_{kn,\varepsilon} \quad \textit{for all } j\in\{1,\dots,k-1\} \textit{ and } \varepsilon\in\{-1,1\}, \tag{4.4.1}$$

then $\dim Sol(\mathsf{R}_2')=1$.

(ii) *If*

$$\sum_{n\in\mathbb{Z}}(c_{kn+j,-1}+c_{kn+j,1})=1 \quad \textit{for every } j\in\{0,\dots,k-1\}, \tag{4.4.2}$$

then $\dim Sol(\mathsf{R}_2')=1$.

In fact, condition (4.4.1) implies condition (4.4.2), but assertion (i) can be extended to the case of equation (R_2) with infinitely many positive $c_{n,\varepsilon}$'s. Clearly, both conditions (4.4.1) and (4.4.2) imply condition (c). In the case where $k=2$ and $c_{n,-1}=0$ for all $n\in\mathbb{Z}$, i.e. in the case of equation (R_3), condition (4.4.1), as well as condition (4.4.2), reduces to the first sum rule condition (3.8.1). Thus Theorem 4.8 generalizes the well known result (having many essentially different proofs) saying that equation (R_3) has a nontrivial L^1-solution under the first sum rule condition (see e.g. [39, 121, 145, 168, 191, 192, 206]). Let us mention that without the nonnegativity assumption the first sum rule condition is not sufficient for the existence of L^1-solutions even for equation (R_3) with finitely many terms (see [37]).

Now, let us pass to necessary and sufficient conditions for $\dim Sol(\mathsf{R}_2')=1$. Obviously, we may characterize the dimension of the space $Sol(\mathsf{R}_2')$ by the existence of an absolutely continuous distributional fixed point of a suitable random affine map (see Proposition 3.5) or via the representation from Proposition 4.1. We will not explain this approach. An interesting criterion for the existence of nontrivial L^1-solutions of equation (R_3) with finitely many nonzero terms can be found in [82, 88, 116, 121, 192]. That criterion uses the joint spectral radius introduced in [172]. Unfortunately, it is inapplicable in the general situation, because we do not know a simple way to compute the joint spectral radius. For more details on joint spectral radius and discussions of this aspect see e.g. [20, 44, 75, 95, 166, 167, 188, 212] (cf. also [183] and the references given there).

A nice criterion for the existence of nontrivial L^1-solutions of equation (R_3) with nonnegative c_n's can be found in [168] (see also [39]). This criterion uses the notion of blocking sets. Let us introduce this notation. Fix an integer number $k\geq 2$. Put $V_0=\{2\pi\}$, $V_N=\big\{2\pi\sum_{n=1}^N d_n/k^n : d_1,\dots,d_{N-1}\in\{0,\dots,k-1\},\ d_N\in\{1,\dots,k-1\}\big\}$ for every $N\in\mathbb{N}$, $V=\bigcup_{N\in\mathbb{N}_0}V_N$, $E=\big\{(v,w):\exists N\in\mathbb{N}_0\,\exists j\in\{0,\dots,k-1\} v\in V_N,\ w=\frac{v+j}{k}\big\}$. It is easy to see that the pair $T=(V,E)$ forms a tree of order k with the root 2π. An infinite path from the root $v_0=2\pi$ is a sequence $(v_n)_{n\in\mathbb{N}_0}$ such that $v_n\in V_n$ and $(v_n,v_{n+1})\in E$ for every $n\in\mathbb{N}_0$. A subset $\mathcal{V}$ of vertices of the tree T is said to be *blocking* if $2\pi\notin\mathcal{V}$, $v\in\mathcal{V}$ if and only if $2\pi-v\in\mathcal{V}$, and any infinite path from the root 2π includes exactly one element of $\mathcal{V}$. It is easy to show that every blocking set is finite.

The functions $m_{-1}, m_1 : \mathbb{R} \to \mathbb{C}$ given by $m_\varepsilon(t) = \sum_{(n,\varepsilon)\in\mathbf{S}} c_{n,\varepsilon} e^{-int}$ are said to be the *masks* or the *characteristic functions* of equation (R_2').

Our first characterization for $\dim Sol(\mathsf{R}_2')= 1$ generalizes results from [39, 168].

THEOREM 4.9 (see [101]). *Fix $\varepsilon \in \{-1, 1\}$ and assume that $m_\varepsilon(0) = 0$. Let*

$$p = \max\{n \in \mathbb{Z} : c_{n,-\varepsilon} \neq 0\} - \min\{n \in \mathbb{Z} : c_{n,-\varepsilon} \neq 0\}.$$

Then $\dim Sol(\mathsf{R}_2')= 1$ if and only if the tree T has a blocking set of cardinality less than or equal to p consisting of roots of equation $m_{-\varepsilon}(t) = 0$.

The following example shows the usefulness of Theorem 4.9.

EXAMPLE 4.10. Consider the refinement equation

$$f(x) = \sum_{(n,-1)\in\mathbf{S}} c_{n,-1} f(-2x - n) \tag{4.4.3}$$

and assume that $\sum_{n\in\mathbb{Z}} c_{2n,-1} = \sum_{n\in\mathbb{Z}} c_{2n+1,-1} = 1$. Then $m_1(0) = 0$, $p \geq 2$ and $m_{-1}(\pi) = \sum_{(n,-1)\in\mathbf{S}} c_{n,-1} e^{-in\pi} = \sum_{n\in\mathbb{Z}} c_{2n,-1} - \sum_{n\in\mathbb{Z}} c_{2n+1,-1} = 0$. Since the set $\{\pi\}$ is blocking, we conclude, by Theorem 4.9, that equation (4.4.3) has exactly one (up to a multiplicative constant) L^1-solution. Observe that the same conclusion follows immediately from Theorem 4.8.

Let us cite one more result from [101], where similar types of characterization for $\dim Sol(\mathsf{R}_2)= 1$ are included.

THEOREM 4.11 (see [101]). *Let $p = \max\{n \in \mathbb{Z} : c_{n,1} \neq 0\} - \min\{n \in \mathbb{Z} : c_{n,1} \neq 0\}$.*

(i) *If $c_{n,1} = c_{n,-1}$ for every $n \in \mathbb{Z}$, then $\dim Sol(\mathsf{R}_2')= 1$ if and only if the tree T has a blocking set of cardinality less than or equal to p consisting of roots of equation*

$$m_{-1}(-t) + m_1(t) = 0. \tag{4.4.4}$$

(ii) *If $c_{n,1} = c_{-n,-1}$ for every $n \in \mathbb{Z}$ and if $\sum_{n\in\mathbb{Z}} c_{kn+j,1} = \sum_{n\in\mathbb{Z}} c_{kn-j,1}$ for every $j \in \{1, \dots, k-1\}$, then $\dim Sol(\mathsf{R}_2')= 1$ if and only if the tree T has a blocking set of cardinality less than or equal to $2p$ consisting of roots of equation* (4.4.4).

5. Extremely non-measurable solutions of refinement type equations. We end this paper with a few words on the existence of extremely non-measurable solutions of refinement type equations.

Assume that $\dim Sol(\mathsf{R}_2)= 1$. Then fix a nonnegative $f \in Sol(\mathsf{R}_2)$ such that $\|f\|_1 = 1$ and choose a representative f_0 of f which is nonnegative everywhere and satisfies equation (R_2) for every $x \in \mathbb{R}$; this is always possible. Next fix $a, b \in [-\infty, +\infty]$ with $a < b$ and put

$$B_a^b = \big\{(x, y) \in \mathbb{R}^2 : af_0(x) \leq y \leq bf_0(x)\big\}.$$

Obviously, $l_2(B_a^b) = b - a > 0$. Now from [8] it follows that there exists a function $g : \mathbb{R} \to \mathbb{R}$ satisfying (R_2) for every $x \in \mathbb{R}$ such that the set $B_a^b \setminus \mathrm{Graph}(g)$ contains neither subset of B_a^b of second category having the property of Baire nor subset of B_a^b of positive inner measure on the plane; we will call such a function *extremely non-measurable*. Moreover, if f_0 is continuous, then the set $\mathrm{Graph}(g)$ is connected. Thus we can formulate the following proposition.

Proposition 5.1 (see [147]; cf. [70]). *If* $\dim Sol(\mathsf{R}_2)= 1$*, then there exist extremely non-measurable solutions of equation* (R_2)*. Moreover, if there exists a continuous function* $f \in Sol(\mathsf{R}_2)$*, then there exist extremely non-measurable solutions of equation* (R_2) *with connected graphs.*

Similar results in this direction can be found in [9] (see also [21, 143]).

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