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A REVIEW OF SELECTED TOPICS IN MAJORIZATION THEORY

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In memory of Professor Ky Fan

Abstract. In this expository paper, some recent developments in majorization theory are reviewed. Selected topics on group majorizations, group-induced cone orderings, Eaton triples, normal decomposition systems and similarly separable vectors are discussed. Special attention is devoted to majorization inequalities. A unified approach is presented for proving majorization relations for eigenvalues and singular values of matrices. Some methods based on the Chebyshev functional and similarly separable vectors are described. Generalizations of Hardy–Littlewood–Pólya Theorem and Schur–Ostrowski Theorem are presented. Generalized Schur-convex functions are investigated. Extensions of Ky Fan inequalities are provided. Applications to Grüss and Ostrowski type inequalities are given.

1. Introduction and summary. The theory of majorization has many various applications in a number of fields, including matrix theory, convex analysis, probability, statistics, geometry, Lie theory, numerical analysis, optimization, etc. (see [2, 3, 4, 15, 16, 17, 18, 35, 44, 45, 51, 52, 70, 71]). The interested reader may consult the book *Inequalities: Theory of Majorization and its Applications* by W. A. Marshall, I. Olkin and B. C. Arnold [39] for the richness of applications in diverse disciplines (see also [5, 6, 27]).

In the literature, special attention is paid to majorization inequalities for linear operators (see [2, 3, 4, 45, 51, 52, 42, 69]). The importance of such results lies in the fact

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that they are related to eigenvalues and singular values of the operators. On the other hand, many existing results depend on the positivity of certain functionals. The aim of this survey article is to demonstrate two general methods for generalizing some classical majorization results. The first is based on the theory of Eaton triples and the second relies on the generalized Chebyshev functional and similarly separable vectors.

In this expository paper, we review some recent developments in the majorization theory. We focus on group majorizations, group-induced cone orderings, normal decomposition systems (ND systems), Eaton triples (E-systems) and similarly separable vectors. We quote the relevant material mainly from [48, 50, 51, 54, 55, 57, 56, 58, 59, 60].

The outline of the paper is as follows. In Section 2 we collect some basic facts from group majorization theory (see [15, 16, 18, 24, 34, 35, 42, 45, 46, 68, 70, 71]). The results are presented in Sections 3–8.

In Section 3 we offer a unified approach to the problem of establishing majorization inequalities concerning eigenvalues and singular values of matrices [45, 47, 51, 52, 53]. In particular, we show a relationship between such inequalities and decomposition statements for matrices as Spectral Decomposition, Singular Value Decomposition Theorem, Autonne Decomposition and Takagi Decomposition [45, 51].

Section 4 contains a discussion of morphisms of E-systems. Firstly, the motivation for this notion is provided. Namely, morphisms are characterized by a G-majorization inequality of the mentioned type. Secondly, a method for constructing morphisms is pointed out via simple morphisms. Lastly, homomorphisms are employed to give a technique of construction of Eaton triples.

Section 5 is devoted to the Chebyshev functional [50, 61]. By making use of similarly separable vectors, some sufficient and necessary conditions are provided for the functional to be nonnegative [48, 50, 57, 60]. The class of separable vectors on $\mathbb{R}^n$ includes many important subclasses such as monotone, monotone in mean, star-shaped, convex n-tuples, etc.

In Section 6 the similarity method is utilized to show some generalizations of Hardy–Littlewood–Pólya Theorem and of Schur–Ostrowski Theorem [57, 59]. Additionally, extended group majorization is studied.

Further applications are given in Section 7. Here Shi type inequalities are investigated [58]. As corollaries, some G-majorization extensions of Ky Fan inequality are presented.

Finally, in Section 8, a new approach to Grüss and Ostrowski type inequalities is shown [57]. By replacing the standard bounding constants by some corresponding bounding functions, some tighter estimates can be provided.

2. Preliminaries

2.1. Majorization and Schur-convex functions. We begin with some notation and terminology.

The *decreasing rearrangement* of a vector $z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$ is defined by

$$z_\downarrow = (z_{[1]}, z_{[2]}, \dots, z_{[n]}),$$

where $z_{[i]}$ denotes the ith largest entry of z, $i = 1, 2, \dots, n$. Thus $z_{[1]} \geq z_{[2]} \geq \dots \geq z_{[n]}$ are the entries of z in decreasing order.

DEFINITION 2.1 (Weak majorization). A vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is said to be *weakly majorized* by a vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, in symbols $x \prec_w y$, if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \qquad \text{for } k = 1, 2, \dots, n. \tag{1}$$

The preorder $\prec_w$ on $\mathbb{R}^n$ is called *weak majorization* [39, p. 12].

DEFINITION 2.2 (Majorization). A vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is said to be *majorized* by a vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, in symbols $x \prec y$, if (1) holds and, in addition,

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

The preorder $\prec$ on $\mathbb{R}^n$ is called *majorization* [39, p. 8].

Below we review some basic properties of the majorization preorder $\prec$. Thus we give motivation for the notion of *group-induced cone ordering* defined and described in the next subsection.

A geometric interpretation of majorization is as follows.

THEOREM 2.3 ([64]). *Let $x, y \in \mathbb{R}^n$. Then*

$$x \prec y \quad \textit{iff} \quad x \in \operatorname{conv} \mathbb{P}_n\, y,$$

where $\mathbb{P}_n$ is the permutation group acting on $\mathbb{R}^n$, and $\operatorname{conv} \mathbb{P}_n\, y$ stands for the convex hull of the set $\mathbb{P}_n\, y = \{py : p \in \mathbb{P}_n\}$.

It is easily seen that the set $D = (\mathbb{R}^n)_\downarrow = \{z_\downarrow : z \in \mathbb{R}^n\}$ is a convex cone in $\mathbb{R}^n$. The following two properties (A1)–(A2) are met:

(A1) $D \cap \mathbb{P}_n z$ is not empty for each $z \in \mathbb{R}^n$, i.e., $\mathbb{R}^n = \bigcup_{p \in \mathbb{P}_n} pD$,
(A2) the *rearrangement inequality* holds:

$$\langle x, py \rangle \le \langle x, y \rangle \qquad \text{for } x, y \in D \text{ and } p \in \mathbb{P}_n,$$

where $\langle \cdot, \cdot \rangle$ means the standard inner product on $\mathbb{R}^n$.

Condition (A1) means that each $z \in \mathbb{R}^n$ has the decomposition

$$z = p z_\downarrow \qquad \text{for some } p \in \mathbb{P}_n.$$

On the other hand, by employing the standard norm $\|z\| = \langle z, z \rangle^{1/2}$ for $z \in \mathbb{R}^n$, condition (A2) can be restated as

$$\|x - y\| \le \|x - py\| \qquad \text{for } x, y \in D \text{ and } p \in \mathbb{P}_n,$$

or, equivalently,

$$\|x - y\| = \min_{p \in \mathbb{P}_n} \|x - py\| \qquad \text{for } x, y \in D.$$

Properties of majorization $\prec$ depend on the geometry of the cone D and its *dual cone*

$$\operatorname{dual} D = \{w \in \mathbb{R}^n : \langle w, z \rangle \ge 0 \qquad \text{for } z \in D\}.$$

For example, for $z \in D$ one has $z_\downarrow = z$ and $z_{[i]} = z_i$, $i = 1, 2, \dots, n$. Therefore, if $x, y \in D$ then Definition 2.2 simplifies to

$$x \prec y \quad \text{iff} \quad \sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i \qquad \text{for } k = 1, 2, \dots, n, \qquad \text{and} \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

In other words, for $x, y \in D$,

$$x \prec y \quad \text{iff} \quad \langle x, s_k \rangle \le \langle y, s_k \rangle \qquad \text{for } k = 1, 2, \dots, n, n+1,$$

where

$$s_k = (\underbrace{1, \dots, 1}_{k \text{ times}}, \underbrace{0, \dots, 0}_{n-k \text{ times}}) \qquad \text{for } k = 1, 2, \dots, n, \qquad \text{and} \qquad s_{n+1} = -s_n = (\underbrace{-1, \dots, -1}_{n \text{ times}})$$

are the vectors spanning the convex cone D, i.e.,

$$D = \operatorname{cone}\{s_k : k = 1, 2, \dots, n, n+1\}.$$

Therefore the majorization preorder $\prec$, restricted to the convex cone D, is a *cone preorder* induced by the dual cone of D, that is, for $x, y \in D$,

$$x \prec y \quad \text{iff} \quad y - x \in \operatorname{dual} D.$$

Here

$$\operatorname{dual} D = \operatorname{cone}\{r_k : k = 1, 2, \dots, n-1\},$$
$$r_k = (\underbrace{0, \dots, 0}_{k-1 \text{ times}}, 1, -1, \underbrace{0, \dots, 0}_{n-k-1 \text{ times}}) \qquad \text{for } k = 1, 2, \dots, n-1.$$

The above symbol $\operatorname{cone} A$ denotes the convex cone of all nonnegative finite linear combinations of vectors in subset A of a linear space.

DEFINITION 2.4 (Schur-convexity, Schur-concavity). Let $A \subset \mathbb{R}^n$ be a (nonempty) symmetric set (i.e., $px \in A$ whenever $x \in A$ and $p \in \mathbb{P}_n$).

A function $f : A \to \mathbb{R}$ is said to be *Schur-convex* (resp. *Schur-concave*) on A, if for $x, y \in A$,

$$x \prec y \qquad \text{implies} \qquad f(x) \le (\ge) f(y).$$

Schur-convex (resp. Schur-concave) functions on $A = \mathbb{R}^n$ are simply called Schur-convex (resp. Schur-concave). For abbreviation, Schur-convex (resp. Schur-concave) functions are called *S-convex* (resp. *S-concave*).

It is well-known that a Schur-convex function $f : A \to \mathbb{R}$ is necessarily *symmetric*, i.e.,

$$f(px) = f(x) \qquad \text{for } x \in A \text{ and } p \in \mathbb{P}_n.$$

2.2. *G*-majorization and group-induced cone orderings. Throughout $(V, \langle \cdot, \cdot \rangle)$ is a finite-dimensional real inner product space. The symbol $O(V)$ stands for the orthogonal group acting on V.

DEFINITION 2.5 (Group majorization). Let G be a subgroup of $O(V)$. Given vectors $x, y \in V$, we write $x \prec_G y$ if x lies in the convex hull $\operatorname{conv} Gy$ of the G-orbit $Gy = \{gy : g \in G\}$, i.e.,

$$x \prec_G y \quad \text{iff} \quad x \in \operatorname{conv} Gy.$$

The preorder $\prec_G$ is called *group majorization with respect to* G, in short, G*-majorization* (see [15], [39, p. 589]).

For instance, if $G = \mathbb{P}_n$ is the group of $n \times n$ permutation matrices acting on $V = \mathbb{R}^n$, then $\prec_G$ is the majorization preorder $\prec$ on $\mathbb{R}^n$ [39, p. 10, p. 34, p. 162].

If $x \prec_G y$ and $y \prec_G x$ then we write $x \equiv_G y$.

The G-majorization $\prec_G$ is G-invariant on V in the sense that for any $x, y \in V$,

$$x \prec_G y \qquad \text{iff} \qquad g_1 x \prec_G g_2 y \quad \text{for } g_1, g_2 \in G.$$

Definition 2.6 (GIC ordering). G-majorization $\prec_G$ induced by a compact group $G \subset O(V)$ is said to be a *group-induced cone ordering* (in short, *GIC ordering*) if there exists a closed convex cone $D \subset V$ such that

(A1) $D \cap Gz$ is not empty for each $z \in V$,
(A2) $\langle x, gy \rangle \leq \langle x, y \rangle$ for $x, y \in D$ and $g \in G$.

Any GIC ordering $\prec_G$, restricted to its convex cone D, is the cone ordering on D induced by $C = \operatorname{dual} D$. That is, for $x, y \in D$, the following statements are equivalent:

(i) $x \prec_G y$,
(ii) $\langle y - x, s \rangle \geq 0$ for $s \in D$,
(iii) $\langle y - x, s_i \rangle \geq 0$ for $i = 1, \ldots, k$, provided $D = \operatorname{cone}\{s_1, s_2, \ldots, s_k\}$.

Definition 2.7 (G-increasing function). Let $A \subset V$ be a G-invariant set, i.e., $gx \in A$ whenever $x \in A$ and $g \in G$.

A function $f : A \to \mathbb{R}$ is said to be G*-increasing* on A, if for $x, y \in A$,

$$x \prec_G y \qquad \text{implies} \qquad f(x) \leq f(y).$$

G-increasing functions $f : V \to \mathbb{R}$ on $A = V$ are simply called G*-increasing.*

Definition 2.8 (G-invariant function). A function $f : A \to \mathbb{R}$ is said to be G*-invariant*, if A is G-invariant and

$$f(gx) = f(x) \qquad \text{for } g \in G \text{ and } x \in A.$$

Each G-increasing function is necessarily G-invariant.

2.3. Normal decomposition systems and Eaton triples. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real inner product space and let $G \subset O(V)$ be a closed group acting on V.

If $\prec_G$ is a GIC ordering then for each $x \in V$ the intersection $D \cap Gx$ is a singleton set consisting of the unique vector denoted by $x_\downarrow$ [45, p. 14].

Definition 2.9 (Normal map). If axioms (A1)–(A2) hold for some closed convex cone $D \subset V$, then the map

$$(\cdot)_\downarrow : V \ni x \to x_\downarrow \in D,$$

where $\{x_\downarrow\} = D \cap Gx$, is called *normal map.*

The map $(\cdot)_\downarrow$ is G-invariant and idempotent. Its range is the convex cone D. The restriction of $(\cdot)_\downarrow$ to D is the identity. For each $x \in V$, the vectors $x_\downarrow$ and x are equivalent in the sense that $x_\downarrow \prec_G x$ and $x \prec_G x_\downarrow$.

Under axioms (A1)–(A2), for $x, y \in V$ we have

$$\begin{aligned} y \prec_G x \quad &\text{iff} \quad y_\downarrow \prec_G x_\downarrow \quad \text{iff} \quad \langle z, y_\downarrow \rangle \le \langle z, x_\downarrow \rangle \quad \text{for} \quad z \in D, \\ y \prec_G x \quad &\text{iff} \quad \langle z, gy \rangle \le \langle z, x_\downarrow \rangle \quad \text{for} \quad z \in D \quad \text{and} \quad g \in G. \end{aligned} \tag{2}$$

In light of (A1)–(A2), each vector x in V has its *canonical* (*normal*) *decomposition*:

$$x = gx_\downarrow \quad \text{for some} \quad g \in G.$$

Definition 2.10 (Normal decomposition system). If axioms (A1)–(A2) hold for closed convex cone $D \subset V$ then the triple $(V, G, (\cdot)_\downarrow)$ is called a *normal decomposition system* (in short, ND system) [34, 35].

Definition 2.11 (Eaton triple). If axioms (A1)–(A2) hold for closed convex cone $D \subset V$ then the triple (V, G, D) is called an *Eaton triple* (in short, E-system) [15, 16, 70].

The above-mentioned notions play a unifying role in statistics, matrix theory, Lie theory, etc. (see [15, 18, 34, 35, 42, 47, 70]).

2.4. Examples of Eaton triples. An important class of examples is provided by finite reflection groups according to the following definition and theorem.

Definition 2.12 (Reflection group). A group $G \subset O(V)$ is said to be a *reflection group* if G is the closure of a subgroup of $O(V)$ generated by some set of the reflections

$$S_r x = x - 2\,\frac{\langle x, r\rangle}{\langle r, r\rangle}\, r \qquad \text{for } x \in V,$$

where $0 \ne r \in V$ [18, 30].

Theorem 2.13 ([18, Lemma 4.1], [68, Theorem 4.1]). *Let $G \subset O(V)$ be a finite group. Then the following three statements are equivalent*:

(i) *G is a finite reflection group.*
(ii) *The G-majorization $\prec_G$ is a GIC ordering for some closed convex cone $D \subset V$.*
(iii) *The triple (V, G, D) is an Eaton triple for some closed convex cone $D \subset V$.*

We need some notation (see [51]).

$\mathbb{R}^n_+$ = the convex cone of nonnegative vectors in $\mathbb{R}^n$,
$\mathbb{R}^n_\downarrow$ = the convex cone of nonincreasing vectors in $\mathbb{R}^n$,
$\mathbb{R}^n_{+\downarrow}$ = the convex cone of nonnegative nonincreasing vectors in $\mathbb{R}^n$,
$\mathbb{O}_n$ = the group of $n \times n$ real orthogonal matrices,
$\mathbb{DO}_n$ = the group of $n \times n$ diagonal orthogonal matrices,
$\mathbb{P}_n$ = the group of $n \times n$ permutation matrices,
$\mathbb{GP}_n(\mathbb{R})$ = the group of $n \times n$ generalized permutation matrices, i.e., real matrices with exactly one nonzero entry with absolute value 1 in each row and column.

Example 2.14 ([15, p. 16]). It is easily seen that (V, G, D) is an Eaton triple for $V = \mathbb{R}^n$, $G = \mathbb{DO}_n$ and $D = \mathbb{R}^n_+$. Moreover, G is a finite reflection group, and $y \prec_G x$ means $|y| \le |x|$ for $x, y \in \mathbb{R}^n$, where $x_\downarrow = |x| = (|x_1|, \dots, |x_n|)^T$.

Example 2.15 ([15, p. 16]). It is known that if $V = \mathbb{R}^n$, $G = \mathbb{P}_n$ and $D = \mathbb{R}^n_\downarrow$, then (V, G, D) is an Eaton triple, G is a finite reflection group, and $y \prec_G x$ means $y \prec x$ for $x, y \in \mathbb{R}^n$. Furthermore, $x_\downarrow = (x_{[1]}, \dots, x_{[n]})^T$.

EXAMPLE 2.16 ([15, p. 16]). Replacing V, G, and D with $\mathbb{R}^n$, $\mathbb{GP}_n(\mathbb{R})$ and $\mathbb{R}^n_{+\downarrow}$, respectively, one obtains Eaton triple (V, G, D) with finite reflection group G. Here $y \prec_G x$ reduces to $|y| \prec_w |x|$ for $x, y \in \mathbb{R}^n$, and, in addition, $x_\downarrow = (|x|_{[1]}, \ldots, |x|_{[n]})^T$.

In forthcoming matrix examples, we need further notation (for the field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$).

$\mathbb{M}_n(\mathbb{F})$ = the vector space of $n \times n$ matrices over $\mathbb{F}$,
$\mathbb{H}_n$ = the (real) vector space of $n \times n$ Hermitian matrices,
$\mathbb{S}_n(\mathbb{F})$ = the vector space of $n \times n$ symmetric matrices over $\mathbb{F}$,
$\mathbb{K}_n(\mathbb{F})$ = the vector space of $n \times n$ skew-symmetric matrices over $\mathbb{F}$,
$\mathbb{D}_n(\mathbb{F})$ = the vector space of $n \times n$ diagonal matrices over $\mathbb{F}$,
$\mathbb{GP}_n(\mathbb{F})$ = the group of $n \times n$ generalized permutation matrices over $\mathbb{F}$, i.e., matrices with exactly one nonzero entry with magnitude 1 in each row and column,
$\mathbb{U}_n$ = the group of $n \times n$ unitary matrices,
$\mathbb{DU}_n$ = the group of $n \times n$ diagonal unitary matrices,
X^* = the conjugate transpose of matrix X,
$\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))$ = the vector of eigenvalues of Hermitian matrix X stated in nonincreasing order: $\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(X)$,
$s(X) = (s_1(X), s_2(X), \ldots, s_n(X))$ = the vector of singular values of matrix X, i.e., $s(X) = \lambda(X^*X)^{1/2}$ with $s_1(X) \geq s_2(X) \geq \ldots \geq s_n(X)$,
$d(X)$ = the vector of diagonal entries of matrix X,
$\operatorname{diag} z$ = the diagonal matrix with the entries of a vector $z \in \mathbb{R}^n$ on the main diagonal,
$\operatorname{Re} X$ = the real part of matrix $X = (x_{i,j})$, i.e., $\operatorname{Re} X = (\operatorname{Re} x_{i,j})$,
$U_1(\cdot)U_2$ = the matrix operator of the form $X \to U_1XU_2$, where X, U_1 and U_2 are matrices.

We now present two important examples of Eaton triples related to the eigenvalues of an Hermitian matrix and to the singular values of a complex matrix, respectively.

EXAMPLE 2.17 (see [15, p. 17], [35, pp. 943–944], [55, p. 619]). Consider $V = \mathbb{H}_n$ with the inner product defined by

$$\langle X, Y \rangle = \operatorname{Re} \operatorname{tr} XY \qquad \text{for } X, Y \in \mathbb{H}_n.$$

Let G be the group of all linear operators of the form

$$X \to UXU^* \qquad \text{for } X \in \mathbb{H}_n \text{ with } U \in \mathbb{U}_n.$$

Then (V, G, D) is an E-system for

$$D = \{\operatorname{diag}(z_1, \ldots, z_n) \in \mathbb{D}_n : z_1 \geq \ldots \geq z_n\}.$$

Indeed, axiom (A1) reduces to the Spectral Theorem, and axiom (A2) becomes the Fan–Theobald's trace inequality [19, 72]. Furthermore, we have

$$X_\downarrow = \operatorname{diag} \lambda(X) \text{ for } X \in \mathbb{H}_n,$$
$$Y \prec_G X \text{ iff } \lambda(Y) \prec \lambda(X) \text{ for } X, Y \in \mathbb{H}_n$$

(see (2) and [15, p. 17]). In consequence, $\prec_G$ on $\mathbb{D}_n(\mathbb{R})$ may be identified with the classical majorization $\prec$ on $\mathbb{R}^n$.

Schur's inequality ([66], [39, p. 300]) for an $n \times n$ Hermitian matrix X says that

$$d(X) \prec \lambda(X). \tag{3}$$

Example 2.18 (see [15, pp. 17–18], [35, pp. 944–945], [55, p. 619]). Take $V = \mathbb{M}_n(\mathbb{C})$ with real inner product given by

$$\langle X, Y\rangle = \operatorname{Re}\operatorname{tr} XY^* \qquad \text{for } X, Y \in \mathbb{M}_n(\mathbb{C}),$$

and let G be the group of all linear operators of the form

$$X \to U_1 X U_2 \qquad \text{for } X \in \mathbb{M}_n(\mathbb{C}), \text{ where } U_1, U_2 \in \mathbb{U}_n.$$

Put

$$D = \{\operatorname{diag}(z_1, \ldots, z_n) \in \mathbb{D}_n : z_1 \geq \ldots \geq z_n \geq 0\}.$$

Here (A1) is the Singular Values Decomposition Theorem [39, p. 771], and (A2) is the trace inequality of von Neumann [39, p. 789]. So, (V, G, D) is an E-system. In addition, we have

$$X_\downarrow = \operatorname{diag} s(X) \qquad \text{for } X \in \mathbb{M}_n(\mathbb{C}),$$
$$Y \prec_G X \quad \text{iff} \quad s(Y) \prec_w s(X) \quad \text{for } X, Y \in \mathbb{M}_n(\mathbb{C})$$

(see (2) and [15, pp. 17–18]).

It is well-known by Ky Fan's inequality [20] that for an $n \times n$ complex matrix X,

$$d(X) \prec_w s(X). \tag{4}$$

3. G-majorization inequalities for linear operators. In this section we aim to show some majorization inequalities generated by certain linear operators. To this end we employ Eaton triples and normal maps. Such an approach gives a better understanding of many results on eigenvalues and singular values of certain classes of matrices.

As previously, it is assumed that $(V, \langle\cdot,\cdot\rangle)$ is an inner product space, G is a closed subgroup of the orthogonal group $O(V)$, and $D \subset V$ is a closed convex cone.

3.1. G-majorization and orthoprojectors. We begin our discussion with G-majorization inequality (6) which generalizes, among others, Schur's inequality (3) and Ky Fan's inequality (4).

Theorem 3.1 ([51, Theorem 2.1]). *Let (V, G, D) be an Eaton triple. Assume W is a linear subspace of V, H is a subset of G, and E is a subset of D. Let P be the orthoprojector from V onto W and Q be the orthoprojector from* span D *onto* span E.

If $QD = E$ and

$$W = \bigcup_{h \in H} hE, \tag{5}$$

then

$$Px \prec_G Qx_\downarrow \qquad \textit{for} \quad x \in V. \tag{6}$$

Example 3.2. Let V, G and D be defined as in Example 2.17. Setting

$$W = \mathbb{S}_n(\mathbb{R}), \qquad H = \{U(\cdot)U^T : U \in \mathbb{O}_n\} \qquad \text{and} \qquad E = D$$

(cf. [34, Example 7.4]), we obtain $PX = \operatorname{Re} X$ for $X \in \mathbb{H}_n$, and Q is the identity on span D.

Condition (5) is fulfilled by Spectral Decomposition for matrices in $\mathbb{S}_n(\mathbb{R})$. From (6) we deduce that (cf. [2, p. 111])

$$\lambda(\operatorname{Re} X) \prec \lambda(X) \qquad \text{for} \quad X \in \mathbb{H}_n.$$

EXAMPLE 3.3. Assume V, G and D are defined as in Example 2.18. For $k \in \{1, 2, \ldots, n\}$, let

$$W = \mathbb{M}_k(\mathbb{C}) \oplus 0_{n-k}, \qquad H = \{U_1(\cdot)U_2 : U_1, U_2 \in \mathbb{U}_k \oplus I_{n-k}\},$$
$$\text{and} \qquad E = \{Z \in \mathbb{D}_k(\mathbb{R}) \oplus 0_{n-k} : z_{11} \geq \ldots \geq z_{kk} \geq 0\}.$$

Condition (5) amounts to Singular Value Decomposition for matrices in $\mathbb{M}_k(\mathbb{C})$. The orthoprojector P is given by

$$PX = X_{11} \oplus 0_{n-k} \qquad \text{for} \quad X \in \mathbb{M}_n(\mathbb{C}),$$

where X_{11} is the $k \times k$ principal submatrix of X. Additionally, Q is the restriction of P to $\mathbb{D}_n(\mathbb{R})$. Consequently, inequality (6) gives

$$s(X_{11}) \prec_w (s_1(X), \ldots, s_k(X)) \qquad \text{for} \quad X \in \mathbb{M}_n(\mathbb{C}).$$

EXAMPLE 3.4. Let $V = \mathbb{M}_n(\mathbb{R})$,

$$G = \{U_1(\cdot)U_2 : U_1, U_2 \in \mathbb{O}_n\}, \qquad D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_{+\downarrow}\}, \qquad W = \mathbb{K}_n(\mathbb{R}),$$
$$H = \{UU_0(\cdot)U^T : U \in \mathbb{O}_n\} \qquad \text{and} \qquad E = \{s_1 I_2 \oplus \ldots \oplus s_k I_2 : s_1 \geq \ldots \geq s_k \geq 0\},$$

where $n = 2k$ is even, $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, U_0 is the $n \times n$ block-diagonal matrix $B \oplus \ldots \oplus B$ with $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Condition (5) is satisfied by Autonne Decomposition (cf. [34, Example 7.5]).

It is clear that

$$PX = \frac{X - X^T}{2} \qquad \text{for} \quad X \in \mathbb{M}_n(\mathbb{R}),$$
$$QX = \frac{s_1 + s_2}{2} I_2 \oplus \ldots \oplus \frac{s_{n-1} + s_n}{2} I_2 \qquad \text{for} \quad X = \operatorname{diag}(s_1, \ldots, s_n) \in \mathbb{D}_n(\mathbb{R}).$$

By utilizing inequality (6) we obtain

$$s\Big(\frac{X - X^T}{2}\Big) \prec_w \Big(\frac{s_1 + s_2}{2}, \frac{s_1 + s_2}{2}, \ldots, \frac{s_{n-1} + s_n}{2}, \frac{s_{n-1} + s_n}{2}\Big)$$

for $X \in \mathbb{M}_n(\mathbb{R})$, where $s_i = s_i(X)$, $i = 1, 2, \ldots, n$, (cf. [2, p. 109]).

3.2. Reduced triple of an Eaton triple.

We introduce the notion of a reduced triple.

DEFINITION 3.5 (Reduced triple). Given an Eaton triple (V, G, D), set

$$V_0 = \operatorname{span} D \quad \text{and} \quad G_0 = \{g|_{V_0} : g \in G \quad \text{and} \quad gV_0 = V_0\}.$$

If (V_0, G_0, D) is an Eaton triple (under the inherited inner product), then it is called the *reduced triple* of (V, G, D) [70].

It is important that if (V_0, G_0, D) is the reduced triple of (V, G, D) then G_0 is a finite reflection group acting on V_0 and the following reduction holds:

$$y \prec_G x \quad \text{iff} \quad y_\downarrow \prec_G x_\downarrow \quad \text{iff} \quad y_\downarrow \prec_{G_0} x_\downarrow \qquad \text{for } x, y \in V$$

(see [45, Theorem 3.2], [18, Lemma 4.1, (35)], [68, Theorem 4.1]).

EXAMPLE 3.6. If V, G, D are as in Example 2.17, then the reduced triple is (V_0, G_0, D) for $V_0 = \mathbb{D}_n(\mathbb{R})$, $G_0 = \{S(\cdot)S^T : S \in \mathbb{P}_n\}$ and $D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_\downarrow\}$. In consequence, (V_0, G_0, D) can be identified with the E-system $(\mathbb{R}^n, \mathbb{P}_n, \mathbb{R}^n_\downarrow)$.

EXAMPLE 3.7. In the case of Example 2.18, the reduced triple (V_0, G_0, D) of (V, G, D) consists of $V_0 = \mathbb{D}_n(\mathbb{R})$,

$$G_0 = \{CS(\cdot)S^T : S \in \mathbb{P}_n,\ C \in \mathbb{DO}_n\} \text{ and } D = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_{+\downarrow}\}.$$

Additionally, the reduced triple can be identified with the triple $(\mathbb{R}^n, \mathbb{GP}_n(\mathbb{R}), \mathbb{R}^n_{+\downarrow})$.

THEOREM 3.8 ([51, Theorem 2.7]). *Let (V, G, D) and (W, H, E) be Eaton triples with reduced systems (V_0, G_0, D) and (W_0, H_0, E), respectively. Assume that $W \subset V$, $H \subset G$ and $D \subset E$. Let P stand for the orthoprojector from V onto W.*

If $W_0 = V_0$ then there exists a subset $\widehat{G} \subset G$ such that $W = \bigcup_{g \in \widehat{G}} gD$, and

$$Px \prec_G x_\downarrow \quad \text{for} \quad x \in V.$$

EXAMPLE 3.9. Let V, G and D be as in Example 2.18. Setting

$$W = \mathbb{H}_n, \quad H = \{U(\cdot)U^* : U \in \mathbb{U}_n\} \quad \text{and} \quad E = \{Z \in \mathbb{D}_n(\mathbb{R}) : d(Z) \in \mathbb{R}^n_\downarrow\}$$

leads to

$$PX = \frac{X + X^*}{2} \quad \text{for} \quad X \in \mathbb{M}_n(\mathbb{C}).$$

Application of Theorem 3.8 gives Fan–Hoffman's inequality (see [2, p. 109], [39, p. 327]):

$$s\Big(\frac{X + X^*}{2}\Big) \prec_w s(X) \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

EXAMPLE 3.10. Letting (V, G, D) to be as in Example 2.17, for $k \in \{1, 2, \dots, n\}$ we consider

$$W = \mathbb{H}_k \oplus \mathbb{H}_{n-k}, \qquad H = \{U(\cdot)U^* : U \in \mathbb{U}_k \oplus \mathbb{U}_{n-k}\}$$
$$\text{and} \qquad E = \{Z \in \mathbb{D}_n(\mathbb{R}) : z_{11} \geq \ldots \geq z_{kk},\ z_{k+1\,k+1} \geq \ldots \geq z_{nn}\}.$$

Then we obtain

$$PX = X_{11} \oplus X_{22} \qquad \text{for} \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbb{H}_n,$$

where X_{11} and X_{22} are of sizes $k \times k$ and $(n-k) \times (n-k)$, respectively.

By making use of Theorem 3.8 we get Ky Fan's inequality (see [39, p. 308], cf. also [2, p. 97]):

$$(\lambda(X_{11}), \lambda(X_{22})) \prec \lambda(X) \qquad \text{for} \quad X \in \mathbb{H}_n.$$

3.3. Subsystems of E-systems. We are interested in the notion of a subsystem of an E-system and its role in deriving G-majorization inequalities.

DEFINITION 3.11 (Subsystem of E-system). Let (V, G, D) be an E-system. Assume H is a closed subgroup of G, W is an H-invariant subspace of V, and E is a closed convex subcone of D. The triple (W, H, E) is called a *subsystem* of (V, G, D), if (W, H, E) is an E-system (under the inherited inner product). For notation simplicity, we write (W, H, E) in place of $(W, H|_W, E)$ (cf. [34, 35]).

A special class of subsystems is formed by the reduced triples. In Theorem 3.12 we characterize arbitrary subsystems of an E-system (cf. [45, Theorem 3.2]).

THEOREM 3.12 ([51, Theorem 3.1]). *Let (V, G, D) be an E-system. Suppose H is a closed subgroup of G, W is an H-invariant subspace of V, and $E \subset W$ is a closed convex subcone of D. Let P be the orthoprojector from V onto W, and Q be the orthoprojector from* $\operatorname{span} D$ *onto* $\operatorname{span} E$. *Assume $QD = E$.*

Then the following statements are equivalent:

(a) *(W, H, E) is a subsystem of (V, G, D).*

(b) *The following inequality holds*

$$Px \prec_H Qx_\downarrow \qquad \text{for } x \in V. \tag{7}$$

(c) *The following inclusion holds*

$$W \subset \bigcup_{g \in G} gE,$$

and, in addition, the orderings $\prec_H$ and $\prec_G$ coincide on W, i.e.

$$y \prec_H x \quad \text{iff} \quad y \prec_G x \quad \text{for} \quad x, y \in W.$$

(d) *The following decomposition holds*

$$W = \bigcup_{h \in H} hE. \tag{8}$$

EXAMPLE 3.13. We take V, G and D to be as in Example 2.18, and we set

$$W = \mathbb{D}_n(\mathbb{C}), \quad H = \{CS(\cdot)S^T : S \in \mathbb{P}_n,\ C \in \mathbb{DU}_n\} \quad \text{and} \quad E = D.$$

Then (8) holds by Polar Decomposition for diagonal matrices. Moreover,

$$PX = \operatorname{diag}(x_{11}, \ldots, x_{nn}) \qquad \text{for} \quad X = (x_{ij}) \in \mathbb{M}_n(\mathbb{C}).$$

Applying (7) yields the following Ky Fan's result [39, p. 314]:

$$(|x_{11}|, \ldots, |x_{nn}|) \prec_w (s_1(X), \ldots, s_n(X)) \qquad \text{for} \quad X \in \mathbb{M}_n(\mathbb{C}).$$

3.4. *G*-doubly stochastic operators. Here we show that G-doubly stochastic operators induce some subsystems of E-systems.

DEFINITION 3.14 (G-doubly stochastic operator). Let (V, G, D) be an E-system. A linear operator $L : V \to V$ is called *G-doubly stochastic* if

$$Lx \prec_G x \quad \text{for} \quad x \in V. \tag{9}$$

It can be proved that L is G-doubly stochastic iff so is its adjoint L^* defined by

$$\langle Lx, y\rangle = \langle x, L^* y\rangle \quad \text{for all } x, y \in V. \tag{10}$$

EXAMPLE 3.15. If $V = \mathbb{R}^n$, $G = \mathbb{P}_n$ and $D = \mathbb{R}^n_\downarrow$ as in Example 2.15, and if L is an $n \times n$ matrix, then (9) amounts to $Lx \prec x$ for $x \in \mathbb{R}^n$, which is equivalent to the conditions $L \geq 0$, $Lv = v$ and $L^T v = v$, where $v = (1, \ldots, 1)^T \in \mathbb{R}^n$ (see [3, p. 169, Theorem 3.1]). In other words, L is a matrix with nonnegative entries and with row and column sums equal to 1. Such matrices L are said to be *doubly stochastic*. Clearly, L is doubly stochastic if and only if so is L^T.

Given an E-system (V, G, D), a point $x \in V$ is said to be *regular* provided $x \in V_r = \bigcup_{g \in G} g \operatorname{ri} D$, where $\operatorname{ri}(\cdot)$ denotes "the relative interior of".

The set V_r is dense in V, i.e., $\operatorname{cl} V_r = V$, where $\operatorname{cl}(\cdot)$ means "the closure of".

THEOREM 3.16 ([51, Theorem 4.1]). *Suppose (V, G, D) is an E-system. Let W be a subspace of V such that $D \subset W$, and let $H = \{h \in G : hD \subset W\}$. Then the following statements are equivalent*:

(i) $W = \bigcup_{h \in H} hD$.

(ii) *There exists a G-doubly stochastic operator $L : V \to V$ such that $W = \{x \in V : Lx = x\}$, and, in addition, the set $V_r \cap W$ is dense in W.*

Under condition (i)*, the set H is a group if and only if $H = \{h \in G : hW = W\}$. In this case, the triple (W, H, D) is a subsystem of (V, G, D).*

In some situations, the set H in Theorem 3.16 can be replaced by a set $H_0 \subset H$.

EXAMPLE 3.17. Define V, G and D as in Example 2.18. Set $W = \mathbb{S}_n(\mathbb{C})$. Let H be defined as in Theorem 3.16, and let $H_0 = \{h_0 = U(\cdot)U^T : U \in \mathbb{U}_n\}$. Then $W = \bigcup_{h_0 \in H_0} h_0 D$ by the Takagi Decomposition for complex symmetric matrices (see [28, Cor. 4.4.4], [33, Theorem 2], cf. also [34, Example 3.5]). It can be established that

$$H = \big\{h = U(\cdot)AU^T : U \in \mathbb{U}_n,\ A = \operatorname{diag}(a_1, \ldots, a_n),\ |a_i| = 1\big\}.$$

It follows that condition (i) in Theorem 3.16 is fulfilled. Consequently, part (ii) of this theorem holds. For instance, for the operator $L : V \to V$ given by $LX = X^T$ one has $W = \{X \in V : LX = X\}$ and L is G-doubly stochastic, since $LX \prec_G X$ for $X \in V$ by $s(X^T) = s(X)$ [29, p. 154].

On the other hand, the orthoprojector from V onto W is given by

$$PX = \frac{X + X^T}{2} \qquad \text{for} \quad X \in \mathbb{M}_n(\mathbb{C}).$$

By virtue of Theorems 3.16, 3.1 and 3.12 via the Takagi Decomposition one obtains

$$s\Big(\frac{X + X^T}{2}\Big) \prec_w s(X) \qquad \text{for} \quad X \in \mathbb{M}_n(\mathbb{C}).$$

4. Morphisms of E-systems.

In this section we study properties and applications of *morphisms* of E-systems.

Unless otherwise stated, V and W are finite-dimensional real inner product spaces, and G and H are closed subgroups of the orthogonal groups $O(V)$ and $O(W)$, respectively.

4.1. Motivation.

In [35] Lewis introduced the notion of isomorphic ND systems.

DEFINITION 4.1 (Isomorphic ND systems). Two ND systems $(V, G, (\cdot)_\downarrow)$ and $(W, H, (\cdot)_\downarrow)$ are said to be *isomorphic* if there exist an inner product space isomorphism $K : V \to W$ and a group isomorphism $\varphi : G \to H$ such that $(Kx)_\downarrow = Kx_\downarrow$ and $Kgx = \varphi(g)Kx$ for $x \in V$ and $g \in G$ [35, p. 931].

In order to motivate our next definition, we give an example.

EXAMPLE 4.2. Let (V, G, D) be the E-system defined in Example 2.18. It is known that

$$s(A \circ X) \prec_w s(A) \circ s(X) \qquad \text{for } A, X \in \mathbb{M}_n(\mathbb{C}), \tag{11}$$

where $\circ$ denotes the Hadamard (entrywise) product of matrices in $\mathbb{M}_n(\mathbb{C})$ and of vectors in $\mathbb{R}^n$ (see [32, p. 168]).

Assume A is a diagonal matrix with decreasingly ordered positive diagonal entries. Define the linear operator $K_A : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ by

$$K_A X = A \circ X \quad \text{for } X \in \mathbb{M}_n(\mathbb{C}).$$

Let $(W, H, E) = (V, G, D)$. Then $K_A D \subset E$ and inequality (11) leads to

$$(K_A x)_\downarrow \prec_H K_A x_\downarrow \quad \text{for } x \in V. \tag{12}$$

Likewise, the following result holds for the conventional product:

$$s(AX) \prec_w s(A) \circ s(X) \quad \text{for } A, X \in \mathbb{M}_n(\mathbb{C}) \tag{13}$$

(see [32, p. 168]). Taking

$$K_A X = AX \qquad \text{for } X \in \mathbb{M}_n(\mathbb{C})$$

with diagonal matrix A as above, we deduce from (13) that (12) is satisfied.

4.2. Morphisms and simple morphisms

Definition 4.3 (Morphism). Given two E-systems $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$, a linear operator K is said to be an *$\mathcal{E}, \mathcal{F}$-morphism* if

$$KD \subset E \quad \text{and} \quad (Kx)_\downarrow \prec_H Kx_\downarrow \quad \text{for } x \in V. \tag{14}$$

The set of all morphisms of E-systems $\mathcal{E}$ and $\mathcal{F}$ is denoted by $\mathrm{Mor}\,(\mathcal{E}, \mathcal{F})$.

The second part of condition (14) can be restated in the following equivalent form:

$$y \prec_G x \quad \text{implies} \quad Ky \prec_H Kx \quad \text{for } x \in D \text{ and } y \in V. \tag{15}$$

An important class of $\mathcal{E}, \mathcal{F}$-morphisms are simple morphisms.

Definition 4.4 (Simple morphism). Given two E-systems $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$, a linear operator $K : V \to W$ is said to be a *simple morphism*, if

$$KD \subset E \quad \text{and} \quad Kx \equiv_H Kx_\downarrow \qquad \text{for } x \in V,$$

or equivalently,

$$KD \subset E \quad \text{and} \quad Kgx \equiv_H Kx \qquad \text{for } x \in D \text{ and } g \in G.$$

The set of all simple morphisms of E-systems $\mathcal{E}$ and $\mathcal{F}$ is denoted by $\mathrm{SMor}\,(\mathcal{E}, \mathcal{F})$.

The set $\mathrm{Mor}\,(\mathcal{E}, \mathcal{F})$ of $\mathcal{E}, \mathcal{F}$-morphisms is a closed convex cone. The set $\mathrm{Mor}\,\mathcal{E} = \mathrm{Mor}\,(\mathcal{E}, \mathcal{E})$ is a selfadjoint semigroup.

Hereafter $K^* : W \to V$ is the *dual operator* (adjoint) of K (see (10)).

Theorem 4.5 ([55, Theorem 2.1]). *Assume $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$ are E-systems, and $K : V \to W$ is a linear operator. The following two conditions are equivalent*:

(i) *$KD \subset E$, and $Kgx \prec_H Kx$ for $x \in D$ and $g \in G$.*
(ii) *$K^*E \subset D$, and $K^*hz \prec_G K^*z$ for $z \in E$ and $h \in H$.*

Theorem 4.5 asserts that K is a morphism if and only if K^* is so. Furthermore,

$$K \in \mathrm{Mor}\,(\mathcal{E},\mathcal{F}) \quad \text{implies} \quad K^*K \in \mathrm{Mor}\,(\mathcal{E}) \quad \text{and} \quad KK^* \in \mathrm{Mor}\,(\mathcal{F}).$$

The next result suggests that by employing some easily checkable classes of morphisms K^*, one can get other morphisms K.

Theorem 4.6 ([55, Theorem 2.2]). *Assume $\mathcal{E} = (V, G, D)$ and $\mathcal{F} = (W, H, E)$ are E-systems, and $K : V \to W$ is a linear operator. Suppose that $\mathcal{F}_0 = (W_0, H_0, E_0)$ is a subsystem of $\mathcal{F}$ such that $KV \subset W_0$.*

If $K^* \in \mathrm{Mor}\,(\mathcal{F}_0, \mathcal{E})$, i.e., $K^*E_0 \subset D$ and

$$K^*h_0z \prec_G K^*z \quad \text{for } z \in E_0 \text{ and } h_0 \in H_0,$$

then $K \in \mathrm{Mor}\,(\mathcal{E}, \mathcal{F}_0)$, i.e., $KD \subset E_0$ and

$$Kgx \prec_{H_0} Kx \quad \text{for } x \in D \text{ and } g \in G. \tag{16}$$

In particular, if $K^* \in \mathrm{SMor}\,(\mathcal{F}_0, \mathcal{E})$, i.e., $K^*E_0 \subset D$ and

$$\text{for } z \in E_0 \text{ and } h_0 \in H_0 \text{ there exists } g \in G \text{ such that } K^*h_0z = gK^*z, \tag{17}$$

then $K \in \mathrm{Mor}\,(\mathcal{E}, \mathcal{F}_0)$.

For instance, if $\mathcal{E} = \mathcal{F}$ and the restriction of K^* to W_0 is the identity, then (16) *holds.*

4.3. Homomorphisms. Throughout this subsection, W_0 is a finite-dimensional real inner product space, H_0 is a closed subgroup of the orthogonal group $O(W_0)$, and E_0 is a closed convex cone included in W_0. We define $\|w\| = \langle w, w\rangle^{1/2}$ for $w \in W_0$.

Our aim is to provide some sufficient conditions for triple $\mathcal{F}_0 = (W_0, H_0, E_0)$ to be an E-system. We begin with axiom (A1).

Theorem 4.7 ([55, Theorem 3.1]). *The following three statements are equivalent*:

(i) *Axiom* (A1) *is satisfied for $\mathcal{F}_0 = (W_0, H_0, E_0)$.*
(ii) *For any $w \in W_0$ there exists $u \in E_0$ such that $w \prec_{H_0} u$ and $\|w\| = \|u\|$.*
(iii) *There exist a triple $\mathcal{E} = (V, G, D)$ (not necessarily an E-system) and a linear operator $K : V \to W_0$ such that the following conditions are satisfied*:

$$W_0 = KV, \qquad E_0 = KD \quad \text{and} \quad H_0 \text{ is a closed subgroup of } O(W_0). \tag{18}$$

$$\text{For } z \in V \text{ and } x \in D,\ z \prec_G x \text{ implies } Kz \prec_{H_0} Kx. \tag{19}$$

$$\begin{gathered}\text{For } w \in W_0 \text{ there exist } z \in V \text{ and } x \in D \text{ such that}\\ z \prec_G x,\ w = Kz \text{ and } \|w\| = \|Kx\|.\end{gathered} \tag{20}$$

In the rest of this subsection, we investigate systems $\mathcal{F}_0 = (W_0, H_0, E_0)$ of form (18), where $K : V \to W_0$ is a linear operator and $\mathcal{E} = (V, G, D)$ is a triple (not necessarily an E-system). In light of Theorem 4.7, we must concentrate on operators satisfying (19)–(20).

According to (14) and (15), we say that K is an $\mathcal{E}, \mathcal{F}_0$-*morphism* if (19) is met. (Here we do not assume that $\mathcal{E}$ and $\mathcal{F}_0$ are E-systems.)

Definition 4.8 (Radial morphism). An $\mathcal{E}, \mathcal{F}_0$-morphism K is said to be $\mathcal{E}, \mathcal{F}_0$-*radial morphism* if (20) holds.

Note that each simple morphism is a radial morphism.

Now we focus on axiom (A2) for the triple $\mathcal{F}_0 = (W_0, H_0, E_0)$.

THEOREM 4.9 ([55, Theorem 3.3]). *Let $\mathcal{E} = (V, G, D)$ be an E-system and let K be an $\mathcal{E}, \mathcal{F}_0$-radial morphism.*

The following two conditions are equivalent:

(i) *Axiom* (A2) *is satisfied for $\mathcal{F}_0 = (W_0, H_0, E_0)$.*
(ii) *The operator $K^* : W_0 \to V$ is an $\mathcal{F}_0, E$-morphism, i.e., $K^* E_0 \subset D$ and*

$$K^* h_0 u \prec_G K^* u \quad \textit{for } h_0 \in H_0 \textit{ and } u \in E_0.$$

For triples $\mathcal{F}_0 = (W_0, H_0, E_0)$ and $\mathcal{F} = (W, H, E)$, we write $\mathcal{F}_0 \subset \mathcal{F}$ if

$$W_0 \subset W, \quad E_0 \subset E \quad \text{and} \quad H_0 \subset \{h \in H : hW_0 \subset W_0\}|_{W_0}.$$

COROLLARY 4.10 ([55, Corollary 3.4]). *Let $\mathcal{E} = (V, G, D)$ be an E-system and let K be an $\mathcal{E}, \mathcal{F}_0$-radial morphism.*

If there exists an E-system $\mathcal{F} = (W, H, E)$ such that $\mathcal{F}_0 \subset \mathcal{F}$, then $\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system.

DEFINITION 4.11 (Homomorphism). An $\mathcal{E}, \mathcal{F}_0$-*morphism* K is called an $\mathcal{E}, \mathcal{F}_0$-*homomorphism* if for $u \in E_0$ and $w \in W_0$, $w \equiv_{H_0} u$ implies that there exist $x \in D$ and $z \in V$ such that $z \prec_G x$, $u = Kx$ and $w = Kz$.

Theorem 4.12 provides further sufficient conditions for $\mathcal{F}_0 = (W_0, H_0, E_0)$ to be an E-system (cf. [49, Theorems 3.4 and 3.9]).

THEOREM 4.12 ([55, Theorem 3.5]). *Let $\mathcal{E} = (V, G, D)$ be an E-system and let K be an $\mathcal{E}, \mathcal{F}_0$-radial homomorphism.*

If $K^* K D \subset D$, then $\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system.

Now we consider partial isometries.

DEFINITION 4.13 (Partial isometry). A linear operator $K : V \to W$ is said to be a *partial isometry* if $KK^*K = K$.

In the event that K is a partial isometry, the operator $KK^* : W \to W$ is the orthoprojector from W onto the subspace $KK^*W = KV = W_0$.

THEOREM 4.14 ([55, Theorem 3.6]). *Let $K : V \to W$ be a partial isometry with finite-dimensional real inner product spaces V and W, and let $D \subset V$ be a closed convex cone. Let $\mathcal{F}_0 = (W_0, H_0, E_0)$, where $W_0 = KV$, $E_0 = KD$ and H_0 is a closed subgroup of $O(W_0)$. Suppose that $\mathcal{F} = (W, H, E)$ is an E-system such that $\mathcal{F}_0 \subset \mathcal{F}$ and $KK^*E = E_0$.*

Then the following statements are equivalent:

(i) *$\mathcal{F}_0 = (W_0, H_0, E_0)$ is an E-system.*
(ii) *$KK^*w \prec_{H_0} KK^*w_\downarrow$ for $w \in W$, where $(\cdot)_\downarrow$ stands for the normal map of $\mathcal{F}$.*
(iii) *$W_0 \subset \bigcup_{h \in H} hE_0$, and, in addition, the operator KK^* is an $\mathcal{F}, \mathcal{F}_0$-radial homomorphism.*

4.4. Applications to matrices. We illustrate Theorem 4.6 for the matrix E-system $\mathcal{E} = (V, G, D)$ defined in Example 2.18 and their subsystems. Let $\mathcal{F} = (W, H, E)$ with $W = V$, $H = G$ and $E = D$.

Some subsystems $\mathcal{F}_0 = (W_0, H_0, E_0)$ of $\mathcal{F}$ are collected below (see [15, 16, 35, 70]).

(I) $W_0 = \mathbb{M}_n(\mathbb{R})$ = the space of $n \times n$ real matrices,
H_0 = the group of orthogonal equivalences $U_1(\cdot)U_2$ with $U_1, U_2 \in \mathbb{O}_n$,
$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

(II) $W_0 = \mathbb{S}_n(\mathbb{C})$ = the space of $n \times n$ complex symmetric matrices,
H_0 = the group of unitary congruences $U(\cdot)U^T$ with $U \in \mathbb{U}_n$,
$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

(III) W_0 = the space of $n \times n$ matrices of the form $\left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $k \times k$ matrix $X \in \mathbb{M}_k(\mathbb{C})$ and $1 \leq k \leq n$,
H_0 = the group of unitary similarities $U_1(\cdot)U_2$ with U_1 and U_2 being the matrix of the form $\left(\begin{smallmatrix} U & 0 \\ 0 & I_{n-k} \end{smallmatrix}\right)$ for some $k \times k$ unitary matrix U,
$E_0 = \{\operatorname{diag}(s_1, \ldots, s_k, 0, \ldots, 0) : s_1 \geq \ldots \geq s_k \geq 0\}$.

(IV) $W_0 = \mathbb{D}_n(\mathbb{C})$ = the space of $n \times n$ complex diagonal matrices,
H_0 = the group of equivalences $U_1(\cdot)U_2$ with $U_1, U_2 \in \mathbb{GP}_n(\mathbb{C})$,
$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

(V) $W_0 = \mathbb{D}_n(\mathbb{R})$ = the space of $n \times n$ real diagonal matrices,
H_0 = the group of equivalences $U_1(\cdot)U_2$ with $U_1, U_2 \in \mathbb{GP}_n(\mathbb{R})$,
$E_0 = \{\operatorname{diag}(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}$.

COROLLARY 4.15 ([55, Corollary 4.1]). *For any of the above subsystems $\mathcal{F}_0$, let $K : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a linear operator such that $K\mathbb{M}_n(\mathbb{C}) \subset W_0$.*

(i) *If the restriction $K^*|_{W_0}$ is a simple morphism of $\mathcal{F}_0$ and $\mathcal{E}$, i.e., if* (17) *is satisfied and $K^*E_0 \subset D$, then K is a morphism of $\mathcal{E}$ and $\mathcal{F}_0$, i.e., $KD \subset E_0$ and*
$$s(Kx) \prec_w s(Kx_\downarrow) \qquad \text{for } x \in \mathbb{M}_n(\mathbb{C}), \tag{21}$$
where $x_\downarrow = \operatorname{diag} s(x)$.

(ii) *If the restriction $K^*|_{W_0}$ is the identity on W_0, then inequality* (21) *holds.*

(iii) *If K is symmetric ($K^* = K$) and $D \subset W_0$, and if the restriction $K|_{W_0}$ is the identity, then inequality* (21) *holds in the form*
$$s(Kx) \prec_w s(x) \qquad \text{for } x \in \mathbb{M}_n(\mathbb{C}). \tag{22}$$

For the subsystem defined in (IV), (22) generalizes the classical Ky Fan's inequality (4).

The next result is a consequence of Theorem 4.14.

COROLLARY 4.16 ([55, Corollary 4.2]). *Let $K : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a partial isometry. Let $W_0 = KV$, $E_0 = KD$ and H_0 be a closed subgroup of $O(W_0)$. Suppose that $KK^*D = E_0$. Then*
$$\mathcal{F}_0 = (W_0, H_0, E_0) \text{ is an E-system} \quad \text{iff} \quad KK^*w \prec_{H_0} KK^*w_\downarrow \text{ for } w \in \mathbb{M}_n(\mathbb{C}),$$
where $w_\downarrow = \operatorname{diag} s(w)$.

5. Chebyshev functional and its applications

5.1. Motivation

DEFINITION 5.1 (Synchronous functions). Two functions $f, g : [a, b] \to \mathbb{R}$ are said to be *synchronous* if

$$[f(t) - f(u)][g(t) - g(u)] \geq 0 \qquad \text{for } t, u \in [a, b].$$

The celebrated *Chebyshev integral inequality* says that if real functions $f, g \in L^2_{[a,b]}$ are synchronous then

$$(b - a) \int_a^b f(t)g(t)\, dt - \int_a^b f(t)\, dt \cdot \int_a^b g(t)\, dt \geq 0. \tag{23}$$

Note that the coefficient $b - a$ in (23) satisfies $\int_a^b v^2(t)\, dt = b - a$ with $v(t) = 1$, $t \in [a, b]$.

It is obvious that each two nonincreasing functions $f, g : [a, b] \to \mathbb{R}$ are synchronous. Thus Chebyshev inequality (23) is true for two nonincreasing functions.

There are many inequalities similar to (23), e.g., Andersson's result [1] asserts that if $f, g : [0, 1] \to \mathbb{R}$ and $f(0) = g(0) = 0$ and both f and g are increasing and convex then

$$\frac{3}{4} \int_0^1 f(t)g(t)\, dt - \int_0^1 f(t)\, dt \cdot \int_0^1 g(t)\, dt \geq 0.$$

Fink [22] proved that

$$\frac{\left(\int_0^1 t\, d\sigma(t)\right)^2}{\int_0^1 t^2\, d\sigma(t)} \int_0^1 f(t)g(t)\, d\sigma(t) - \int_0^1 f(t)\, d\sigma(t) \cdot \int_0^1 g(t)\, d\sigma(t) \geq 0 \tag{24}$$

for functions

$$f, g \in \left\{h \in C^1_{[0,1]} : h(0) = 0 \quad \text{and } h(t)/t \text{ is increasing on } (0, 1]\right\}$$

and measures σ such that

$$\int_0^s t\, d\sigma(t) \geq 0 \quad \text{and} \quad \int_s^1 t\, d\sigma(t) \geq 0 \quad \text{for } s \in [0, 1], \quad \text{and} \quad \int_0^1 t\, d\sigma(t) > 0.$$

A discrete counterpart of (24) is the following Toader's weighted inequality [73]:

$$\frac{\left(\sum_{i=1}^n i v_i\right)^2}{\sum_{i=1}^n i^2 v_i} \sum_{i=1}^n x_i y_i v_i - \sum_{i=1}^n x_i v_i \cdot \sum_{i=1}^n y_i v_i \geq 0 \tag{25}$$

provided that $v_i > 0$, $i = 1, 2, \ldots, n$, and x and y are star-shaped, i.e., the sequences $i \to x_i/i$ and $i \to y_i/i$ are nondecreasing (see [51, p. 239]).

DEFINITION 5.2 (Synchronous n-tuples). Two n-tuples $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ are said to be *synchronous* if

$$(x_i - x_j)(y_i - y_j) \geq 0 \qquad \text{for } i, j \in \{1, 2, \ldots, n\}.$$

A discrete version of (23) is as follows. If $x, y \in \mathbb{R}^n$ are synchronous then

$$n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \cdot \sum_{i=1}^n y_i \geq 0. \tag{26}$$

This is the *Chebyshev sum inequality*.

The coefficient n in (26) satisfies $\sum_{i=1}^n v_i^2 = n$, where $v_i = 1$ for $i = 1, 2, \ldots, n$, and the unital vector $v = (1, \ldots, 1) \in \mathbb{R}^n$ is a common fixed point for all operators in the group $\mathbb{P}_n$ of $n \times n$-permutation matrices, i.e., $v \in \{a \in \mathbb{R}^n : pa = a \quad \text{for } p \in \mathbb{P}_n\}$.

In Subsections 5.2–5.3 we show methods for establishing inequalities similar to (23)–(26) with the help of the Chebyshev functional, Eaton triples and the notion of similarly separable vectors.

Throughout this section, unless otherwise stated, V is a finite-dimensional real linear space with an inner product $\langle \cdot, \cdot \rangle$, and $O(V)$ denotes the orthogonal group acting on V.

5.2. Chebyshev functional and E-systems. Here we present Chebyshev type inequalities in the framework of E-systems.

Definition 5.3 (Chebyshev functional). The *Chebyshev functional* is defined by

$$T_v(x, y) = \langle v, v \rangle \langle x, y \rangle - \langle x, v \rangle \langle y, v \rangle \quad \text{for } x, y \in V, \tag{27}$$

where $0 \neq v \in V$ [50, p. 535].

Definition 5.4 (G-synchronous vectors). Given an Eaton triple (V, G, D), two vectors $x, y \in V$ are said to be *G-synchronous* if there exists $g \in G$ such that $x, y \in gD$.

We define

$$M_G(V) = \{a \in V : ga = a \quad \text{for } g \in G\}.$$

Theorem 5.5 ([50, Theorem 3.1]). *Let (V, G, D) be an Eaton triple with one-dimensional subspace $M_G(V) = \operatorname{span} v$, where $0 \neq v \in V$.*

If $x, y \in V$ are G-synchronous then the following Chebyshev type inequality holds:

$$\langle v, v \rangle \langle x, y \rangle - \langle x, v \rangle \langle y, v \rangle \geq 0. \tag{28}$$

In the forthcoming examples we give some interpretations of inequality (28).

Example 5.6. Let $(V, G, D) = (\mathbb{R}^n, \mathbb{P}_n, D)$, where $D = \{a = (a_1, \ldots, a_n) \in \mathbb{R}^n : a_1 \geq a_2 \geq \ldots \geq a_n\}$. Then $M_G(V) = \operatorname{span} v$ for $v = (1, \ldots, 1) \in \mathbb{R}^n$.

Let $((x_1, y_1), \ldots, (x_n, y_n))$ be a sample of size n from distribution of a two-dimensional random vector. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are synchronous then x and y are positively correlated, i.e.,

$$S_{x,y} = \frac{1}{n} \sum_{i=1}^n x_i y_i - \frac{1}{n} \sum_{i=1}^n x_i \cdot \frac{1}{n} \sum_{i=1}^n y_i \geq 0$$

(see (26)–(28)).

Example 5.7 ([50, pp. 540–541]). Let (V, G, D) be the Eaton triple defined in Example 2.17. Then $M_G(V) = \operatorname{span} I_n$, where I_n denotes the n-by-n identity matrix. The G-synchronicity of Hermitian matrices X and Y means that X and Y are *simultaneously diagonalizable*. Additionally, $X_\downarrow = \operatorname{diag} \lambda(X)$, where $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$ is the vector of the eigenvalues of a Hermitian matrix X with $\lambda_1(X) \geq \ldots \geq \lambda_n(X)$.

In light of Theorem 5.5 the following Chebyshev type inequality holds:

$$n \sum_{i=1}^n \lambda_i(X) \lambda_i(Y) \geq \operatorname{tr} X \operatorname{tr} Y \quad \text{for } X, Y \in \mathbb{H}_n.$$

5.3. Generalized Chebyshev functional. Motivated by Theorem 5.5, we study a generalization of Chebyshev functional (27).

DEFINITION 5.8 (Generalized Chebyshev functional). Assume $(V, \langle\cdot,\cdot\rangle)$ is an inner product space. The *generalized Chebyshev functional* is defined by

$$T_{v,w}(x,y) = \langle v,w\rangle\langle x,y\rangle - \langle x,v\rangle\langle y,w\rangle \quad \text{for } x,y \in V, \tag{29}$$

where $0 \neq v, w \in V$.

We are interested in sufficient and necessary conditions for the functional (29) to be nonnegative. As will be seen below, the key property is the *separability* of some vectors. Separable vectors are natural generalizations of monotone, monotone in mean, star-shaped and convex sequences, etc.

DEFINITION 5.9 (Separable vector). Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and $e = (e_1,\dots,e_n) \in V^n$, $v \in V$ and $\mu \in \mathbb{R}$. Let J_1 and J_2 be index sets with $J_1 \cup J_2 = \{1,2,\dots,n\}$.

A vector $z \in V$ is said to be *μ, v-separable* on J_1 and J_2 with respect to e, if

$$\langle z-\mu v, e_i\rangle \geq 0 \quad \text{for } i \in J_1, \quad \text{and} \quad \langle z-\mu v, e_j\rangle \leq 0 \quad \text{for } j \in J_2 \tag{30}$$

(see [51, p. 235]).

Note that if $\langle v, e_i\rangle > 0$ for $i = 1,2,\dots,n$ then the μ, v-separability of vector z is equivalent to

$$\frac{\langle z,e_i\rangle}{\langle v,e_i\rangle} \geq \mu \geq \frac{\langle z,e_j\rangle}{\langle v,e_j\rangle} \qquad \text{for } i \in J_1 \text{ and } j \in J_2. \tag{31}$$

The three examples below include interpretations of the notion of separability in $V = \mathbb{R}^n$ with the standard inner product

$$\langle a,b\rangle = \sum_{k=1}^{n} a_k b_k \qquad \text{for } a = (a_1,a_2,\dots,a_n) \text{ and } b = (b_1,b_2,\dots,b_n). \tag{32}$$

EXAMPLE 5.10. If $z = (z_1,\dots,z_n) \in \mathbb{R}^n$, $v = (1,1,\dots,1) \in \mathbb{R}^n$ and $e = (e_1,\dots,e_n)$ is the standard basis of $\mathbb{R}^n$, i.e., $e_k = (\underbrace{0,\dots,0}_{k-1\ times},1,0,\dots,0) \in \mathbb{R}^n$ for $k = 1,2,\dots,n$, then condition (31) means that

$$z_i \geq \mu \geq z_j \qquad \text{for } i \in J_1 \text{ and } j \in J_2.$$

EXAMPLE 5.11 ([59, p. 937]). Remind that a vector $z = (z_1,z_2,\dots,z_n) \in \mathbb{R}^n$ is said to be *star-shaped* if

$$\frac{z_1}{1} \leq \frac{z_2}{2} \leq \dots \leq \frac{z_n}{n}. \tag{33}$$

Choose $v = (1,2,\dots,n)$. Then (33) can be rewritten as

$$\frac{\langle z,e_1\rangle}{\langle v,e_1\rangle} \leq \frac{\langle z,e_2\rangle}{\langle v,e_2\rangle} \leq \dots \leq \frac{\langle z,e_n\rangle}{\langle v,e_n\rangle}.$$

For arbitrary $m \in \{0,1,\dots,n\}$, take μ to be any number between $\frac{\langle z,e_m\rangle}{\langle v,e_m\rangle}$ and $\frac{\langle z,e_{m+1}\rangle}{\langle v,e_{m+1}\rangle}$ (with the convention that $\frac{\langle z,e_0\rangle}{\langle v,e_0\rangle} = -\infty$ and $\frac{\langle z,e_{n+1}\rangle}{\langle v,e_{n+1}\rangle} = \infty$). Then (31) is fulfilled for

$J_1 = \{m+1, m+2, \dots, n\}$ and $J_2 = \{1, 2, \dots, m\}$. In other words, z is μ, v-separable on J_1 and J_2 with respect to e for each $m \in \{0, 1, \dots, n\}$.

EXAMPLE 5.12 ([59, p. 937]). We show that the majorization relation $x \prec y$ corresponds to the μ-separability of the vector $y - x$ for $\mu = 0$ and $J_1 = \{1, 2, \dots, n\}$ and $J_2 = \{n\}$.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ with $x_1 \geq x_2 \geq \dots \geq x_n$ and $y_1 \geq y_2 \geq \dots \geq y_n$. Suppose that x is majorized by y. That is,

$$\sum_{i=1}^{k} (y_i - x_i) \geq 0 = \sum_{i=1}^{n} (y_i - x_i) \qquad \text{for } k = 1, 2, \dots, n. \tag{34}$$

Put $v = (1, \dots, 1) \in \mathbb{R}^n$ and $e_k = (\underbrace{1, \dots, 1}_{k \text{ times}}, 0, \dots, 0) \in \mathbb{R}^n$ for $k = 1, 2, \dots, n$. Then (34) reads

$$\langle y - x, e_k \rangle \geq 0 = \langle y - x, e_n \rangle \qquad \text{for } k = 1, 2, \dots, n.$$

With $\mu = 0$ and J_1 and J_2 defined above, one has

$$\langle y - x - \mu v, e_i \rangle \geq 0 = \langle y - x - \mu v, e_j \rangle \qquad \text{for } i \in J_1 \text{ and } j \in J_2.$$

Therefore the difference $z = y - x$ is $0, v$-separable on J_1 and J_2 w.r.t. e (see (30)).

We define

$$S_e(v; J_1, J_2) = \{z \in V : \ z \text{ is } v\text{-separable on } J_1 \text{ and } J_2 \text{ w.r.t. } e \text{ for some } \mu \in \mathbb{R}\}.$$

The role of the separability in determining the sign of the generalized Chebyshev functional (29) is shown in the following result.

THEOREM 5.13 ([48, Theorem 3.5]). *Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional real inner product space with two dual bases $e = (e_1, \dots, e_n)$ and $d = (d_1, \dots, d_n)$ in V, i.e., $\langle e_i, d_j \rangle = \delta_{ij}$ (Kronecker delta), $i, j \in J = \{1, 2, \dots, n\}$.*

Let $y, w, v \in V$ with $\langle w, v \rangle > 0$. Suppose that J_1 and J_2 are index sets with $J_1 \cup J_2 = J$. Then the following two statements are equivalent:

(i) *The generalized Chebyshev inequality*

$$\langle w, v \rangle \langle z, y \rangle - \langle z, w \rangle \langle y, v \rangle \geq 0 \tag{35}$$

holds for all $z \in S_e(v; J_1, J_2)$.

(ii) *The vector y is η, w-separable on J_1 and J_2 w.r.t. d, where $\eta = \langle y, v \rangle / \langle w, v \rangle$.*

The next definition is inspired by Theorem 5.13.

DEFINITION 5.14 (Similar separability). Let $e = (e_1, \dots, e_n)$ and $d = (d_1, \dots, d_n)$ be two sequences of vectors in V. Assume $v, w \in V$ and $\mu, \eta \in \mathbb{R}$.

Two vectors $a, b \in V$ are said to be *similarly separable* w.r.t. $(\mu, v, e; \eta, w, d)$ if there exist index sets J_1 and J_2 with $J_1 \cup J_2 = \{1, 2, \dots, n\}$ such that

(i) a is μ, v-separable on J_1 and J_2 w.r.t. e, and
(ii) b is η, w-separable on J_1 and J_2 w.r.t. d

(see [59, p. 937]).

Corollary 5.15 ([54, Lemma 2.1]). *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real inner product space with two dual bases $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ in V. Let $z, y, w, v \in V$ with $\langle w, v\rangle > 0$. Define $\eta = \langle y, v\rangle / \langle w, v\rangle$.*

If the vectors $z, y \in V$ are similarly separable w.r.t. $(\mu, v, e; \eta, w, d)$ for some $\mu \in \mathbb{R}$, then the generalized Chebyshev inequality (35) *holds.*

It is interesting that the generalized Chebyshev inequality (35) contains Cauchy–Schwarz inequality. Namely, if $y = z$, $w = v$ and $d = e$, then the similar separability of z, y holds automatically for $\mu = \eta$. In this case (35) reduces to the C–S inequality

$$\|z\|^2\|v\|^2 \geq \langle z, v\rangle^2 \qquad \text{for } z, v \in V.$$

6. Applications of similar separability. In this section we apply Theorem 5.13 to give some generalizations of the classical Hardy–Littlewood–Pólya (H–L–P) Theorem for convex functions and of Schur–Ostrowski Theorem for differentiable Schur-convex functions. In addition, we deal with extended G-majorization.

6.1. Generalized H–L–P theorems. We begin with a result of H–L–P giving relationship between majorization and convexity.

Theorem 6.1 ([27]). *Let $x, y \in \mathbb{R}^n$ with $x_i, y_i \in I$, where $I \subset \mathbb{R}$ is an interval.*

The following two statements are equivalent:

(i) $y \prec x$.

(ii) *The following inequality holds for all continuous convex functions $f : I \to \mathbb{R}$:*

$$\sum_{k=1}^{n} f(y_k) \leq \sum_{k=1}^{n} f(x_k).$$

Some extensions of Theorem 6.1 can be found in [11, 44].

For a given positive vector $(p_1, \ldots, p_n) \in \mathbb{R}^n_+$, we introduce inner product on $\mathbb{R}^n$ by

$$\langle a, b\rangle = \sum_{k=1}^{n} a_k b_k p_k \qquad \text{for } a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \in \mathbb{R}^n. \tag{36}$$

Theorem 6.2 ([54, Theorem 2.2]). *Assume $f : I \to \mathbb{R}$ is a convex function on the open interval $I \subset \mathbb{R}$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $p = (p_1, \ldots, p_n)$, where $x_i, y_i \in I$, $p_i > 0$ for $i \in J = \{1, \ldots, n\}$.*

Let $\partial f : I \to \mathbb{R}$ be the subdifferential of f, and let $\varphi \in \partial f$. Define

$$\Phi(z) = (\varphi(z_1), \ldots, \varphi(z_n)) \qquad \text{for } z = (z_1, \ldots, z_n) \in I^n.$$

Let e and d be dual bases for $\mathbb{R}^n$ with inner product given by (36)*, and $w, v \in \mathbb{R}^n$ with $\langle w, v\rangle > 0$. Let $\eta = \langle x - y, v\rangle / \langle w, v\rangle$.*

Suppose that there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that

(i) *y is v-separable on J_1 and J_2 w.r.t. e,*

(ii) *$x - y$ is η, w-separable on J_1 and J_2 w.r.t. d, and*

(iii) *Φ preserves v-separability on J_1 and J_2 w.r.t. e.*

Under the above assumptions, the following assertions hold.

(A) *If* $\langle x-y, v\rangle = 0$, *then*

$$\sum_{k=1}^{n} p_k f(y_k) \le \sum_{k=1}^{n} p_k f(x_k). \tag{37}$$

(B) *If* $\langle x-y, v\rangle \ge 0$ *and* $\langle \Phi(y), w\rangle \ge 0$, *then* (37) *holds.*

6.2. Generalization of Schur–Ostrowski Theorem. Characterizations of differentiable S-convex and G-increasing functions are provided below in Schur–Ostrowski's Theorem 6.3 and in Eaton–Perlman's Theorem 6.4, respectively.

THEOREM 6.3 ([66, 62]). *Assume that F is a symmetric real function having a differential on $\mathbb{R}^n$. Then a necessary and sufficient condition that F be a Schur-convex function on $\mathbb{R}^n$ is*

$$(z_i - z_j)\Big(\frac{\partial F}{\partial z_i}(z) - \frac{\partial F}{\partial z_j}(z)\Big) \ge 0 \qquad \textit{for } z \in \mathbb{R}^n \textit{ and } i, j = 1, 2, \dots, n. \tag{38}$$

Statement (38) is called *Schur–Ostrowski's condition* (in short, *S–O condition*).

THEOREM 6.4 ([18]). *Let G be a reflection group acting on $\mathbb{R}^n$. Assume that F is a G-invariant real function possessing a differential on $\mathbb{R}^n$. Then a necessary and sufficient condition that F be G-increasing on $\mathbb{R}^n$ is*

$$\langle z, r\rangle \cdot \langle \nabla F(z), r\rangle \ge 0 \qquad \textit{for } z \in \mathbb{R}^n \textit{ and } r \in \mathbb{R}^n \textit{ such that } S_r \in G,$$

where $\nabla F(z)$ denotes the gradient of F at z.

Let $(V, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$.

For a function $F : A \to \mathbb{R}$ with convex $A \subset V$, the symbol $\nabla_h F(z)$ stands for the directional derivative of F in the direction $h \in V$ at the point z, and $\nabla F(z)$ stands for the *gradient* of F at z.

DEFINITION 6.5 (Generalized Schur–Ostrowski's condition). Given a convex set $A \subset V$ and $x, y \in A$ and $v \in V$, a differentiable function $F : A \to \mathbb{R}$ is said to satisfy *generalized Schur–Ostrowski condition* (*GSOC*) if for each $\mu \in \mathbb{R}$ and $z \in [x, y]$ there exists $\widetilde{\mu} \in \mathbb{R}$ such that

$$\langle z - \mu v, e_i\rangle \cdot \langle \nabla F(z) - \widetilde{\mu} v, e_i\rangle \ge 0 \qquad \text{for } i = 1, 2, \dots, n$$

(see [59]).

DEFINITION 6.6 (Function class $\mathcal{S}(A, x, y)$ satisfying GSOC). Given a convex set $A \subset V$ and $x, y \in A$ and $v, w \in V$, by $\mathcal{S}(A, x, y)$ we denote the class of all differentiable functions $F : A \to \mathbb{R}$ satisfying the generalized Schur–Ostrowski condition (GSOC) and such that the maps $[0,1] \ni t \to \nabla_{y-x} F(x + t(y-x))$ and $[0,1] \ni t \to \nabla_w F(x + t(y-x))$ are integrable on $[0,1]$ (see [59, p. 939]).

Members of the class $\mathcal{S}(A, x, y)$ are called *generalized Schur-convex functions.*

In the forthcoming results we generalize the sufficiency part of Theorems 6.3 and 6.4.

THEOREM 6.7 ([59, Theorem 4]). *Let W be a finite-dimensional subspace of V and $e = (e_1, \dots, e_n)$ and $d = (d_1, \dots, d_n)$ be dual bases in W. Let $A \subset V$ be a convex set and let $x, y \in A$ and $w, v \in V$ with $\langle w, v\rangle > 0$.*

Suppose that for some index sets J_1 *and* J_2 *with* $J_1 \cup J_2 = \{1, 2, \dots, n\}$ *and for some* $\mu_1, \mu_2 \in \mathbb{R}$ *and* $\eta = \langle y - x, v\rangle / \langle w, v\rangle$,

$$\langle x - \mu_1 v, e_i\rangle > 0 \quad \textit{for } i \in J_1 \quad \textit{and} \quad \langle x - \mu_1 v, e_j\rangle < 0 \quad \textit{for } j \in J_2, \tag{39}$$

$$\langle y - \mu_2 v, e_i\rangle > 0 \quad \textit{for } i \in J_1 \quad \textit{and} \quad \langle y - \mu_2 v, e_j\rangle < 0 \quad \textit{for } j \in J_2, \tag{40}$$

$$\langle y - x - \eta w, d_i\rangle \ge 0 \quad \textit{for } i \in J_1 \quad \textit{and} \quad \langle y - x - \eta w, d_j\rangle \le 0 \quad \textit{for } j \in J_2. \tag{41}$$

Let $F \in \mathcal{S}(A, x, y)$. *Assume* $y - x - \eta w \in W$ *and* $\nabla F(z) - \widetilde{\mu} v \in W$ *for* $z \in [x, y]$.

Under the above assumptions, the following three assertions hold.

(A) $F(y) - F(x) \ge \dfrac{\langle y - x, v\rangle}{\langle w, v\rangle} \displaystyle\int_0^1 \langle \nabla F(x + th), w\rangle \, dt.$

(B) *If* $\langle y - x, v\rangle = 0$ *then* $F(x) \le F(y)$.

(C) *If* $\langle y - x, v\rangle \ge 0$ *and* $\langle \nabla F(z), w\rangle \ge 0$ *for* $z \in [x, y]$, *then* $F(x) \le F(y)$.

Theorem 6.7 can be simplified as follows.

COROLLARY 6.8 ([59, Corollaries 9-10]). *The assertions* (A), (B) *and* (C) *of Theorem* 6.7 *are still true if the conditions* (39), (40) *and* (41) *are replaced by*

$$\frac{\langle x, e_1\rangle}{\langle v, e_1\rangle} > \frac{\langle x, e_2\rangle}{\langle v, e_2\rangle} > \dots > \frac{\langle x, e_n\rangle}{\langle v, e_n\rangle}, \tag{42}$$

$$\frac{\langle y, e_1\rangle}{\langle v, e_1\rangle} > \frac{\langle y, e_2\rangle}{\langle v, e_2\rangle} > \dots > \frac{\langle y, e_n\rangle}{\langle v, e_n\rangle}, \tag{43}$$

$$\frac{\langle y - x, d_1\rangle}{\langle w, d_1\rangle} \ge \frac{\langle y - x, d_2\rangle}{\langle w, d_2\rangle} \ge \dots \ge \frac{\langle y - x, d_n\rangle}{\langle w, d_n\rangle} \tag{44}$$

(*provided* v *is e-positive and* w *is d-positive*).

An application of Theorem 6.7 to GIC orderings is included in the next result.

THEOREM 6.9 ([59, Theorem 13]). *Let* $\prec_G$ *be a GIC ordering on* V *induced by compact group* $G \subset O(V)$ *and closed convex cone* $D \subset V$.

Let V_0 *be a subspace of* V *and* $(e_1, \dots, e_n)$ *and* $(s_1, \dots, s_n)$ *be dual bases in* V_0 *such that* $C = \operatorname{cone}\{e_1, \dots, e_{n_1}\}$ *and* $D = \operatorname{cone}\{s_1, \dots, s_{n_2}\}$ *are dual cones in* V_0 *with* $n_1 \le n \le n_2$, *where* $s_{n+1}, \dots, s_{n_2} \in V$.

Assume $w, v \in V$ *with* $\langle w, v\rangle > 0$. *Let* $x_0, y_0 \in V$ *satisfy* $x_0, y_0 \in \operatorname{ri} D$, $x_0 \prec_G y_0$, *and* $\langle x_0, v\rangle = \langle y_0, v\rangle$.

Put $x = x_0 + u + \alpha w$ *and* $y = y_0 + u + \beta w$, *where* $u \in V$ *and* $\alpha, \beta \in \mathbb{R}$ *are such that* $u + \alpha w \in \operatorname{span} v$ *and* $u + \beta w \in \operatorname{span} v$.

Let $F \in \mathcal{S}(A, x, y)$, *where* $A = \operatorname{ri} D + \operatorname{span} v$.

Under the above assumptions, the following three assertions hold.

(A) $F(y) - F(x) \ge (\beta - \alpha) \int_0^1 \langle \nabla F(x + th), w\rangle \, dt.$

(B) *If* $\alpha = \beta$ *then* $F(x) \le F(y)$.

(C) *If* $\alpha \le \beta$ *and* $\langle \nabla F(z), w\rangle \ge 0$ *for* $z \in [x, y]$, *then* $F(x) \le F(y)$.

6.3. Extended *G*-majorization. For classical majorization $\prec$ on $\mathbb{R}^n$ the following criterions are well-known.

THEOREM 6.10 ([39, Theorem B.1, p. 186], [75, Lemma 4]). *Let* $x, y \in \mathbb{R}^n$, $y_1 \geq y_2 \geq \ldots \geq y_n$, $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. *If there exists* m ($1 \leq m \leq n$, $m \in \mathbb{N}$) *such that*

$$y_i \leq x_i \text{ for } i = 1, 2, \ldots, m \text{ and } y_j \geq x_j \text{ for } j = m+1, m+2, \ldots, n,$$

then

$$y \prec x.$$

THEOREM 6.11 ([40, Theorem 2.4]). *If* $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *with* $x_i > 0$, $1 \leq i \leq n$, *and* $y_1 \geq y_2 \geq \ldots \geq y_n > 0$ *and*

$$\frac{x_1}{y_1} \geq \ldots \geq \frac{x_n}{y_n},$$

then

$$\frac{y}{\sum_{j=1}^n y_j} \prec \frac{x}{\sum_{j=1}^n x_j}.$$

We extend Theorems 6.10 and 6.11 from the classical majorization $\prec$ to group-induced cone orderings.

THEOREM 6.12 ([60, Theorem 3.1]). *Assume that* $\prec_G$ *is a GIC ordering on* V *induced by finite group* G *and convex cone* $D_0 = \operatorname{cone}\{s_1, s_2, \ldots, s_q\}$ *with* $s_1, s_2, \ldots, s_q \in V$. *Set* $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$, *where* $s_1, s_2, \ldots, s_n$ *form a basis of* V, $n \leq q$.

Let $x, y, v \in V$ *with* $\langle x, v\rangle y \in D_0$ (*e.g.*, $\langle x, v\rangle \geq 0$ *and* $y \in D_0$). *Suppose that for* $k = 1, 2, \ldots, n$ *there exist dual bases* $e = (e_1, \ldots, e_n)$ *and* $d = (d_1, \ldots, d_n)$ *of* V, *and index sets* J_1 *and* J_2 *with* $J_1 \cup J_2 = \{1, 2, \ldots, n\}$ *such that*

(i) x *is* η, y*-separable on* J_1 *and* J_2 *w.r.t.* e *with* $\eta = \langle x, v\rangle / \langle y, v\rangle$ *and* $\langle y, v\rangle > 0$,
(ii) s_k *is* v*-separable on* J_1 *and* J_2 *w.r.t.* d.

If $n < q$, *assume additionally that*

$$\big\langle \langle x, v\rangle y, s_k \big\rangle \leq \big\langle \langle y, v\rangle x, s_k \big\rangle \quad \text{for } k = n+1, \ldots, q. \tag{45}$$

Then the following G*-majorization inequality holds*:

$$\langle x, v\rangle y \prec_G \langle y, v\rangle x. \tag{46}$$

If $\langle x, v\rangle \langle y, v\rangle > 0$ *then* (46) *becomes*

$$\frac{y}{\langle y, v\rangle} \prec_G \frac{x}{\langle x, v\rangle}.$$

THEOREM 6.13 ([60, Theorem 3.4]). *Assume that* $\prec_G$ *is a GIC ordering on* V *induced by finite group* G *and convex cone* $D_0 = \operatorname{cone}\{s_1, s_2, \ldots, s_q\}$ *with* $s_1, s_2, \ldots, s_q \in V$. *Set* $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$, *where* $s_1, s_2, \ldots, s_n$ *form a basis of* V, $n \leq q$.

Let $x, y, v \in V$ *be such that* $\langle x, v\rangle > 0$, $\langle y, v\rangle > 0$ *and* $y \in D_0$.

Suppose that there exist a basis $e = (e_1, e_2, \ldots, e_n)$ *and an index* $m \in \{0, 1, 2, \ldots, n\}$ *such that*

(i) x *is* η, y*-separable on* J_1 *and* J_2 *w.r.t.* e, *where* $\eta = \frac{\langle x, v\rangle}{\langle y, v\rangle}$ *and* $J_1 = \{1, 2, \ldots, m\}$, $J_2 = \{m+1, m+2, \ldots, n\}$,
(ii) *the vectors* $s_1, s_2, \ldots, s_n$, $e_1, e_2, \ldots, e_n$ *and* v *satisfy conditions*

$$s_i \in \operatorname{cone}\{e_1, \ldots, e_m\} \quad \text{and} \quad v - s_j \in \operatorname{cone}\{e_{m+1}, \ldots, e_n\} \quad \text{for } i \in J_1 \text{ and } j \in J_2.$$

If $n < q$, assume additionally that (45) *holds. Then*

$$\frac{y}{\langle y, v\rangle} \prec_G \frac{x}{\langle x, v\rangle}. \tag{47}$$

If in addition $\langle x, v\rangle = \langle y, v\rangle$ then

$$y \prec_G x. \tag{48}$$

COROLLARY 6.14 ([60, Corollary 3.5]). *Under the hypotheses of Theorem* 6.13, *let conditions* (i)–(ii) *of Theorem* 6.13 *be replaced by the following statement*:

There exists a basis $e = (e_1, e_2, \ldots, e_n)$ of V such that

(i) $\dfrac{\langle x, e_1\rangle}{\langle y, e_1\rangle} \geq \dfrac{\langle x, e_2\rangle}{\langle y, e_2\rangle} \geq \ldots \geq \dfrac{\langle x, e_n\rangle}{\langle y, e_n\rangle}$ (*with positive denominators*),

(ii) *the vectors $s_1, s_2, \ldots, s_n$, $e_1, e_2, \ldots, e_n$ and v satisfy conditions*

$$s_i = e_1 + e_2 + \ldots + e_i \quad \textit{for} \quad i = 1, 2, \ldots, n, \qquad \textit{and} \qquad v = s_n.$$

Then inequality (47) *holds.*

By comparing (47) and (48) we can call the relation (47) *extended G-majorization* of y and x with respect to v.

7. Shi and Ky Fan type inequalities. H.-N. Shi [67] proved the following majorization result (see also [26, Lemmas 2.2–2.3], [39, p. 9], [74, Lemma 2]).

THEOREM 7.1 ([67, pp. 81–83]). *If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, $\sum_{i=1}^n x_i = s > 0$, $c \geq s$, then*

$$\frac{cv + x}{nc + s} \prec \frac{x}{s} \qquad \textit{and} \qquad \frac{cv - x}{nc - s} \prec \frac{x}{s},$$

where $v = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

We denote by $A(z_1, z_2, \ldots, z_n)$ and $G(z_1, z_2, \ldots, z_n)$ the *arithmetic* and *geometric means*, respectively, of positive numbers $z_1, z_2, \ldots, z_n$.

The following Ky Fan's result is of interest.

THEOREM 7.2 ([5, p. 5]). *Let $x_i \in (0, 1/2]$ $(i = 1, 2, \ldots, n)$. Then the following Ky Fan's inequality holds*:

$$\frac{G(x_1, x_2, \ldots, x_n)}{G(1 - x_1, 1 - x_2, \ldots, 1 - x_n)} \leq \frac{A(x_1, x_2, \ldots, x_n)}{A(1 - x_1, 1 - x_2, \ldots, 1 - x_n)}.$$

Our purpose in this section is to give an extension of Theorems 7.1 and 7.2 from the classical majorization preorder $\prec$ to a class of group-induced cone orderings $\prec_G$. As applications, some Ky Fan type inequalities are established.

7.1. G-majorization inequalities of Shi type. In the forthcoming theorems we provide extensions of the above result of Shi (see also [39, p. 9]).

THEOREM 7.3 ([58, Theorem 7]). *Let (V, G, D) be an Eaton triple and let $M_G(V) = \operatorname{span} v$ for some nonzero $v \in V$. Then*

$$\frac{cv + x}{\langle cv + x, v\rangle} \prec_G \frac{x}{\langle x, v\rangle} \qquad \textit{for } x \in V \textit{ with } \langle x, v\rangle > 0,\ c \geq 0.$$

We need some notation.

Definition 7.4. A sequence $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ is said to be *relatively convex* (resp., *relatively concave*) with respect to sequence $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ if

$$\begin{vmatrix} 1 & b_k & a_k \\ 1 & b_l & a_l \\ 1 & b_m & a_m \end{vmatrix} \geq (\leq)\, 0,$$

whenever $k, l, m \in \{1, 2, \dots, n\}$ and $b_k \leq b_l \leq b_m$ (cf. [44, p. 2]).

Theorem 7.5 ([58, Theorem 11]). *Let (V, G, D) be an Eaton triple with $M_G(V) = \operatorname{span} v$ for some nonzero $v \in V$, $\|v\| > 1$. Let $D = \operatorname{cone}\{t_1, \dots, t_n, t_{n+1}\}$ with $t_0 = 0$, $t_n = v$, $t_{n+1} = -t_n$ and $1 \leq \langle t_i, v\rangle \leq \langle t_n, v\rangle$ for $i = 1, \dots, n$, and let $g_0 \in G$ be such that $g_0 D = -D$ and $t_{n-j} = v - g_0^{-1} t_j$ for $j = 1, 2, \dots, n$.*

Assume that $x \in V$ and $c \in \mathbb{R}$ are such that $c \geq \langle x, v\rangle > 0$.

Let $a = (a_0, a_1, \dots, a_n)$ and $b = (b_0, b_1, \dots, b_n)$, where $a_i = \langle t_i, x_\downarrow\rangle$ and $b_i = \langle t_i, v\rangle$ for $i = 0, 1, 2, \dots, n$. Suppose that a is relatively concave with respect to b. Then

$$\frac{cv - x}{\langle cv - x, v\rangle} \prec_G \frac{x}{\langle x, v\rangle}.$$

7.2. Generalization of Ky Fan inequality. An application of Theorems 7.3 and 7.5 leads to the following corollaries.

Corollary 7.6 ([58, Corollary 12]). *Under the assumptions of Theorem 7.3, let $\varphi : A \to (0, +\infty)$ be a positively homogeneous G-increasing (resp. G-decreasing) function on G-invariant set $A \subset V$.*

If $x \in A$ and $c \geq 0$ with $\langle x, v\rangle > 0$, then

$$\frac{\langle x, v\rangle}{\langle cv + x, v\rangle} \leq (\geq) \frac{\varphi(x)}{\varphi(cv + x)}. \tag{49}$$

Corollary 7.7 ([58, Corollary 13]). *Under the assumptions of Theorem 7.5, let $\varphi : A \to (0, +\infty)$ be a positively homogeneous G-increasing (resp. G-decreasing) function on G-invariant set $A \subset V$.*

If $x \in A$ and $c \in \mathbb{R}$ with $0 < \langle x, v\rangle \leq c$, then

$$\frac{\langle x, v\rangle}{\langle cv - x, v\rangle} \leq (\geq) \frac{\phi(x)}{\phi(cv - x)}. \tag{50}$$

See [26, 36, 67, 76] and references therein for a number of results of type (49)–(50) for S-convex or S-concave functions φ.

Example 7.8. Let $V = \mathbb{R}^n$, $G = \mathbb{P}_n$, $D = \mathbb{R}^n_\downarrow$, $v = (1, 1, \dots, 1)$, $c = 1$ and

$$\varphi(z) = \left(\prod_{i=1}^n z_i\right)^{1/n} \qquad \text{for } z = (z_1, z_2, \dots, z_n) \in (0, +\infty)^n.$$

Then inequality (49) of Corollary 7.6 leads to the following *Xia-Chu's inequality*:

$$\frac{A(x_1, \dots, x_n)}{A(1 + x_1, \dots, 1 + x_n)} \geq \frac{G(x_1, \dots, x_n)}{G(1 + x_1, \dots, 1 + x_n)} \qquad \text{for } 0 < x_i,\ i = 1, \dots, n,$$

where $A(\cdot)$ and $G(\cdot)$ denote the arithmetic and geometric means of $(\cdot)$ [76, Corollary 4.3].

Using (50) of Corollary 7.7 yields the following variant of *Ky Fan's inequality*:

$$\frac{A(x_1,\dots,x_n)}{A(1-x_1,\dots,1-x_n)} \geq \frac{G(x_1,\dots,x_n)}{G(1-x_1,\dots,1-x_n)}$$

Whenever $0 < \sum_{i=1}^n x_i \leq 1$, $i = 1,\dots,n$.

8. Applications to Grüss and Ostrowski type inequalities. We denote by $L^p_{[a,b]}$ $(1 \leq p < \infty)$ the space of p-power integrable functions on interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p\,dt\right)^{1/p}.$$

The symbol $L^\infty_{[a,b]}$ stands for the space of all essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in[a,b]} |f(x)|.$$

For two real functions $f,g:[a,b]\to\mathbb{R}$ such that $f,g,fg \in L^1_{[a,b]}$, the *Chebyshev functional* is defined by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

Chebyshev [8] proved that if $f',g' \in L^\infty_{[a,b]}$, then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2\|f'\|_\infty\,\|g'\|_\infty.$$

Grüss [25] showed that

$$|T(f,g)| \leq \frac{1}{4}(\beta_0-\alpha_0)(\delta_0-\gamma_0) \tag{51}$$

provided two bounded integrable functions $f,g:[a,b]\to\mathbb{R}$ satisfy

$$\alpha_0 \leq f(t) \leq \beta_0 \quad \text{and} \quad \gamma_0 \leq g(t) \leq \delta_0 \quad \text{for } t\in[a,b].$$

See [10, 21, 31, 37] for generalizations and extensions of (51).

8.1. Grüss type inequalities. Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ endowed with the corresponding norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. A *Grüss type inequality* estimates from above the quantity $|\langle x,y\rangle - \langle x,v\rangle\langle v,y\rangle|$ with $x,y,v\in V$, $\|v\|=1$ (cf. [11, 10, 65]).

Remind that if $K\subset V$ is a convex cone, then $\leq_K$ is cone preorder on V defined by

$$y \leq_K x \quad \text{iff} \quad x-y\in K.$$

THEOREM 8.1 ([56, Theorem 4.2]). *Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and let $v\in V$ with $\|v\|=1$. Assume that $x,y,\alpha,\beta,\gamma,\delta\in V$ are vectors such that*

(a) *$\alpha+\beta\in\operatorname{span} v$ and $\gamma+\delta\in\operatorname{span} v$,*
(b) *$\alpha\leq_{K_1} x \leq_{\operatorname{dual} K_1} \beta$ and $\gamma \leq_{K_2} y \leq_{\operatorname{dual} K_2} \delta$ for some convex cones $K_1,K_2\subset V$.*

Then we have the inequality

$$\big|\langle x,y\rangle - \langle x,v\rangle\langle v,y\rangle\big| \leq \frac{1}{4}\|\beta-\alpha\|\,\|\delta-\gamma\|.$$

If the vectors $\alpha, \beta, \gamma, \delta$ are proportional to v, Theorem 8.1 reduces to [11, Theorem 1].

Remind that a function $\varphi : [a, b] \to \mathbb{R}$ is said to be a *constant function*, if there exists a constant $c \in \mathbb{R}$ such that $\varphi(t) = c$ for $t \in [a, b]$.

COROLLARY 8.2 ([56, Corollary 4.5]). *Let $f, g, \alpha, \beta, \gamma, \delta \in L^2_{[a,b]}$ be functions such that*

(a) *$\alpha + \beta$ and $\gamma + \delta$ are constant functions,*
(b) *$\alpha(t) \le f(t) \le \beta(t)$ and $\gamma(t) \le g(t) \le \delta(t)$ for $t \in [a, b]$, or, more generally,*

$$\int_a^b (\beta(t) - f(t))(f(t) - \alpha(t))\, dt \ge 0 \quad \text{and} \quad \int_a^b (\delta(t) - g(t))(g(t) - \gamma(t))\, dt \ge 0.$$

Then we have the inequality

$$|T(f, g)| \le \frac{1}{4(b-a)} \left(\int_a^b (\beta(t) - \alpha(t))^2\, dt \right)^{1/2} \left(\int_a^b (\delta(t) - \gamma(t))^2\, dt \right)^{1/2}. \tag{52}$$

For many functions f and g with appropriate choice of functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$, inequality (52) provides a tighter estimate than (51) (see [56, Example 4.6]).

8.2. Ostrowski–Grüss type inequalities. A related result to (51) is the following Ostrowski's inequality [43, p. 468].

THEOREM 8.3 ([43, p. 468]). *If $f:[a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \frac{\left(x - (a+b)/2\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \qquad \text{for } x \in [a, b]. \tag{53}$$

Dragomir and Wang [14] showed Ostrowski–Grüss type inequality as follows.

THEOREM 8.4 ([14]). *If $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative such that*

$$\alpha_0 \le f'(t) \le \beta_0 \qquad \text{for } t \in [a, b],$$

then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \le \frac{1}{4} (b-a)(\beta_0 - \alpha_0) \qquad \text{for } x \in [a, b]. \tag{54}$$

Some improvements of (54) can be found in [9, 41].

Grüss and Ostrowski type inequalities have many applications (see [10, 13, 14, 38, 63]).

Here we present an extension of Grüss' inequality in L^p-spaces (cf. [7, Theorems 2 and 3], [12, p. 2]).

THEOREM 8.5 ([57, Theorem 2.1]). *Let $f, \alpha, \beta \in L^p_{[a,b]}$ and $g \in L^q_{[a,b]}$ ($\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p \le \infty$) be functions such that*

(a) *$\alpha + \beta$ is a constant function, and*
(b) *$\alpha(t) \le f(t) \le \beta(t)$ for $t \in [a, b]$.*

Then we have the inequality

$$|T(f,g)| \le \frac{1}{2(b-a)} \|\beta - \alpha\|_p \cdot \left\| g - \frac{1}{b-a}\int_a^b g(t)\,dt \right\|_q . \tag{55}$$

THEOREM 8.6 ([57, Theorem 2.4]). *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. Suppose that $f', \alpha, \beta \in L^p_{[a,b]}$ $(1 \le p \le \infty)$ are functions such that*

(a) *$\alpha + \beta$ is a constant function, and*
(b) *$\alpha(t) \le f'(t) \le \beta(t)$ for $t \in [a,b]$.*

Then for $x \in [a,b]$

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} \right| \le \begin{cases} \frac{1}{4}\|\beta-\alpha\|_p \frac{(b-a)^{1/q}}{(q+1)^{1/q}} & \text{if } 1 \le q < \infty, \\ \frac{1}{4}\|\beta-\alpha\|_1 & \text{if } q = \infty, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

COROLLARY 8.7 ([57, Corollary 2.5]). *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. Suppose that $\alpha_0, \beta_0 \in \mathbb{R}$ are numbers such that $\alpha_0 \le f'(t) \le \beta_0$ for $t \in [a,b]$.*

Then for $x \in [a,b]$ and $1 \le q \le \infty$

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a} \right| \le \begin{cases} \frac{1}{4(q+1)^{1/q}}(\beta_0-\alpha_0)(b-a) & \text{if } 1 \le q < \infty, \\ \frac{1}{4}(\beta_0-\alpha_0)(b-a) & \text{if } q = \infty. \end{cases} \tag{56}$$

The case $q = 1$ of Corollary 8.7 yields a result of Cheng [9, Theorem 1.5] with the factor $\frac{1}{8}$ on the right-hand side of (56) (cf. [38, Theorem 3]). If $q = 2$ then Corollary 8.7 becomes a result of Matić *et al.* [41, Theorem 6] with the factor $\frac{1}{4\sqrt{3}}$ on the right-hand side of (56). Finally, the case $q = \infty$ leads to inequality (54) of Dragomir and Wang [14, Theorem 2.1].

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