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D'ALEMBERT'S FUNCTIONAL EQUATION ON GROUPS

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Abstract. Given a (not necessarily unitary) character $\mu : G \to (\mathbb{C} \setminus \{0\}, \cdot)$ of a group G we find the solutions $g : G \to \mathbb{C}$ of the following version of d'Alembert's functional equation

$$g(xy) + \mu(y)g(xy^{-1}) = 2g(x)g(y),\ x, y \in G. \tag{$*$}$$

The classical equation is the case of $\mu = 1$ and $G = \mathbb{R}$. The non-zero solutions of $(*)$ are the normalized traces of certain representations of G on $\mathbb{C}^2$. Davison proved this via his work [20] on the pre-d'Alembert functional equation on monoids.

The present paper presents a detailed exposition of these results working directly with d'Alembert's functional equation. In the process we find for any non-abelian solution g of $(*)$ the corresponding solutions $w : G \to \mathbb{C}$ of

$$w(xy) + w(yx) = 2w(x)g(y) + w(y)g(x),\ x, y \in G. \tag{$**$}$$

A novel feature is our use of the theory of group representations and their matrix-coefficients which simplifies some arguments and relates the results to harmonic analysis on groups.

1. Introduction. The continuous solutions $g : \mathbb{R} \to \mathbb{C}$ of d'Alembert's functional equation on $\mathbb{R}$:

$$g(x + y) + g(x - y) = 2g(x)g(y) \text{ for all } x, y \in \mathbb{R}, \tag{1}$$

are known. Apart from the trivial solution $g = 0$ they are $g_\lambda(x) = \cos(\lambda x)$, $x \in \mathbb{R}$, where the parameter λ ranges over $\mathbb{C}$. That $g(x) = \cos x$ is a solution of (1) explains why the equation is also called the *cosine functional equation*. It has other well known solutions than $\cos x$, for example $g(x) = \cos(ix) = \cosh x$.

We will discuss generalizations of (1) in which the underlying space $\mathbb{R}$ is replaced by a topological group G. More precisely we seek continuous, complex-valued solutions

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$g : G \to \mathbb{C}$ of *d'Alembert's classical functional equation*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y) \text{ for all } x, y \in G. \tag{2}$$

Actually we work in an even more general setting, because we shall consider *d'Alembert's functional equation*

$$g(xy) + \mu(y)g(xy^{-1}) = 2g(x)g(y) \text{ for all } x, y \in G, \tag{3}$$

where $\mu : G \to \mathbb{C}^*$ is a continuous character on G. d'Alembert's classical functional equation is the special case of $\mu = 1$. (3) with $\mu \neq 1$ occurs in the recent literature. See [20, Proposition 2.11] and [42, Lemma IV.4], and Section 7. On groups the study of (3) is equivalent to the study of the pre-d'Alembert functional equation ([20, Proposition 2.11]).

There has been quite a development of the theory of d'Alembert's functional equations (2) and (3) during the last ten years, in particular on non-abelian groups, as shown in works by Dilian Yang about compact groups [47, 49, 48], the present author [41, 45] for step 2 nilpotent groups, Friis [35] for results on Lie groups and Davison [19, 20] for general groups, even monoids. We feel that the theory is quite satisfactory at the present stage. The purpose of this paper is to describe the theory with detailed proofs, making it accessible to mathematicians interested in functional equations. In the non-abelian case an important new feature of our exposition here is the use of the theory of matrix-coefficients of group representations that enables us to simplify and clarify some arguments in the literature. In the abelian case we go via the sine addition formula which is not the standard approach found in text books, see for example [2].

Our main results are Theorem 6.1 and Corollary 10.2, that characterize the solutions of d'Alembert's functional equation (3) in the abelian and non-abelian case, respectively. In both cases any solution $g \in C(G) \setminus \{0\}$ has the form $g = \frac{1}{2} \operatorname{tr} \rho$, where ρ is a continuous representation of G on $\mathbb{C}^2$. This was shown for compact groups by Yang [47] and in general by Davison [20]. We prove that g is abelian if and only if ρ is reducible. In the process we characterize the solutions of $(**)$ for given non-abelian g as certain elements in the space of matrix-coefficients of ρ.

We have tried to make the exposition self-contained by collecting the few facts needed about matrix-coefficients of group representations in appendices.

For other approaches and generalizations see Section 12.

2. Notation and terminology. Throughout this paper G is a topological group with neutral element e. The discrete topology on G is allowed. $C(G)$ denotes the space of continuous, complex-valued functions on G. Usually we write the group operation multiplicatively, but if the group is abelian we mainly use $+$. The multiplicative group $\mathbb{C} \setminus \{\cdot\}$ is denoted by $\mathbb{C}^*$. We let $\mu : G \to \mathbb{C}^*$ be a fixed continuous character on G.

Let $F : G \to \mathbb{C}$. For $x \in G$ we define the Cauchy difference $F_x : G \to \mathbb{C}$ by $F_x(y) := F(xy) - F(x)F(y)$. The right translate of F by $x \in G$ is the function $[R(x)F](y) := F(yx)$, $y \in G$. We write $\check{F}(y) := F(y^{-1})$, $y \in G$.

By a vector space we mean a complex vector space.

Definition 2.1. A *d'Alembert function* on the topological group G is a function $g \in C(G)$ such that g satisfies (3) and $g(e) = 1$.

3. On group characters

DEFINITION 3.1. A (*group*) *character* on G is a homomorphism $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group $\mathbb{C}^* := (\mathbb{C} \setminus \{0\}, \cdot)$ of non-zero complex numbers.

Euler's formula

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{4}$$

is a presage for much of the theory. It says that cosine is a mean of the two characters $x \mapsto e^{\pm ix}$. In this paper we describe how Euler's formula extends from $\cos x$ to solutions of d'Alembert's functional equation (3).

LEMMA 3.2. *If $\chi : G \to \mathbb{C}$ satisfies $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in G$, then either χ is identically 0 or it is a character. If it is a character, then $\chi(e) = 1$.*

THEOREM 3.3 (Artin). *The characters on G form a linearly independent set in the vector space of complex-valued functions on G.*

Proof. See [29, Lemma 29.41]. ∎

COROLLARY 3.4. *Let $\chi_1, \chi_2, \chi_3, \chi_4$ be characters on G. If $\chi_1 + \chi_2 = \chi_3 + \chi_4$, then $\chi_1 = \chi_3$ or $\chi_1 = \chi_4$, so the decomposition $\chi_1 + \chi_2$ into characters is unique up to interchange of χ_1 and χ_2.*

4. Abelian functions. It is usually simpler to treat functions on abelian than on non-abelian groups. Fortunately, the group commutativity can sometimes be compensated by properties of the functions under scrutiny. Let us mention two properties:

If a function f on G is constant on each coset $x[G, G]$, $x \in G$, then f may be viewed as (we write is) a function on the abelian group $G/[G, G]$. We say that a function is *abelian*, if it is a function on $G/[G, G]$. This is equivalent to *Kannappan's condition* $f(xyz) = f(xzy)$ for all $x, y, z \in G$. If f is abelian, then $f(x_1x_2 \cdots x_n) = f(x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n)})$ for all $x_1, x_2, \ldots, x_n \in G$, all permutations π of n elements and all $n = 1, 2, 3, \ldots$. In other words, if the argument of f is a product then its factors can be permuted arbitrarily without changing the value of f on it. We say that a function is *non-abelian* if it is not abelian.

A function $f : G \to \mathbb{C}$ is *central*, if $f(xy) = f(yx)$ for all $x, y \in G$. Any abelian function is central, but the converse does not hold: the trace is a central, but not abelian, function on matrices.

5. Relations to the sine addition formula. In our discussion of abelian d'Alembert functions we shall use properties of the solutions of the *sine addition formula*

$$f(xy) = f(x)g(y) + f(y)g(x) \text{ for all } x, y \in G. \tag{5}$$

THEOREM 5.1. *The pairs of functions $g, f \in C(G)$ satisfying (5) can be described as follows*:

(a) *$f = 0$ and g arbitrary in $C(G)$.*
(b) *$g = \chi/2$ and $f = c\chi$, where $\chi \in C(G)$ is a character on G and $c \in \mathbb{C} \setminus \{0\}$ a constant.*

(c) *There exist two different characters* $\chi_1, \chi_2 \in C(G)$ *on* G *and a constant* $c \in \mathbb{C} \setminus \{0\}$ *such that*

$$g = \frac{\chi_1 + \chi_2}{2} \text{ and } f = c(\chi_1 - \chi_2).$$

(d) *There exist a character* $\chi \in C(G)$ *on* G *and an additive function* $a \in C(G)$, $a \neq 0$, *such that* $g = \chi$ *and* $f = \chi a$.

The main idea of Theorem 5.1 can be found in Glaeser [28] from 1951. It has been rediscovered several times.

The *symmetrized sine addition formula*

$$w(xy) + w(yx) = 2w(x)g(y) + 2w(y)g(x), \ x, y \in G, \tag{6}$$

where $w, g \in C(G)$ are the unknown functions, reduces to the sine addition formula when w is central. Proposition 5.2(d) connects (6) to d'Alembert's functional equation.

PROPOSITION 5.2. *Let* $g : G \to \mathbb{C}$ *be a solution of* (3).

(a) *If* $g \neq 0$, *then* $g(e) = 1$.
(b) *The invariance formula* $\mu(x)g(x^{-1}) = g(x)$ *holds for all* $x \in G$.
(c) g *is central.*
(d) *The pair* (g_x, g) *is a solution of* (6) *for any* $x \in G$.

Proof. (a) Put $y = e$ in (3).

(b) is trivial if $g = 0$, so we assume that $g \neq 0$. We get (b) when we put $x = e$ in (3) and apply (a).

(c) Using (b) for the fourth equality sign the computation

$$\begin{aligned} g(xy) + \mu(y)g(xy^{-1}) &= 2g(x)g(y) = 2g(y)g(x) = g(yx) + \mu(x)g(yx^{-1}) \\ &= g(yx) + \mu(x)\mu(yx^{-1})g((yx^{-1})^{-1}) = g(yx) + \mu(y)g(xy^{-1}) \end{aligned}$$

shows g is central.

(d) For any $x, y, z \in G$ we get from (3) that

$$\begin{aligned} g(xyz) + \mu(z)g(xyz^{-1}) &= 2g(xy)g(z), \\ g(xyz^{-1}) + \mu(yz^{-1})g(xzy^{-1}) &= 2g(x)g(yz^{-1}), \text{ and} \\ g(xzy^{-1}) + \mu(y^{-1})g(xzy) &= 2g(xz)g(y^{-1}). \end{aligned}$$

When we then multiply the middle identity by $\mu(z)$, the third by $\mu(y)$ and subtract the middle one from the sum of the two others we find

$$g(xyz) + g(xzy) = 2g(xy)g(z) - 2g(x)g(yz^{-1})\mu(z) + 2g(xz)\mu(y)g(y^{-1}),$$

from which small reformulations based on (b) and (c) reveal that (g_x, g) is a solution of (6). ∎

6. Abelian solutions. The formula (7) below generalizes Euler's formula (4) for cos to abelian d'Alembert functions.

THEOREM 6.1. *Let $\mu : G \to \mathbb{C}^*$ be a character on G and $g : G \to \mathbb{C}$ be a non-zero abelian solution of* (3). *Then there exists a character χ of G such that*

$$g = \frac{\chi + \mu\check{\chi}}{2}\,. \tag{7}$$

The character χ in decomposition (7) *of g is unique, except χ can be replaced by $\mu\check{\chi}$.*

Conversely, any function g of the form (7), *where χ is a character, is a non-zero abelian solution of* (3).

Proof. According to Proposition 5.2(d) the pair (g_x, g) is a solution of (6) for any $x \in G$. Since g is abelian it is a solution of (5) with $f = g_x$, so Theorem 5.1 applies.

Assume that there exists an $x \in G$ such that $g_x \neq 0$. This means Theorem 5.1(a) does not occur. Nor does (b), because $g(e) = 1$. In the remaining two cases of Theorem 5.1 $g = (\chi_1 + \chi_2)/2$ for some characters $\chi_1, \chi_2 \in C(G)$. When we substitute this into (3) we get after some elementary computations that

$$\chi_1(x)\big[\mu(y)\check{\chi}_1(y) - \chi_2(y)\big] + \chi_2(x)\big[\mu(y)\check{\chi}_2(y) - \chi_1(y)\big] = 0 \quad \forall x, y \in G.$$

If $\chi_1 \neq \chi_2$, then χ_1 and χ_2 are linearly independent (Lemma 3.3), so the coefficients to $\chi_1(x)$ and $\chi_2(x)$ are 0. In particular $\mu(y)\check{\chi}_1(y) - \chi_2(y) = 0$, which gives the desired formula (7). If $\chi_1 = \chi_2$ we divide by $\chi_1(x) = \chi_2(x)$ and get again the desired result.

If $g_x = 0$ for all $x \in G$, then g is multiplicative and hence a character (Lemma 3.2), say $g = \chi$. Now $g = (\chi + \chi)/2$, so the arguments above may be reused with $\chi_1 = \chi_2 = \chi$.

The uniqueness of χ is due to Corollary 3.4.

The statement about the converse is a simple computation, so we skip it. ∎

Kannappan [31] was the first to derive Theorem 6.1 for a general group (in 1968). He had $\mu = 1$. The proofs in [31] are different from the ones found above.

Any non-abelian solution g is the normalized trace of a representation of G on $\mathbb{C}^2$ (Corollary 10.2). This is the case here as well, because

$$g = \frac{1}{2}\operatorname{tr}\begin{pmatrix} \chi & 0 \\ 0 & \mu\check{\chi} \end{pmatrix},$$

but here the representation is reducible in contrast to the non-abelian case.

It may happen for a non-abelian group that all classical d'Alembert functions on it are abelian. According to [45, Theorem 4.2] this is so for all nilpotent groups which are generated by their squares. An example of such a group is the Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

7. Non-abelian examples. With the complete description of abelian d'Alembert functions in Theorem 6.1 in mind an obvious question is whether all d'Alembert functions are abelian. Kannappan posed the question in [32, Section I] from 1971, but it took a while to answer it. The answer is no. The examples g_0 and g_1 below of non-abelian d'Alembert functions were published respectively by Corovei [18, Example p. 105–106] in 1977 and Ng [1, Remark 5] in 1989.

Lemma 7.1. *The normalized trace*

$$t(x) := \frac{1}{2}\operatorname{tr}(x),\ \ x \in GL(2,\mathbb{C}), \tag{8}$$

is a continuous solution of (3) *with* $\mu(x) = \det x$ *on* $GL(2,\mathbb{C})$.

Proof. By the Cayley–Hamilton theorem any $x \in M(2\times 2,\mathbb{C})$ is annulled by its characteristic polynomial, so that $x^2 - (\operatorname{tr} x)x + (\det x)I = 0$. Multiplying this by x^{-1} we get $x + (\det x)x^{-1} = (\operatorname{tr} x)I$. The rest of the proof is a straightforward substitution of t into (3). ∎

It follows that the restrictions of t to $SL(2,\mathbb{C})$ and $SU(2)$ are classical d'Alembert function on $SL(2,\mathbb{C})$ and $SU(2)$. We shall see below that t and the restrictions are non-abelian.

Examples of non-abelian d'Alembert functions related to t can be found in the literature. To get some of them we let $\mathbb{H} = \{a+bi+cj+dk \mid a,b,c,d\in\mathbb{R}\}$ denote the division algebra of the quaternions, introduce the *group of unit quaternions* $\{a+bi+cj+dk \in \mathbb{H} \mid a^2+b^2+c^2+d^2 = 1\}$ and a certain subgroup of it, viz. the quaternion group $Q_8 := \{\pm1,\pm i,\pm j,\pm k\}$. The map

$$a+bi+cj+dk \mapsto \begin{pmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{pmatrix} \tag{9}$$

is an algebra-isomorphism of $\mathbb{H}$ into the algebra of 2×2 matrices with complex entries, both viewed as algebras over the field $\mathbb{R}$. It is well known that the restriction Φ of the map to the group of unit quaternions is a topological isomorphism of it onto $SU(2)$.

Φ transfers $g := t|_{SU(2)}$ from $SU(2)$ to the d'Alembert function $g_1 = g\circ\Phi$ on the group of unit quaternions. A computation produces the explicit formula $g_1(a+bi+cj+dk) = a$ for g_1.

Restricting g_1 from the group of unit quaternions to the quaternion group Q_8 gives a d'Alembert function, call it g_0, on Q_8. g_0 is not abelian, because $g_0(ijk) = g_0(-1) = -1$, while $g_0(ikj) = g_0(1) = 1$.

Since g_0 is non-abelian, so is any extension of g_0 to a group containing Q_8. In particular g_1 is non-abelian on the group of unit quaternions. Consequently so is g on $SU(2)$ and t on $SL(2,\mathbb{C})$ and $GL(2,\mathbb{C})$.

8. Technical results. In this section we collect hard technical results that are needed in our discussion of non-abelian d'Alembert functions. They are due to Davison [20]. Although the results are short to formulate, some proofs are regrettably rather tricky, long and computational.

All the results involve a certain function Δ that thus plays a key role for non-abelian d'Alembert functions. We define Δ and derive some of its properties.

Definition 8.1. Given a solution of g d'Alembert's functional equation (3) we define the corresponding function $\Delta = \Delta_g : G\times G \to \mathbb{C}$ by $\Delta(x,y) :=\ g_x(x)g_y(y) - g_x(y)^2$ for $x,y\in G$.

THEOREM 8.2. *Let g be a solution of d'Alembert's functional equation* (3) *and let* Δ *be the function from Definition* 8.1.

(a) $2\Delta(x,y) = g(x^2y^2) - g((xy)^2)$ *for all* $x, y \in G$.
(b) g *is abelian if and only if* $\Delta = 0$.
(c) *Let* $a, b \in G$. *If* $\Delta(a,b) \neq 0$, *then* $w = 0$ *is the only solution* w *of the symmetrized sine addition formula* (6) *for which* $w(a) = w(b) = w(ab) = 0$.

Proof. We use several times that $w(y^2) = 2w(y)g(y)$ for all $y \in G$ for any solution w of (6) (to get this formula put $x = y$ in (6)). In particular $g_x(y^2) = 2g_x(y)g(y)$ for all $y \in G$.

(a) The proposition holds trivially if $g = 0$, so we may assume $g \neq 0$, so that $g(e) = 1$. We compute the two terms in $g(x^2y^2) - g((xy)^2)$ separately. We use that g is central, so that $g_x(y) = g_y(x)$, and that $g_x(y^2) = 2g_x(y)g(y)$ for all $x, y \in G$. And also use the formula $g_x(x) = g(x)^2 - \mu(x)$ for all $x \in G$.

$$\begin{aligned} g(x^2y^2) &= g_{x^2}(y^2) + g(x^2)g(y^2) = 2g_{x^2}(y)g(y) + g(x^2)g(y^2) \\ &= 2g_y(x^2)g(y) + g(x^2)g(y^2) = 4g_y(x)g(x)g(y) + g(x^2)g(y^2) \\ &= 4g_x(y)g(x)g(y) + (g_x(x) + g(x)^2)(g_y(y) + g(y)^2), \end{aligned}$$

while

$$\begin{aligned} g((xy)^2) &= -\mu(xy) + 2g(xy)^2 = -\mu(x)\mu(y) + 2[g_x(y) + g(x)g(y)]^2 \\ &= -\mu(x)\mu(y) + 2g_x(y)^2 + 2g(x)^2g(y)^2 + 4g_x(y)g(x)g(y) \\ &= -[g(x)^2 - g_x(x)][g(y)^2 - g_y(y)] \\ &\quad + 2g_x(y)^2 + 2g(x)^2g(y)^2 + 4g_x(y)g(x)g(y), \end{aligned}$$

from which a small computation shows that $g(x^2y^2) - g((xy)^2) = 2g_x(x)g_y(y) - 2g_x(y)^2$ as desired.

(b) It is trivial that g abelian implies $\Delta = 0$. It is left to show that $\Delta = 0$ implies that g is abelian, so assume that $\Delta = 0$. If we replace y by yz and zy respectively in the expression for $\Delta(x,y) = 0$ we get that $g_x(x)g_{yz}(yz) - g_x(yz)^2 = 0$ and $g_x(x)g_{zy}(zy) - g_x(zy)^2 = 0$. From g being central we find that $g_{yz}(yz) = g_{zy}(zy)$, so that $g_x(yz)^2 = g_x(zy)^2$ or $[g_x(yz) - g_x(zy)][g_x(yz) + g_x(zy)] = 0$. Using the definition of g_x on the first factor and the functional equation on the second factor we obtain that

$$[g(xyz) - g(xzy)][g_x(y)g(z) + g_x(z)g(y)] = 0 \text{ for all } x, y, z \in G. \tag{10}$$

Interchange of x and y (resp. x and z) only changes the first factor by a sign, so we find that

$$[g(xyz) - g(xzy)][g_x(y)g(z) + g_y(z)g(x)] = 0, \text{ and} \tag{11}$$

$$[g(xyz) - g(xzy)][g_y(z)g(x) + g_x(z)g(y)] = 0. \tag{12}$$

Subtracting (11) from the sum of (10) and (12) we obtain that

$$[g(xyz) - g(xzy)]g(x)g_y(z) = 0 \text{ for all } x, y, z \in G. \tag{13}$$

Now an observation: if $g_c(c) = 0$ for a $c \in G$ then $g(xyc) = g(xcy)$ for all $x, y \in G$. Proof: for any $x \in G$ we have

$$0 = \Delta(x,c) = g_x(x)g_c(c) - g_x(c)^2 = -g_x(c)^2 = -[g(xc) - g(x)g(c)]^2,$$

so that $g(xc) = g(x)g(c)$. Finally for any $x, y \in G$ we get $g(xyc) = g(xy)g(c) = g(yx)g(c) = g(yxc) = g(xcy)$.

Going back to (13): if $g_y(z) = 0$ then we read from $0 = \Delta(y,z) = g_y(y)g_z(z) - g_y(z)^2 = g_y(y)g_z(z)$ that either $g_y(y) = 0$ or $g_z(z) = 0$. In either case $g(xyz) - g(xzy) = 0$ by the observation. Thus (13) reduces to

$$[g(xyz) - g(xzy)]g(x) = 0 \text{ for all } x, y, z \in G. \tag{14}$$

The factor $g(x)$ in (14) may be replaced by $g(y)$ and $g(z)$ (symmetry!).

We shall show that the set $D := \{(x,y,z) \in G \times G \times G \,|\, g(xyz) \neq g(xzy)\}$ is empty. Let us note that for any $(x,y,z) \in D$ we have

$$g(x) = g(y) = g(z) = 0, \tag{15}$$

$$g(xy) \neq 0, \text{ and } g(xyz) \neq 0, \tag{16}$$

where the first statement comes from (14) and the second one from the observation above: $0 \neq g_x(y) = g(xy) - g(x)g(y) = g(xy) - 0 = g(xy)$. About the third one we see from Proposition 5.2(d) that $g(xyz) + g(xzy) = 0$. Thus if $g(xyz) = 0$ then so is $g(xzy)$, and hence $g(xyz) = g(xzy)$, contradicting that $g(xyz) \neq g(xzy)$.

Assuming there exists an element $(x_0, y_0, z_0) \in D$ we shall arrive at a contradiction. If $(x_0, y_0, x_0y_0z_0) \in D$, then $g(x_0y_0z_0) = 0$ by (15), contradicting (16). Hence $(x_0, y_0, x_0y_0z_0) \notin D$, so that $g(x_0y_0(x_0y_0z_0)) = g(x_0(x_0y_0z_0)y_0)$. The left hand side is (use the formula $g_x(y^2) = 2g_x(y)g(y)$ in the computations)

$$\begin{aligned} g((x_0y_0)^2z_0) &= g_{z_0}((x_0y_0)^2) + g(z_0)g((x_0y_0)^2) \\ &= 2g_{z_0}(x_0y_0)g(x_0y_0) + 0 \cdot g((x_0y_0)^2) \\ &= 2[g(z_0x_0y_0) - g(z_0)g(x_0y_0)]g(x_0y_0) \\ &= 2g(z_0x_0y_0)g(x_0y_0) = 2g(x_0y_0z_0)g(x_0y_0) \neq 0 \end{aligned}$$

with the $\neq 0$ due to (16). We finally obtain the desired contradiction, because the right hand side vanishes:

$$\begin{aligned} g(x_0^2y_0z_0y_0) &= g_{y_0z_0y_0}(x_0^2) + g(y_0z_0y_0)g(x_0^2) \\ &= 2g_{y_0z_0y_0}(x_0)g(x_0) + g(z_0y_0^2)g(x_0^2) = 0 + [g_{z_0}(y_0^2) + g(z_0)g(y_0^2)]g(x_0^2) \\ &= g_{z_0}(y_0^2)g(x_0^2) = 2g_{z_0}(y_0)g(y_0)g(x_0^2) = 0. \end{aligned}$$

(c) From $w(ab) + w(ba) = 2w(a)g(b) + w(b)g(a)$ we see that also $w(ba) = 0$. Putting $x = a$ and $y = b$ in the following formula we find that $0 = w(z)\Delta(a,b)$ for all $z \in G$, and so, since $\Delta(a,b) \neq 0$, that $w = 0$:

$$\begin{aligned} w(yx)g(xyz) + w(xy)g(xzy) = w(x)[g(xyzy) + g(xz)\mu(y)] \\ + w(y)[g(yxzx) + g(yz)\mu(x)] + 2w(z)\Delta(x,y). \end{aligned} \tag{17}$$

In the rest of the proof we derive the formula (17). We recall that $\mu(x) = 2g(x)^2 - g(x^2)$ and that $g(xy) = g(yx)$.

From (6) we get two identities

$$w(xyz) + w(yzx) = 2w(x)g(yz) + 2w(yz)g(x),$$
$$w(yzx) + w(zxy) = 2w(y)g(zx) + 2w(zx)g(y).$$

The first one comes about when we replace y by yz in (6). The second one is the first except for a cyclic permutation of x, y, z. When we add the two identities and use (6) we obtain

$$w(yzx) + w(xy)g(z) + w(z)g(xy) = w(x)g(yz) + w(yz)g(x) + w(y)g(zx) + w(zx)g(y).$$

When we in the just derived identity interchange x and y we get

$$w(xzy) + w(yx)g(z) + w(z)g(xy) = w(y)g(xz) + w(xz)g(y) + w(x)g(yz) + w(zy)g(x).$$

We add the last two identities and find

$$\begin{aligned} w(yzx) + w(xzy) + [w(xy) + w(yx)]g(z) + 2w(z)g(xy) \\ = 2w(x)g(yz) + 2w(y)g(xz) + g(x)[w(yz) + w(zy)] + g(y)[w(xz) + w(zx)], \end{aligned}$$

that we by help of (6) reduce to

$$w(yzx) + w(xzy) = 2w(x)g(yz) + 2w(y)g(xz) + 2w(z)[2g(x)g(y) - g(xy)]. \tag{18}$$

When we in (18) put $y = x$ we get the identity

$$w(xzx) = \mu(x)w(z) + 2w(x)g(xz). \tag{19}$$

In (19) we replace z by yzy and get

$$w(xyzyx) = \mu(x)w(yzy) + 2w(x)g(xyzy),$$

in which we interchange x and y and get

$$w(yxzxy) = \mu(y)w(xzx) + 2w(y)g(yxzx).$$

When we add the last two identities and apply (19) to the right hand side we find that

$$\begin{aligned} w(xyzyx) + w(yxzxy) = 2w(x)[g(xyzy) + g(xz)\mu(y)] \\ + 2w(y)[g(yxzx) + g(yz)\mu(x)] + 2w(z)\mu(x)\mu(y). \end{aligned}$$

When we compute the left hand side by help of (18), more precisely, when we replace x by yx and y by xy in (18), we get

$$\begin{aligned} 2w(yx)g(xyz) + 2w(xy)g(yxz) + 2w(z)[2g(xy)^2 - g(x^2y^2)] \\ = 2w(x)[g(xyzy) + g(xz)\mu(y)] + 2w(y)[g(yxzx) + g(yz)\mu(x)] \\ + 2w(z)\mu(x)\mu(y). \end{aligned}$$

Due to (a) this is a reformulation of (17). ∎

Corollary 8.3. *If $\Delta(a,b) \neq 0$, then the three functions*

$$w_1 := \frac{g_b(b)}{\Delta(a,b)}\, g_a - \frac{g_a(b)}{\Delta(a,b)}\, g_b + \frac{g(b)}{\Delta(a,b)}\, \frac{g_{ab} - g_{ba}}{2}\,,$$
$$w_2 := -\frac{g_a(b)}{\Delta(a,b)}\, g_a + \frac{g_a(a)}{\Delta(a,b)}\, g_b + \frac{g(a)}{\Delta(a,b)}\, \frac{g_{ab} - g_{ba}}{2}\,,$$
$$w_3 := -\frac{1}{\Delta(a,b)}\, \frac{g_{ab} - g_{ba}}{2} = -\frac{1}{\Delta(a,b)}\, \frac{(R_{ab} - R_{ba})g}{2}$$

in $\mathrm{span}\{g_x \,|\, x \in G\}$ *have the properties*

$$w_1(a) = 1,\ w_1(b) = 0,\ w_1(ab) = 0,$$
$$w_2(a) = 0,\ w_2(b) = 1,\ w_2(ab) = 0,$$
$$w_3(a) = 0,\ w_3(b) = 0,\ w_3(ab) = 1.$$

9. The space of translates. In this section we let $g \in C(G)$ be a non-abelian solution of (3). Theorem 8.2(b) allows us to fix $a, b \in G$ such that $\Delta(a,b) \neq 0$.

In the abelian case the solutions of d'Alembert's functional equation were described by group characters (Theorem 6.1), i.e. by 1-dimensional representations of the group. In analogy with non-abelian harmonic analysis it is to be expected that the group characters shall be replaced by characters of irreducible representations. This is indeed so. We proceed to describe how to obtain the spaces on which the representations work. They are subspaces of the space $T(g) \subseteq C(G)$ spanned by all right translates of g, i.e. of

$$T(g) := \mathrm{span}\{R(x)g \,|\, x \in G\}, \tag{20}$$

while the corresponding representations are restrictions of the right regular representation to these subspaces. A pleasant surprise is that $T(g)$ is closely related to the symmetrized sine addition formula (6), more precisely to the function space

$$W(g) := \{w \in C(G) \,|\, (w, g) \text{ satisfies } (6)\}. \tag{21}$$

We derive some properties first of $T(g)$ and then of the restriction of the right regular representation R to $T(g)$.

Lemma 9.1.

(a) $W(g) = \mathrm{span}\{g_x \,|\, x \in G\}$.
(b) $T(g) = \mathbb{C}g + W(g)$, *where the sum is direct.*
(c) $\dim T(g) = 4$ *and* $\dim W(g) = 3$.
(d) $\{f \in T(g) \,|\, f$ *is central*$\} = \mathbb{C}g$.

Proof. (a) We have seen in Lemma 5.2(d) that $g_x \in W(g)$ for all $x \in G$, so we have the inclusion $\mathrm{span}\{g_x \,|\, x \in G\} \subseteq W(g)$. To prove the converse inclusion we let $w \in W(g)$ be arbitrary. w and $w(a)w_1 + w(b)w_2 + w(ab)w_3$ assume the same values at a, b and ab, so they coincide everywhere on G according to Theorem 8.2(c). Thus $w = w(a)w_1 + w(b)w_2 + w(ab)w_3 \in \mathrm{span}\{g_x \,|\, x \in G\}$.

(b) From $R(e)g = g$ and $g_x = R(x)g - g(x)g$ we see that $T(g) = \mathbb{C}g + \mathrm{span}\{g_x \,|\, x \in G\}$, proving equality in (b). It is left to show that the sum is direct. This is trivial if $g = 0$.

And if $g \neq 0$ then $g(e) = 1$ (Lemma 5.2(a)) and so $w(e) = 0$ for all $w \in W(g)$ (put $x = y = e$ in (6)).

(c) A basis of $T(g)$ is $\{g, w_1, w_2, w_3\}$.

(d) Since g is central it suffices by (b) to show that any central $w \in W(g)$ is 0. For such $w \in W(g)$ we get that $w(xy) = w(x)g(y) + w(y)g(x)$, which means that (w, g) is a solution of the sine addition formula. If $w \neq 0$ then g is abelian according to Theorem 5.1. And that contradicts our assumption about g being non-abelian, so $w = 0$. ∎

PROPOSITION 9.2.

(a) *$T(g)$ is invariant under both the right regular representation R and under the left regular representation L.*
(b) *$R|_{T(g)}$ is a continuous representation of G on $T(g)$.*
(c) *If $W \neq \{0\}$ is an R-invariant subspace of $T(g)$, then the character $\chi_{R|W}$ of the representation $R|W$ is $\chi_{R|W} = \dim(W)\, g$.*
(d) *There are no 1-dimensional, R-invariant subspaces of $T(g)$.*

Proof. (a) By definition $T(g)$ is invariant under R. It is also invariant under left translations, because g is central (Proposition 5.2(c)).

(b) is Proposition C.2(a) below.

(c) Let W be an R-invariant subspace of $T(g)$. The character $\chi_{R|W}$ is in $C(R|_W)$, hence in $C(R|_{T(g)})$ which equals $T(g)$ by Proposition C.2(b). Thus $\chi_{R|_W} \in T(g)$. Characters being central, Lemma 9.1(d) tells us that $\chi_{R|_W} = \alpha g$ for some constant $\alpha \in \mathbb{C}$. Finally, $\alpha = \alpha g(e) = \chi_{R|_W}(e) = \mathrm{tr}((R|_W)(e)) = \mathrm{tr}\, I_W = \dim W$.

(d) We shall arrive at a contradicting assuming the existence of a 1-dimensional, R-invariant subspace W of $T(g)$. From (c) we read that $g = \chi_{R|W}$. A character of a one-dimensional representation of a group G is a group character (a homomorphism into $\mathbb{C}^*$) and hence abelian. This implies g is abelian, contradicting that it is not. ∎

10. Non-abelian d'Alembert functions. In this section we let $g \in C(G)$ be a non-abelian solution of (3).

A conditioned reflex for mathematicians doing representation theory is to study irreducible subrepresentations of any given representation. So that is what we do for the representation $R|_{T(g)}$ of G on $T(g)$. Note that $\dim T(g) = 4$ (Lemma 9.1(c)), so that we are concerned with finite-dimensional vector spaces, not infinite-dimensional.

Choose an R-invariant subspace $W \neq \{0\}$ of $T(g)$ such that $\rho := R|_W$ is irreducible. Such a subspace exists (take W as an R-invariant subspace of minimal dimension ≥ 1). Let χ_ρ denote the character of ρ, and $C(\rho)$ the space of matrix-coefficients of ρ. For any $A \in \mathcal{L}(W)$ we define $c_A \in C(\rho)$ by $c_A(x) := \frac{1}{2}\mathrm{tr}(A\rho(x))$, $x \in G$. Finally we put $\mathrm{sl}(W) := \{A \in \mathcal{L}(W) \mid \mathrm{tr}\, A = 0\}$.

THEOREM 10.1.

(a) $\dim W = 2$.
(b) *ρ is an irreducible, continuous representation of G on W.*
(c) *$\chi_\rho = 2g$, $\det\rho = \mu$ and $C(\rho) = T(g)$.*

(d) *The map* $A \mapsto c_A$ *is an isomorphism of* $\mathcal{L}(W) = \mathbb{C}I \oplus \mathrm{sl}(W)$ *onto* $T(g) = \mathbb{C}g \oplus W(g)$, *mapping* $\mathbb{C}I$ *onto* $\mathbb{C}g$ *and* $\mathrm{sl}(W)$ *onto* $W(g)$.

(e) g *is bounded* $\iff$ ρ *is bounded* $\iff$ ρ *is equivalent to a unitary representation.*

(f) ρ *is uniquely determined up to equivalence by* g.

Proof. (b) Proposition 9.2(b) tells us that ρ is a continuous representation of G on W.

(a) From Proposition C.2(b) we get $C(\rho) = C(R|_W) \subseteq C(R|_{T(g)}) = T(g)$. Using Corollary A.7(b), Proposition 9.2(d) and that $\dim T(g) = 4$ we now infer

$$\dim T(g) \geq \dim C(\rho) = (\dim \rho)^2 = (\dim W)^2 \geq 2^2 = 4 = \dim T(g),$$

which proves not just (a), but also the statement $C(\rho) = T(g)$ in (c).

(c) That $\chi_\rho = 2g$ is immediate from Proposition 9.2(c). Using that and the easily verified fact that $(\mathrm{tr}\, A)^2 - \mathrm{tr}(A^2) = 2 \det A$ for all 2×2 matrices A we find that

$$\mu(x) = 2g(x)^2 - g(x^2) = \frac{1}{2}\left[(\mathrm{tr}\, \rho(x))^2 - \mathrm{tr}(\rho(x)^2)\right] = \det \rho(x).$$

(d) $f \in T(g) = C(\rho)$ has the desired form according to Proposition A.2(c), and the injectivity follows from Corollary A.7(a).

That the map sends $\mathbb{C}I$ onto $\mathbb{C}g$ is obvious from $\chi_\rho = 2g$. For the final statement of (d) it suffices by dimension arguments to prove that $\mathrm{tr}\, A = 0 \Rightarrow f := \mathrm{tr}(A\rho(\cdot)) \in W(g)$, so assume that $\mathrm{tr}\, A = 0$. Writing $f = \alpha g + w$, where $\alpha \in \mathbb{C}$ and $w \in W(g)$ (Lemma 9.1(a)) it now suffices to prove that $\alpha = 0$. Evaluating at $x = e$ we find that $\alpha = \alpha g(e) + w(e) = f(e) = \frac{1}{2}\mathrm{tr}(A\rho(e)) = \frac{1}{2}\mathrm{tr}(A) = 0$.

(e) If ρ is bounded, then so is $g = \frac{1}{2}\chi_\rho$. Assuming conversely that g is bounded we shall prove that the matrix-coefficients of ρ are bounded. If g is bounded, then so are all its translates, and so $T(g) = \mathrm{span}\{R(x)g \mid x \in G\}$ consists of bounded functions. But $C(\rho) = T(g)$.

When ρ is bounded, a result by Weil (see [30, 22.23(c)]) says that there exists an invertible matrix A such that $A^{-1}\rho(x)A$ is unitary for all $x \in G$.

(f) follows from Bourbaki [14, Proposition 2 of Chap. VIII, §13, no. 3], because g determines the character of ρ. ∎

We write down our version of the main result of Davison [20] that characterizes non-abelian d'Alembert functions. For any 2×2-matrix A we let c_A denote the matrix-coefficient $c_A(x) := \frac{1}{2}\mathrm{tr}(A\rho(x))$, $x \in G$.

Corollary 10.2. *If* $g \in C(G)$ *is a non-abelian d'Alembert function on* G, *then there exists a continuous, irreducible representation* ρ *of* G *on* $\mathbb{C}^2$ *such that* $g = \frac{1}{2}\mathrm{tr}\, \rho$ *and* $\mu = \det \rho$. *If* g *is bounded, then* ρ *may be taken unitary.* g *determines* ρ *up to equivalence.*

Furthermore, $W(g) = \{c_A \mid A \in \mathrm{sl}(2, \mathbb{C})\}$.

Proof. The corollary is immediate from Theorem 10.1. ∎

We finally note a partial converse of Corollary 10.2.

Lemma 10.3. *If* ρ *is a two-dimensional representation of* G, *then* $g := \frac{1}{2}\mathrm{tr}\, \rho$ *is a non-zero solution of* (3) *with* $\mu = \det \rho$.

Proof. Proceed as in the proof of Lemma 7.1. ∎

11. On the irreducibility of the representation ρ. We have seen that any non-zero solution g of d'Alembert's functional equation (3) may be written in the form $g = \frac{1}{2}\operatorname{tr}\rho$, where ρ is a two-dimensional representation of G (end of Section 6 and Corollary 10.2). The converse holds as well (Lemma 10.3). The following proposition throws light on the role of the irreducibility of ρ.

Proposition 11.1. *Let ρ be any two-dimensional representation of G. Then $g = \frac{1}{2}\operatorname{tr}\rho$ is non-abelian if and only if ρ is irreducible.*

Proof. We will prove that g is abelian if and only if ρ is not irreducible. If g is abelian we get in the notation of Theorem 6.1 that

$$\frac{1}{2}\operatorname{tr}\rho = g = \frac{\chi + \mu\check{\chi}}{2}\,.$$

Here ρ cannot be irreducible, because matrix-coefficients of inequivalent representations form direct sum by [14, Proposition 2 of Chap. VIII, §13, no. 3].

Assume conversely that ρ is not irreducible. Then the representation space W has an invariant 1-dimensional subspace $\mathbb{C}v$. Let $\{v, w\}$ be a basis of W. The matrix of ρ with respect to that basis has the form

$$\rho(x) = \begin{pmatrix} \chi(x) & a(x) \\ 0 & \sigma(x) \end{pmatrix}.$$

Since ρ is a representation χ and σ are multiplicative and so abelian. Hence so is $g = \frac{1}{2}\operatorname{tr}\rho = (\chi + \sigma)/2$. ∎

If g is non-abelian then the corresponding representation ρ is unique up to equivalence (Theorem 10.1(f)). In general this is not so: if $a : G \to \mathbb{C}$ is an additive function, $a \neq 0$, then the representations

$$\rho_1(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_2(x) := \begin{pmatrix} 1 & a(x) \\ 0 & 1 \end{pmatrix}$$

have the same trace, but they are inequivalent.

12. Generalizations of d'Alembert's functional equation. Generalizations and variations of d'Alembert's classical functional equation (2) are of current interest.

See Gajda's paper and [27] and Fechner's sequel [26]. Or Badora [9] and its generalization [24] by Elqorachi, Akkouchi, Bakali and Bouikhalene. Or Chojnacki [15].

The Moroccan school has a number of papers for various kinds of spherical functions: [3], [6], [7], [5], [4], [10], [12], [11], [13], [21], [22], [23], [24], [25]. See also [38].

Peter Friis [35] has results about d'Alembert's long functional equation. So does [40].

Yang [47] solves d'Alembert's classical functional equation on compact groups and in doing so she treats several important examples like $SU(n)$. See Székelyhidi [46] for an approach via spectral synthesis.

An and Yang [8] solve various functional equations on compact groups.

Davison [20] solves the pre-d'Alembert functional equation on monoids and not just groups. On groups it reduces to d'Alembert's functional equation (3) according to [20, Proposition 2.11].

In [39] Sinopoulos has found the solutions $g : S \to F$ of the functional equation $g(x+y)+g(x+\sigma(y)) = 2g(x)g(y)$, $x, y \in S$, where $(S,+)$ is a commutative semigroup, σ an endomorphism of S such that $\sigma \circ \sigma = I$ and F is a quadratically closed (commutative) field of characteristic different from 2. The solutions are $g = (\chi+\chi\circ\sigma)/2$, where $\chi : G \to F$ is multiplicative. S need not possess a neutral element.

For d'Alembert's functional equations on hypergroups see Roukbi and Zeglami [36] and Orosz [34].

For operator-valued solutions see the following papers and the references therein: Chojnacki [16, 17] and Stetkær [43, 44].

A. Matrix-coefficients. In this appendix we discuss matrix-coefficients of operator-valued functions, where the operators act on finite-dimensional vector spaces. We can apply the set up and the results to group representations, because they are operator-valued functions of a special form.

In the next few lines we recall some notation and standard facts from linear algebra.

Let W denote an n-dimensional vector space where $1 \le n < \infty$. Its dual vector space is denoted by W^*, and the value of $w^* \in W^*$ on $w \in W$ is denoted by $\langle w^*, w\rangle$. For any $w \in W$ and $w^* \in W^*$ we define $w \otimes w^* \in \mathcal{L}(W)$ by $w \otimes w^*(v) := \langle w^*, v\rangle w$ for $v \in W$.

If $w_1, w_2, \dots, w_n$ is a basis of W and $w_1^*, w_2^*, \dots, w_n^*$ is the dual basis of W^*, then $\{w_i \otimes w_j^* \mid i, j = 1, 2, \dots, n\}$ is a basis of $\mathcal{L}(W)$, so any $L \in \mathcal{L}(W)$ can be decomposed into $L = \sum_{i,j=1}^n L_{ij} w_i \otimes w_j^*$, where $L_{ij} \in \mathbb{C}$. As is easy to check, $\{L_{ij}\}$ is the matrix of the linear operator L with respect to the basis $w_1, w_2, \dots, w_n$ of W. Note the formulas $L_{ij} = \langle w_i^*, Lw_j\rangle$ and $\operatorname{tr}((w_j \otimes w_i^*)L) = L_{ij}$.

Definition A.1. Let W denote an n-dimensional vector space where $1 \le n < \infty$. Let X be a set and $F : X \to \mathcal{L}(W)$. A *matrix-coefficient* of F is a function on X of the form $x \mapsto \langle w^*, F(x)w\rangle$, $x \in X$, where $w^* \in W^*$ and $w \in W$. The vector space of functions on X spanned by the matrix-coefficients of F is called the *space of matrix-coefficients* of F and denoted by $C(F)$.

Proposition A.2. *Let W denote an n-dimensional vector space where $1 \le n < \infty$. Let X be a set and $F : X \to \mathcal{L}(W)$. Letting $w_1, w_2, \dots, w_n$ be a basis of W we write $F(x)w_j = \sum_{i=1}^n F_{ij}(x)w_i$, where $F_{ij} : X \to \mathbb{C}$. Then*

(a) *each F_{ij}, $i, j = 1, \dots, n$, is a matrix-coefficient of F,*
(b) $C(F) = \operatorname{span}\{F_{ij} \mid i, j = 1, 2, \dots, n\}$, *so* $\dim C(F) \le n^2$.
(c) *$C(F)$ consists of the functions $x \mapsto \operatorname{tr}(AF(x))$, $x \in X$, where A ranges over $\mathcal{L}(W)$.*
(d) *In particular $x \mapsto \operatorname{tr} F(x)$ belongs to $C(F)$.*

Proof. We leave (a) and (b) to the reader. (c) Let $\{w_1, w_2, \dots, w_n\}$ be a basis of W and let $\{w_1^*, w_2^*, \dots, w_n^*\}$ be its dual basis. Now, $F_{ij}(x) = \operatorname{tr}((w_j \otimes w_i^*)F(x))$, so

$$\begin{aligned} C(F) &= \operatorname{span}\{F_{ij}(\cdot) \mid i, j = 1, 2, \dots, n\} \\ &= \operatorname{span}\{\operatorname{tr}((w_j \otimes w_i^*)F(\cdot)) \mid i, j = 1, 2, \dots, n\} = \operatorname{span}\{\operatorname{tr}(AF(\cdot)) \mid A \in \mathcal{L}(W)\}. \ \blacksquare \end{aligned}$$

Proposition A.2 motivates the terminology matrix-coefficient. Indeed, $\{F_{ij}(x)\}$ is the matrix of the linear operator $F(x) : W \to W$ with respect to the basis $w_1, w_2, \dots, w_n$

of W. The individual matrix-coefficients F_{ij} depend on the choice of the basis $w_1, w_2, \dots, w_n$, but by its very definition the vector space $C(F)$ does not.

The function $x \mapsto \operatorname{tr} F(x) = \sum_{i=1}^n F_{ii}(x)$, $x \in X$, in $C(F)$ does not depend on a choice of basis in W, because tr does not. It plays an important role for d'Alembert functions (see Theorem 10.1).

The following result on invariant subspaces will be used without explicit mentioning:

LEMMA A.3. *Let W denote a finite-dimensional vector space. Let X be a set and $F : X \to \mathcal{L}(W)$. Let furthermore W' be a subspace of W such that $F(x)(W') \subseteq W'$ for all $x \in X$. Define $F' : X \to \mathcal{L}(W')$ by $F'(x)w' := F(x)w'$ for $x \in X$ and $w' \in W'$.*

Then any matrix-coefficient of F' is a matrix-coefficient of F, and so $C(F') \subseteq C(F)$.

Proof. Let $f(x) = \langle \phi, F'(x)w' \rangle$, $x \in X$, be a matrix-coefficient of F'. So $\phi \in (W')^*$ and $w' \in W'$. Extending ϕ from the subspace W' of W to a linear functional $w^* \in W^*$ on W we have $f(x) = \langle w^*, F(x)w' \rangle$, which shows that f is a matrix-coefficient of F. ∎

We now take continuity into account. There is no need to worry about the topology of a finite-dimensional vector space, because it is unique. More precisely formulated ([37, Theorem 1.21(a)]):

THEOREM A.4. *Let V denote an n-dimensional vector space where $n < \infty$. There is exactly one topology on V that makes it a Hausdorff topological vector space. Any isomorphism of $\mathbb{C}^n$ onto V is a homeomorphism, when V is equipped with this topology.*

We shall always equip a finite-dimensional vector space with this topology. The algebraically dual space of V then equals the topologically dual space V^* of continuous linear functionals on V, because this is the case for $\mathbb{C}^n$.

LEMMA A.5. *Let W denote a finite-dimensional vector space. Let X be a topological space and $F : X \to \mathcal{L}(W)$. Then*

(a) *F is continuous if and only if each matrix-coefficient of F is continuous.*
(b) *F is bounded if and only if each matrix-coefficient of F is bounded.*

Proof. Immediate from Proposition A.2, because F is continuous, resp. bounded, if and only if the special matrix-elements F_{ij} are continuous, resp. bounded. After all they are the coordinates of F with respect to the basis $\{w_i \otimes w_j^* \mid i, j = 1, 2, \dots, n\}$ of the vector space $\mathcal{L}(W)$ (we use the notation from the proof of Proposition A.2(c)). ∎

Burnside's theorem, which is a fundamental result in representation theory, allows us to sharpen Proposition A.2(b) and (c), when F is an irreducible representation. Under the circumstances of Corollary A.7 we get equality and not just an inequality in Proposition A.2(b). By definition irreducibility of a set of operators means there are no subspaces except $\{0\}$ and W that are invariant under each element of the set. Burnside's theorem says the following:

THEOREM A.6 (Burnside 1905). *Let W be a finite-dimensional vector space of dimension larger than 1 over an algebraically closed field. If $\mathcal{A}$ is an irreducible algebra of linear transformations of W containing I, then $\mathcal{A} = \mathcal{L}(W)$.*

Proof. [33] gives a short (one page) proof of Burnside's theorem using only elementary linear algebra. ■

COROLLARY A.7. *Let ρ be an irreducible representation of G on a vector space W with $1 \le \dim W < \infty$.*

(a) *Let $A \in \mathcal{L}(W)$. If $\operatorname{tr}(A\rho(x)) = 0$ for all $x \in G$, then $A = 0$.*
(b) *Let $c_A(x) := \frac{1}{2}\operatorname{tr}(A\rho(x))$, $x \in G$. The map $A \mapsto c_A$ is an isomorphism of $\mathcal{L}(W)$ onto the space $C(\rho)$ of matrix-coefficients of ρ. In particular, $\dim C(\rho) = (\dim \rho)^2$.*

Proof. (a) The case of $\dim W = 1$ is trivial, so we may assume that $\dim W \ge 2$. $\operatorname{span}\{\rho(x) \,|\, x \in G\}$ is an algebra containing I, because ρ is a representation of a group, and we may apply Burnside's theorem to it, because ρ is irreducible. Burnside's theorem tells us that $\operatorname{span}\{\rho(x) \,|\, x \in G\}$ is all of $\mathcal{L}(W)$, which implies that $\operatorname{tr}(AX) = 0$ for all $X \in \mathcal{L}(W)$. And then $A = 0$ as desired.

(b) Proposition A.2(c) says that the map $A \mapsto c_A$ is surjective. It is left to show it is injective. So assuming $\operatorname{tr}(A\rho(x)) = 0$ for all $x \in G$ we shall infer that $A = 0$. But that was proved in (a). ■

B. Spaces of functions. The expression $\langle w^*, F(\cdot)w\rangle$ is bilinear in $(w^*, w) \in W^* \times W$, so to produce the space of matrix-coefficients $C(F)$ it suffices to take the span of $\langle w^*, F(\cdot)w\rangle$ where w^* and w range over generating subsets of the vector spaces W^* and W respectively. Proposition A.2(c) is an example with the basis $w_1, w_2, \dots, w_n$ of W as the generating subset of W and the dual basis as generating subset of W^*. This construction of $C(F)$ via a basis and its dual applies to any finite-dimensional vector space W, but sometimes specific generating subsets of W^* and/or W are more pertinent for concrete vector spaces W. For example if W is a space of functions which is the case of interest for us.

DEFINITION B.1. Let W be a subspace of the complex-valued functions on a set Y. For any $y \in Y$ we define the *point-evaluation* $\mathrm{ev}_y \in W^*$ by $\langle \mathrm{ev}_y, f\rangle := f(y)$, $f \in W$.

LEMMA B.2. *If W is a finite-dimensional subspace of the complex-valued functions on a set Y, then $W^* = \operatorname{span}\{\mathrm{ev}_y \,|\, y \in Y\}$.*

$C(F)$ is for any $F : X \to \mathcal{L}(W)$ spanned by the functions $x \mapsto \langle \mathrm{ev}_y, F(x)f\rangle$, $x \in X$, where $f \in W$ and $y \in Y$.

Proof. If $\operatorname{span}\{\mathrm{ev}_y \,|\, y \in Y\}$ were a proper subspace of the finite-dimensional vector space W^* then there would exist an element $w^{**} \in W^{**} \setminus \{0\}$ such that $w^{**}(\mathrm{ev}_y) = 0$ for all $y \in Y$. The natural embedding $i : W \to W^{**}$ is surjective, because $\dim W < \infty$, so $w^{**} = i(f)$ for some $f \in W$. From $w^{**}(\mathrm{ev}_y) = 0$ for all $y \in Y$ we get that $f(y) = 0$ for all $y \in Y$, so that $f = 0$. But $f \neq 0$ because $i(f) = w^{**} \neq 0$, so we have arrived at a contradiction. ■

C. On the regular representations. In this appendix we connect the theory of matrix-coefficients to the right regular representation of G on subspaces of $C(G)$.

Definition C.1. The *right regular representation* R of G on the vector space of complex-valued functions on G is defined by $[R(x)\phi](y) := \phi(yx)$, $y \in G$, for any $x \in G$ and any function $\phi : G \to \mathbb{C}$ on G.

The *left regular representation* L of G on the vector space of complex-valued functions on G is defined by $[L(x)\phi](y) := \phi(x^{-1}y)$, $y \in G$, for any $x \in G$ and any function $\phi : G \to \mathbb{C}$ on G.

Proposition C.2. *Let $W \subseteq C(G)$ be an R-invariant, finite-dimensional subspace of $C(G)$.*

(a) *$R|W$ is a continuous representation of G on W, and $W \subseteq C(R|W)$.*
(b) *If W is also L-invariant, then $W = C(R|W)$.*

Proof. (a) $R|W$ is clearly a representation of G on W. Consider the special matrix-coefficients $x \mapsto \langle \mathrm{ev}_y, R(x)f\rangle = f(yx)$ where $y \in G$ and $f \in W$. They are continuous, because G is a topological group and $f \in C(G)$. Since they span the space of all matrix-coefficients (Lemma B.2), we refer to Lemma A.5(a) to infer the continuity of $R|W$.

From $\langle \mathrm{ev}_e, R(x)f\rangle = f(ex) = f(x)$ we see that any $f \in W$ is a matrix-coefficient, so $W \subseteq C(R|W)$.

(b) According to (a) we shall prove that $W \supseteq C(R|W)$. By Lemma B.2 the matrix-coefficients $x \mapsto [R(x)f](y)$, where $f \in W$ and $y \in G$, span $C(R|W)$, so it suffices to prove that they belong to W. They are functions on G of the form $L(z)f$, because $[R(x)f](y) = f(yx) = [L(y^{-1})f](x)$. Now, $L(z)f \in W$ for any $z \in G$, because W is L-invariant. ∎

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