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COMPACT WIDTHS IN METRIC TREES

ASUMAN GÜVEN AKSOY

Claremont McKenna College, Department of Mathematics
Claremont, CA 91711, USA
E-mail: aaksoy@cmc.edu

KYLE EDWARD KINNEBERG

University of California, Los Angeles, Department of Mathematics
Los Angeles, CA 90095, USA
E-mail: kkinneberg@math.ucla.edu

Abstract. The definition of n-width of a bounded subset A in a normed linear space X is based on the existence of n-dimensional subspaces. Although the concept of an n-dimensional subspace is not available for metric trees, in this paper, using the properties of convex and compact subsets, we present a notion of n-widths for a metric tree, called Tn-widths. Later we discuss properties of Tn-widths, and show that the compact width is attained. A relationship between the compact widths and Tn-widths is also obtained.

1. Introduction. The study of injective envelopes of metric spaces, also known as metric trees ($\mathbb{R}$-trees or T-theory) is motivated by many subdisciplines of mathematics, biology/medicine and computer science. The relationship between metric trees and biology and medicine stems from the construction of phylogenetic trees [26]; and concepts of "string matching" in computer science are closely related with the structure of metric trees [7].

Unlike metric trees, in an ordinary tree all the edges are assumed to have the same length and therefore the metric is not often stressed. However, a metric tree is a generalization of an ordinary tree that allows for different edge lengths. A metric tree is a metric space (M, d) such that for every x, y in M there is a unique arc between x and y isometric to an interval in $\mathbb{R}$. For example, a connected graph without cycles is a metric tree. Metric trees also arise naturally in the study of group isometries of hyperbolic spaces. For metric properties of trees we refer to [12]. Lastly, [22] and [23] explore topological

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characterization of metric trees and prove that for a separable metric space (M, d) the following are equivalent:

- M admits an equivalent metric ρ such that (M, ρ) is a metric tree.
- M is locally arcwise connected and uniquely arcwise connected.

For an overview of geometry, topology, and group theory applications of metric trees, consult Bestvina [8]. For a complete discussion of these spaces and their relation to $CAT(\kappa)$ spaces we refer to [10].

Definition 1.1. Let $x, y \in M$, where (M, d) is a metric space. A *geodesic segment* from x to y, is the image of an isometric embedding $\alpha : [a, b] \to M$ such that $\alpha(a) = x$ and $\alpha(y) = b$. The geodesic segment will be called a *metric segment* and denoted by $[x, y]$ throughout this paper.

Definition 1.2. (M, d), a metric space, is a *metric tree* if and only if for all $x, y, z \in M$, the following holds:

1. there exists a unique metric segment from x to y, and
2. $[x, z] \cap [z, y] = \{z\} \Rightarrow [x, z] \cup [z, y] = [x, y]$.

Note that $\mathbb{R}^n$ with the Euclidean metric satisfies the first condition. It fails, however, to satisfy the second condition. If the metric d is understood, we will denote $d(x, y)$ by xy. We also say that a point z is *between* x and y if $xy = xz + zy$. We will often denote this by xzy. It is not difficult to prove that in any metric space, the elements of a metric segment from x to y are necessarily between x and y, and in a metric tree, the elements between x and y are the elements in the unique metric segment from x to y. Hence, if M is a metric tree and $x, y \in M$, then

$$[x, y] = \{z \in M : xy = xz + zy\}.$$

The following is an example of a metric tree. For more examples see [4].

Example 1.3 (The radial metric). Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by:

$$d(x, y) = \begin{cases} \|x - y\| & \text{if } x = \lambda\, y \text{ for some } \lambda \in \mathbb{R}, \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

We can observe that the d is in fact a metric and that $(\mathbb{R}^2, d)$ is a metric tree.

It is well known that any complete, simply connected Riemannian manifold having non-positive curvature is a $CAT(0)$-space. Other examples include the complex Hilbert ball with the hyperbolic metric (see [16]), Euclidean buildings (see [11]) and classical hyperbolic spaces. If a space is $CAT(\kappa)$ for some $\kappa < 0$ then it is automatically $CAT(0)$-space. Although we will concentrate on metric trees, which is a sub-class of $CAT(0)$-spaces, perhaps it is useful to mention the following:

Proposition 1.4. *If a metric space is $CAT(\kappa)$ space for all κ then it is a metric tree.*

For the proof of the above proposition we refer to [10]. Note that if a Banach space is a $CAT(\kappa)$ space for some κ then it is necessarily a Hilbert space and $CAT(0)$. The property that distinguishes the metric trees from the $CAT(0)$ spaces is the fact that metric trees are hyperconvex metric spaces. Properties of hyperconvex spaces and their relation to metric trees can be found in [3], [6], [17] and [19]. We refer to [9] for the properties of metric segments and to [1] and [2] for the basic properties of complete metric trees. In the following we list some of the properties of metric trees which will be used throughout this paper.

1. (Transitivity of betweenness [9]). Let M be a metric space and let $a, b, c, d \in M$. If abc and acd, then abd and bcd.
2. (Three point property [1]). Let (M, d) be a metric tree and $x, y, z \in M$. There exists $w \in M$ such that $[x, z] \cap [y, z] = [w, z]$ and $[x, y] \cap [w, z] = \{w\}$.
3. (Uniform convexity [1]). A metric tree M is uniformly convex.

2. Kolmogorov n-widths. The following definition due to Kolmogorov [20], gives a measure for the "thickness" or "massivity" of a subset A in a normed linear space X. Kolmogorov n-widths have been widely used in approximation theory (see [24] and references therein). Recently n-width has been utilized as a measure of efficiency in the task of data compression (see [14], [25], [13]). Furthermore, in [4], entropy quantities, other measures of compactness and n-affine Kolmogorov diameter were studied in the context of metric trees.

Definition 2.1. Let A be a subset of a normed linear space X, and let $\mathcal{X}_n$ denote the set of n-dimensional subspaces of X. We define the *Kolmogorov n-width* of A in X to be

$$\delta_n(A, X) = \inf_{X_n \in \mathcal{X}_n} \sup_{a \in A} \inf_{x \in X_n} \|x - a\|$$

The left most infimum is taken over all n-dimensional subspaces X_n of X.

Clearly $\delta_n(A, X)$ gives a measure the extent to which A may be approximated by n-dimensional subspaces of X. Indeed, it is easy to see that if $A \subset X_n$ for some $X_n \in \mathcal{X}_n$, then $\delta_n(A, X) = 0$. A subspace X_n of X of dimension at most n for which

$$\delta_n(A, X) = \sup_{a \in A} \inf_{x \in X_n} \|x - a\|,$$

is called an *optimal subspace* for $\delta_n(A, X)$. Generally it is very difficult to calculate $\delta_n(A, X)$ and determine optimal subspaces X_n of $\delta_n(A, X)$, although a considerable effort has been devoted to it. In many cases one is interested in determining asymptotic behavior of $\delta_n(A, X)$ as $n \to \infty$. Aside from defining $\delta_n(A, X)$, Kolmogorov also computed this quantity for particular spaces. The following is one of his examples:

Example 2.2. Let $\tilde{W}_2^{(r)}$ denote the Sobolev space of 2π-periodic, real-valued, $(r-1)$-times differentiable functions whose $(r-1)$th derivative is absolutely continuous and whose rth derivative is in $L^2 = L^2[0, 2\pi]$. Set

$$\tilde{B}_2^{(r)} = \{f : \ f \in \tilde{W}_2^r, \ \|f^{(r)}\| \leq 1 \}$$

then

$$\delta_0(\tilde{B}_2^{(r)}, L^2) = \infty \text{ while } \delta_{2n-1}(\tilde{B}_2^{(r)}, L^2) = \delta_{2n}(\tilde{B}_2^{(r)}, L^2) = n^{-r}, \quad n = 1, 2, \ldots$$

Furthermore, the optimal subspace for $\delta_{2n}(\tilde{B}_2^{(r)}, L^2)$ is the set of trigonometric polynomials of degree less than or equal to $n-1$; namely,

$$T_{n-1} = \operatorname{span}\{1, \sin x, \cos x, \ldots, \sin(n-1)x, \cos(n-1)x\}.$$

It is natural to ask whether or not we can alter the traditional definition so that n-widths can be defined in metric trees. The obvious replacement for $||x-y||$ is $d(x,y)$. The more difficult alteration, however, is defining "dimension" of a set in a metric space. In the following, we attempt to remedy this problem for metric trees.

2.1. n-widths of metric trees via convexity. We call a subset A of a metric tree (M,d) *convex* if for any $x, y \in A$, the metric segment $[x,y]$ is in A. By definition, every metric tree is convex. The converse is also true: it is easy to see that, if (M,d) is a metric tree and $A \subset M$ is a convex subset of M, then A is a metric tree. For $B \subset M$, the convex hull of B, denoted by $\operatorname{conv}(B)$, is the smallest convex set that contains B, where the order is set inclusion.

Definition 2.3. Let (M,d) be a metric tree, and let $A \subseteq M$. We say that A is Tn-*dimensional* if and only if there exist n points $x_1, \ldots, x_n \in M$ such that

$$A = \operatorname{conv}(x_1, \ldots, x_n)$$

and there do not exist $i \neq j \neq k$ such that $x_i x_j x_k$. Also, we say that A is T*n-*dimensional* if A contains a Tn-dimensional subset but does not contain any Tk-dimensional subsets for all $k > n$.

Note that the restriction $i \neq j \neq k$ tells us that the points $x_1, \ldots, x_n$ are all distinct.

Lemma 2.4. *If (M,d) is a metric tree and A is a subset of M, then*

$$\operatorname{conv}(A) = \{z \in M : xzy \text{ for some } x, y \in A\}.$$

Proof. First, we observe that $C = \{z \in M : xzy$ for some $x, y \in A\}$ is a convex set. Indeed, if $a, b \in C$, then xay and ubw for some $x, y, u, w \in A$. By definition of C, the segment $[y,u]$ is in C, as are the segments $[a,y]$ and $[u,b]$. Since $[a,b] \subseteq [a,y] \cup [y,u] \cup [u,b]$, we know that $[a,b] \subseteq C$. Thus, C is convex. Also, C contains A, so by definition of convex hull, we have that $\operatorname{conv}(A) \subseteq C$.

Now, let $z \in C$. Then there exist some $x, y \in A$ such that xzy. Hence, $z \in [x,y]$ and $[x,y] \subseteq \operatorname{conv}(A)$, so $z \in \operatorname{conv}(A)$. Thus, we indeed have $C \subseteq \operatorname{conv}(A)$ as desired. ∎

An important concept regarding metric trees is that of "final points". We have the following definition and subsequent theorem.

Definition 2.5. Let (M,d) be a metric tree, and let $A \subseteq M$. We call

$$F_A = \{f \in A : f \notin (x,y) \text{ for all } x, y \in A\}$$

the set of *final points* of A. Here, $(x,y) = [x,y] \backslash \{x,y\}$.

Theorem 2.6 ([1]). *A metric tree (M,d) is compact if and only if*

$$M = \bigcup_{f \in F_M} [a,f] \text{ for all } a \in M, \text{ and } \overline{F}_M \text{ is compact.}$$

We now characterize Tn-dimensional subsets of a metric tree, and establish several facts about such subsets. Here, (M, d) will be a metric tree and $\mathcal{X}_n$ will denote the set of all Tn-dimensional subsets of M.

THEOREM 2.7. *Let A be a subset of M. Then A is* Tn*-dimensional if and only if A is a compact metric tree with $F_A = \{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n \in M$.*

Proof. Let A be Tn-dimensional. Then there exist $x_1, \ldots, x_n \in M$ such that $A = \operatorname{conv}(x_1, \ldots, x_n)$ and there do not exist $i \neq j \neq k$ such that $x_i x_j x_k$. By Lemma 2.4, $A = \{z \in M : x_i z x_j \text{ for some } x_i, x_j\}$. Note that A is a metric tree because it is convex. We now show that A is compact.

To do this, we first show that for any $a \in A$, $A = \bigcup_{i=1}^n [a, x_i]$. Let $a \in A$ be fixed. If $z \in \bigcup_{i=1}^n [a, x_i]$, then $z \in [a, x_i]$ for some i. Since A is convex, $[a, x_i] \subseteq A$, so $z \in A$. Now, let $z \in A$. Then there exist i, j such that $z \in [x_i, x_j]$. We know by the three point property that there is some $w \in A$ such that $[a, x_i] \cap [a, x_j] = [a, w]$ and $[x_i, x_j] \cap [a, w] = \{w\}$. Note that $w \in [x_i, x_j]$, so $[x_i, w] \cup [x_j, w] = [x_i, x_j]$. Since $w \in [a, x_i]$ and $w \in [a, x_j]$, we have $[x_i, w] \subseteq [a, x_i]$ and $[x_j, w] \subseteq [a, x_j]$. Thus,

$$z \in [x_i, x_j] = [x_i, w] \cup [x_j, w] \subseteq [a, x_i] \cup [a, x_j] \subseteq \bigcup_{i=1}^n [a, x_i].$$

Hence, $A = \bigcup_{i=1}^n [a, x_i]$, as desired.

We now show that $F_A = \{x_1, \ldots, x_n\}$. Since $A = \{z \in M : x_i z x_j \text{ for some } x_i, x_j\}$, we see that $A = \{z \in M : z \in (x_i, x_j) \text{ for some } x_i, x_j\} \cup \{x_1, \ldots, x_n\}$. Thus, the only possible final points of A are $x_1, \ldots, x_n$. If, for some j, x_j is not a final point, then there must exist some $y, z \in A$ such that $x_j \in (y, z)$. Since $A = \bigcup_{i=1}^n [a, x_i]$ for any $a \in A$, we know that there exist some x_i and x_k such that $y \in [z, x_i]$ and $z \in [y, x_k]$. Therefore, we have $y, z \in [x_i, x_k]$, so $[y, z] \subseteq [x_i, x_k]$. But this implies that $x_j \in (x_i, x_k)$, contrary to our assumption about the set $\{x_1, \ldots, x_n\}$. Hence, $x_j \in F_A$ for all $1 \leq j \leq n$, so $F_A = \{x_1, \ldots, x_n\}$, as claimed. In particular, this implies that $\overline{F}_A = F_A$ is compact. By Theorem 2.6 then, A is compact, so A is a compact tree with $F_A = \{x_1, \ldots, x_n\}$.

Now let A be a compact metric tree with $F_A = \{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n \in M$. Note first that there do not exist $i \neq j \neq k$ such that $x_i x_j x_k$. We want to show that $A = \operatorname{conv}(x_1, \ldots, x_n)$. By definition of final points, we actually have $x_1, \ldots, x_n \in A$, and since A is a metric tree, it is convex. Thus, $\operatorname{conv}(x_1, \ldots, x_n) \subseteq A$. Now, let $z \in A$. Since A is compact, Theorem 2.6 tells us that for all $a \in A$, $A = \bigcup_{i=1}^n [a, x_i]$. Therefore, we have $z \in A = \bigcup_{i=1}^n [x_1, x_i]$, so there is some i for which $z \in [x_1, x_i]$. Thus, $z \in \operatorname{conv}(x_1, \ldots, x_n)$, which implies that $A = \operatorname{conv}(x_1, \ldots, x_n)$. Hence, A is Tn-dimensional, as desired. ∎

LEMMA 2.8. *Every* Tn*-dimensional subset X_n of M contains a* Tm*-dimensional subset for each $1 \leq m \leq n$.*

Proof. Let $X_n = \operatorname{conv}(x_1, \ldots, x_n)$ such that there do not exist $i \neq j \neq k$ where $x_i x_j x_k$. If $1 \leq m \leq n$, let $X_m = \operatorname{conv}(x_1, \ldots, x_m)$, so that X_m is Tm-dimensional and $X_m \subset X_n$. ∎

LEMMA 2.9. *If $X_m \subseteq X_n$ for some $X_m \in \mathcal{X}_m$ and $X_n \in \mathcal{X}_n$, then $m \leq n$.*

Proof. Let $X_m = \operatorname{conv}(x_1, \ldots, x_m)$ and $X_n = \operatorname{conv}(y_1, \ldots, y_n)$. By Theorem 2.7, this implies that $F_{X_m} = \{x_1, \ldots, x_m\}$ and $F_{X_n} = \{y_1, \ldots, y_n\}$. We can assume that $X_m \neq X_n$, so there is some j for which $x_i \neq y_j$ for all $i \in \{1, \ldots, m\}$. Without loss of generality, let this j be 1. By the compactness of X_n, Theorem 2.6 tells us that for any $a \in X_n$,

$$X_n = \bigcup_{i=1}^{n} [a, y_i].$$

Therefore, we have

$$X_m \subseteq \bigcup_{i=1}^{n} [y_1, y_i],$$

so for any $x_j \in F_{X_m}$, we see that $x_j \in [y_1, y_k]$ for some $k \in \{1, \ldots, n\}$. Now define a function $f : F_{X_m} \to F_{X_n}$ by the following:

$$f(x_i) = \begin{cases} y_k & \text{if } x_i \in [y_1, y_k] \text{ and no other element of } F_{X_m} \text{ is in } [x_i, y_k] \\ y_1 & \text{otherwise} \end{cases}$$

and if the first condition holds for more than one k, choose the smallest of such k.

Clearly, f is well-defined for all $x_i \in F_{X_m}$. We want to show that f is an injection. Suppose for a contradiction that $f(x_i) = y_k = f(x_j)$ for some $i \neq j$ in $\{1, \ldots, m\}$ and $k \in \{2, \ldots, n\}$. Then by definition of f, we have $x_i, x_j \in [y_1, y_k]$, which implies that either $x_i \in [x_j, y_k]$ or $x_j \in [x_i, y_k]$. If the former, then we contradict the fact that no element of F_{X_m}, other than x_j, is in $[x_j, y_k]$; and if the latter, we contradict the fact that no element of F_{X_m}, other than x_i, is in $[x_i, y_k]$.

So now suppose that $f(x_i) = y_1 = f(x_j)$ for some $i \neq j$ in $\{1, \ldots, m\}$. Note that $x_i \neq y_1 \neq x_j$, so x_i and x_j are mapped to y_1 by the "otherwise" condition, not by the first condition. We now claim that if some x_k is mapped to y_1 by the "otherwise" condition, then the segment $(x_k, y_1]$ does not contain any elements of X_m. Indeed, since $x_k \in [y_1, y_k]$ for some $k \neq 1$, and since x_k is not mapped to y_k, there must be some $x_l \in (x_k, y_k]$. Now, if there was some $w \in X_m$ in the segment $(x_k, y_1]$ then we would have $x_k \in (x_l, w)$, which contradicts our assumption that x_k is a final point of X_m. Thus, $(x_k, y_1]$ does not contain any elements of X_m.

We therefore know that $(x_i, y_1]$ and $(x_j, y_1]$ contain no elements of X_m. But then by the three point property, there is a $w \in M$ such that $[x_i, y_1] \cap [x_j, y_1] = [w, y_1]$ and $[x_i, x_j] \cap [w, y_1] = \{w\}$. Since $w \in [x_i, x_j]$, $w \in X_m$. If $x_i \neq w$, then $w \in (x_i, y_1]$, and if $x_j \neq w$, then $w \in (x_j, y_1]$. Both possibilities contradict the fact that $(x_i, y_1]$ and $(x_j, y_1]$ contain no elements of X_m. Hence, $x_i = w = x_j$, another contradiction. Therefore, no two distinct elements of F_{X_m} can map to y_1. We can then conclude that f is an injection.

Since F_{X_m} and F_{X_n} are finite sets, the injectivity of f implies that $|F_{X_m}| \leq |F_{X_n}|$, so $m \leq n$. Notice also that the function f is a bijection if and only if $m = n$. ■

Now that we have established some facts about Tn-dimensional subsets of a metric tree, we can give the following definition for the Tn-width.

Definition 2.10. Let A be a subset of a metric tree (M, d), and let $\mathcal{X}_n$ denote the set of Tn-dimensional subsets of M. We define the Tn-*width* of A to be

$$\delta_n^T(A, M) = \inf_{X \in \mathcal{X}_n} \sup_{a \in A} \inf_{x \in X} d(a, x).$$

If M is T*n-dimensional (i.e., M does not contain any Tk-dimensional subsets for $k > n$ but does contain a Tn-dimensional subset), then by convention we say that $\delta_k^T(A, M) = \delta_n^T(A, M)$.

First observe that if A is unbounded, then $\delta_n^T(A, M) = \infty$ for each $n \in \mathbb{N}$. Indeed, this follows directly from the fact that every Tn-dimensional set is bounded. Conversely, it is easy to see that if A is bounded, then $\delta_n^T(A, M) < \infty$ for each n. Therefore, we really will be interested only in the Tn-widths of bounded sets.

Example 2.11. Let $M = \mathbb{R}^k$ endowed with the radial metric. If B_r denotes the (open or closed) ball of radius r in $\mathbb{R}^k$, then $\delta_n^T(B_r, M) = r$ for all $n \in \mathbb{N}$.

To see this, let $n \in \mathbb{N}$. Choose an $X_n \in \mathcal{X}_n$ such that the origin is in X_n. Since $d(a, 0) \leq r$ for any $a \in B_r$, we have $\inf_{x \in X_n} d(a, x) \leq r$. Thus, $\sup_{a \in B_r} \inf_{x \in X_n} d(a, x) \leq r$, so $\delta_n^T(B_r, M) \leq r$.

Now we must show that for each $X_n \in \mathcal{X}_n$, $\sup_{a \in B_r} \inf_{x \in X_n} d(a, x) \geq r$. If $X_n \in \mathcal{X}_n$, then there exist points $x_1, \ldots, x_n$ such that $X_n = \operatorname{conv}(x_1, \ldots, x_n)$. Now, choose a ray v beginning at the origin such that v contains none of the x_i's. This implies that v contains no points in X_n, with the possible exception of the origin. Now, for each $\varepsilon > 0$, we can find a point p_ε in $v \cap B_r$ for which $d(p_\varepsilon, 0) > r - \varepsilon$. Then, if $x \in X_n$, we know that $d(p_\varepsilon, x) = d(p_\varepsilon, 0) + d(x, 0)$ since x and p_ε do not lie on the same ray. Thus, $d(p_\varepsilon, x) > r - \varepsilon$. Hence, for each $\varepsilon > 0$, $\inf_{x \in X_n} d(p_\varepsilon, x) > r - \varepsilon$, so $\sup_{a \in B_r} \inf_{x \in X_n} d(a, x) \geq r$. Therefore, $\delta_n^T(B_r, M) \geq r$.

In the following we first give basic properties of $\delta_n^T(A, M) < \infty$.

Proposition 2.12. *Let $A \subseteq B$ be subsets of M. Then*

1. *For any $n \in \mathbb{N}$, $\delta_n^T(A, M) \leq \delta_n^T(B, M)$.*
2. *The sequence $\{\delta_n^T(A, M)\}_{n \in \mathbb{N}}$ is non-increasing.*

Proof. 1. Let $n \in \mathbb{N}$ such that M has at least one Tn-dimensional subset. Let $X \in \mathcal{X}_n$. Since $A \subseteq B$,

$$\sup_{a \in A} \inf_{x \in X} d(a, x) \leq \sup_{b \in B} \inf_{x \in X} d(b, x).$$

This holds for any $X \in \mathcal{X}_n$, so we have

$$\inf_{X \in \mathcal{X}_n} \sup_{a \in A} \inf_{x \in X} d(a, x) \leq \inf_{X \in \mathcal{X}_n} \sup_{b \in B} \inf_{x \in X} d(b, x).$$

Hence, $\delta_n^T(A, M) \leq \delta_n^T(B, M)$.

If $n \in \mathbb{N}$ such that M has no Tn-dimensional subsets, then there is a $k < n$ such that M is T*k-dimensional. By definition, M has at least one Tk-dimensional subset. Thus, by what we just found, $\delta_k^T(A, M) \leq \delta_k^T(B, M)$. By convention, $\delta_k^T(A, M) = \delta_n^T(A, M)$ and $\delta_k^T(B, M) = \delta_n^T(B, M)$, so $\delta_n^T(A, M) \leq \delta_n^T(B, M)$. Thus, for any $n \in \mathbb{N}$, $\delta_n^T(A, M) \leq \delta_n^T(B, M)$.

2. Suppose that the sequence is increasing somewhere. Then A must be bounded (since otherwise each Tn-width is ∞) and there is an $n \in \mathbb{N}$ such that $\delta_n^T(A, M) < \delta_{n+1}^T(A, M)$. Thus, there is some $X_n \in \mathcal{X}_n$ (say $X_n = \operatorname{conv}(x_1, \ldots, x_n)$) such that

$$\sup_{a \in A} \inf_{x \in X_n} d(a, x) < \sup_{a \in A} \inf_{x \in X_{n+1}} d(a, x)$$

for all $X_{n+1} \in \mathcal{X}_{n+1}$. We claim that for any finite set $\{y_1, \ldots, y_m\}$ in M, the set $Y_m = \operatorname{conv}(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is Tn-dimensional.

Suppose that Y_m is not Tn-dimensional. Since $X_n \subseteq Y_m$, we know by Lemma 2.9 that Y_m is no less than Tn-dimensional. Thus, Y_m is T$(n+k)$-dimensional for some $k \geq 1$. By removing some of the y_i's if $k > 1$, we can produce a set $Y_p = \operatorname{conv}(x_1, \ldots, x_n, y_1, \ldots, y_p)$ such that Y_p is T$(n+1)$-dimensional. Now, since $X_n \subseteq Y_p$, for any $a \in A$, we have

$$\inf_{x \in X_n} d(a,x) \geq \inf_{x \in Y_p} d(a,x).$$

Therefore,

$$\sup_{a \in A} \inf_{x \in X_n} d(a,x) \geq \sup_{a \in A} \inf_{x \in Y_p} d(a,x).$$

But since Y_p is T$(n+1)$-dimensional, this contradicts the fact that

$$\sup_{a \in A} \inf_{x \in X_n} d(a,x) < \sup_{a \in A} \inf_{x \in X_{n+1}} d(a,x)$$

for all $X_{n+1} \in \mathcal{X}_{n+1}$. Hence, Y_m must be Tn-dimensional.

Now, suppose that there exists some $X_{n+1} \in \mathcal{X}_{n+1}$. Then $X_{n+1} = \operatorname{conv}(y_1, \ldots, y_{n+1})$ for some $y_1, \ldots, y_{n+1} \in M$. By what we just established, the set

$$Y_m = \operatorname{conv}(x_1, \ldots, x_n, y_1, \ldots, y_{n+1})$$

is Tn-dimensional. But then $X_{n+1} \subseteq Y_m$, so we have a T$(n+1)$-dimensional set within a Tn-dimensional set, contrary to Lemma 2.9. We can therefore conclude that M does not contain any T$(n+1)$-dimensional sets, so by Lemma 2.8, M does not contain any Tk-dimensional sets for $k > n$. Hence, by definition, M is T*n-dimensional, so by convention, $\delta_n^T(A, M) = \delta_{n+1}^T(A, M)$, a contradiction. Thus, the sequence is non-increasing. ■

2.2. Compact widths. A concept that is related to the n-width is the *compact width.* Given a metric space (M, d), let $\mathcal{X}$ denote the set of compact subsets of M. If A is a subset of M, we define the compact width of A to be

$$a(A, M) = \inf_{X \in \mathcal{X}} \sup_{a \in A} \inf_{x \in X} d(a,x).$$

Observe that like Tn-widths, $a(A, M) = \infty$ if and only if A is unbounded. Indeed, this follows easily from the fact that every compact set in a metric space is bounded. We also have the following lemma.

LEMMA 2.13. *If (M, d) is a metric tree with subset A, then $\delta_n^T(A, M) \geq a(A, M)$ for all $n \in \mathbb{N}$.*

Proof. This is a direct consequence of Theorem 2.7. Since each Tn-dimensional subset of M is compact, $\mathcal{X}_n \subseteq \mathcal{X}$ for all $n \in \mathbb{N}$. Hence, for each n,

$$\delta_n^T(A, M) = \inf_{X \in \mathcal{X}_n} \sup_{a \in A} \inf_{x \in X} d(a,x) \geq \inf_{X \in \mathcal{X}} \sup_{a \in A} \inf_{x \in X} d(a,x) = a(A, M),$$

as desired. ■

DEFINITION 2.14. We say that a metric space (X, d) *has the property P_1* if for every $\varepsilon > 0$ and $r > 0$, there is a $\delta > 0$ such that for each $x, y \in X$, there is a $z \in B(x, \varepsilon)$ for which $B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta)$ if $0 < \theta < \delta$.

Property P_1 was studied by several authors, for example see [5], [21]. The following theorem establishes a relationship between the property P_1 and compact widths.

THEOREM 2.15 ([18]). *Let X be a Banach space. If X has the property P_1, then for each bounded subset A of X, the compact width $a(A, X)$ is attained.*

THEOREM 2.16. *Every metric tree has the property P_1.*

Proof. Let (M, d) be a metric tree, and let $\varepsilon > 0$ and $r > 0$ be given. Choose any $0 < \delta < \varepsilon$. We claim that such δ works regardless of r.

Let $x, y \in M$, and first suppose that $xy \geq \delta$. Choose $z \in [x, y]$ such that $xz = \delta$, and since $\delta < \varepsilon$, we have $z \in B(x, \varepsilon)$. We claim that with this choice of δ and z, we have $B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta)$ for $0 < \theta < \delta$.

Suppose $w \in B(x, r+\delta) \cap B(y, r+\theta)$. By the three point property, there exists a $u \in M$ such that $[x, w] \cap [w, y] = [w, u]$ and $[x, y] \cap [w, u] = \{u\}$. Since $u \in [x, y]$ and $z \in [x, y]$, we know that either $z \in [x, u]$ or $z \in [u, y]$. If $z \in [x, u]$, then we have the following:

$$\begin{aligned} zw &= xw - xz && \text{since } z \in [x,u] \text{ and } u \in [x,w] \text{ implies that } z \in [x,w] \\ &< r + \delta - \delta && \text{since } w \in B(x, r+\delta) \text{ and } xz = \delta \\ &\leq r + \theta && \text{since } \theta > 0. \end{aligned}$$

If $z \in [u, y]$, then we have the following:

$$\begin{aligned} zw &= wy - zy && \text{since } z \in [y,u] \text{ and } u \in [y,w] \text{ implies that } z \in [y,w] \\ &< r + \theta - zy && \text{since } w \in B(y, r+\theta) \\ &\leq r + \theta && \text{since } zy \geq 0. \end{aligned}$$

Therefore, in either case, $w \in B(z, r+\theta)$, so $B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta)$.

Now suppose that $xy < \delta$. In this case, choose $z = y$. Since $\delta < \varepsilon$, we have $z \in B(x, \varepsilon)$, and since $B(y, r+\theta) = B(z, r+\theta)$, we have $B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta)$ for $0 < \theta < \delta$.

Thus, by choosing $0 < \delta < \varepsilon$, for any $x, y \in M$ there is a $z \in M$ for which $B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta)$ for $0 < \theta < \delta$. ∎

COROLLARY 2.17. *For any bounded subset A of a complete metric tree (M, d), the compact width $a(A, M)$ is attained.*

Proof. In [18], Theorem 2.15 above is proved for Banach spaces. However, the proof uses none of the linear structure of a Banach space; it applies equally well to complete metric spaces. ∎

THEOREM 2.18. *For any subset A of a metric tree (M, d),*

$$\lim_{n\to\infty} \delta_n^T(A, M) = a(A, M).$$

Here we take the convention that if M is T**k-dimensional then $\delta_n^T(A, M) = \delta_k^T(A, M)$ for $n > k$.*

Proof. First, observe that if A is unbounded, then $\delta_n^T(A, M) = \infty = a(A, M)$ for all n, so the result holds trivially. Therefore, suppose that A is bounded. Let $\varepsilon > 0$ be given, and let $X \in \mathcal{X}$ such that

$$\sup_{a\in A} \inf_{x\in X} d(a,x) \le a(A,M) + \frac{\varepsilon}{3}\,.$$

We now wish to approximate X by a Tn-dimensional set. By the compactness of X, there exists a finite set of points $x_1,\dots,x_m$ in X such that

$$X \subseteq \bigcup_{i=1}^{m} B\Big(x_i, \frac{\varepsilon}{3}\Big).$$

Let $X_n = \operatorname{conv}(x_1,\dots,x_m)$, so X_n is a Tn-dimensional set for some $n \le m$.

Let $a \in A$. Then there exists a $y \in X$ such that $d(a,y) \le \inf_{x\in X} d(a,x) + \frac{\varepsilon}{3}$. Also, there exists an x_i such that $d(x_i,y) \le \frac{\varepsilon}{3}$. We therefore have

$$d(a,x_i) \le d(a,y) + d(x_i,y) \le \inf_{x\in X} d(a,x) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \inf_{x\in X} d(a,x) + \frac{2\varepsilon}{3}\,.$$

Hence, we know that

$$\inf_{x\in X_n} d(a,x) \le \inf_{x\in X} d(a,x) + \frac{2\varepsilon}{3}$$

for any $a \in A$. This implies that

$$\sup_{a\in A} \inf_{x\in X_n} d(a,x) \le \sup_{a\in A} \inf_{x\in X} d(a,x) + \frac{2\varepsilon}{3} \le a(A,M) + \frac{2\varepsilon}{3} + \frac{\varepsilon}{3}\,,$$

and as a result,

$$\delta_n^T(A,M) = \inf_{X_n\in\mathcal{X}_n} \sup_{a\in A} \inf_{x\in X_n} d(a,x) \le \sup_{a\in A} \inf_{x\in X_n} d(a,x) \le a(A,M) + \varepsilon.$$

Therefore, for every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\delta_n^T(A,M) \le a(A,M) + \varepsilon$.

By Proposition 2.12, the sequence $\{\delta_n^T(A,M)\}_{n\in\mathbb{N}}$ is non-increasing. If $\varepsilon > 0$ is given, choose $N \in \mathbb{N}$ such that $\delta_N^T(A,M) \le a(A,M) + \varepsilon$. We then know that $\delta_n^T(A,M) \le a(A,M) + \varepsilon$ for any $n \ge N$, so

$$\delta_n^T(A,M) - a(A,M) \le \varepsilon.$$

Also by Lemma 2.13, we know that for all $n \in \mathbb{N}$, $\delta_n^T(A,M) \ge a(A,M)$. Hence,

$$|\delta_n^T(A,M) - a(A,M)| = \delta_n^T(A,M) - a(A,M) \le \varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \delta_n^T(A,M) = a(A,M). \ \blacksquare$$

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