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SMOOTHNESS OF GREEN'S FUNCTIONS AND MARKOV-TYPE INEQUALITIES

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Abstract. Let E be a compact set in the complex plane, g_E be the Green function of the unbounded component of $\mathbb{C}_\infty \setminus E$ with pole at infinity and $M_n(E) = \sup \frac{\|P'\|_E}{\|P\|_E}$ where the supremum is taken over all polynomials $P|_E \not\equiv 0$ of degree at most n, and $\|f\|_E = \sup\{|f(z)| : z \in E\}$. The paper deals with recent results concerning a connection between the smoothness of g_E (existence, continuity, Hölder or Lipschitz continuity) and the growth of the sequence $\{M_n(E)\}_{n=1,2,\dots}$. Some additional conditions are given for special classes of sets.

1. Introduction. Let D be a proper subdomain of the Riemann sphere $\mathbb{C}_\infty$, and let $w \in D$ be a fixed point. *A Green function for D with pole at w* (see e.g. [Ra, Def. 4.4.1]) is a map $g_D : D \longrightarrow (-\infty, +\infty]$, such that

- $g_D(\cdot, w)$ is harmonic on $D \setminus \{w\}$, and bounded outside each neighbourhood of w,
- $g_D(w, w) = +\infty$, and

$$\lim_{z \to w} [g_D(z, w) - \log|z|] < \infty \quad \text{for } w = \infty,$$
$$\lim_{z \to w} [g_D(z, w) + \log|z - w|] < \infty \quad \text{for } w \in D \setminus \{\infty\},$$

- there is a polar set $F \subset \partial D$ such that $\partial D \setminus F \neq \emptyset$ and for each $z_0 \in \partial D \setminus F$ we have $\lim\limits_{z \to z_0} g_D(z, w) = 0$.

If the boundary of D is not polar (i.e. its logarithmic capacity is positive: $\mathrm{cap}(\partial D) > 0$) then there exists a unique Green function g_D for D and $g_D(z, w) > 0$ for any $z, w \in D$ (see e.g. [Ra, Th. 4.4.2, 3]).

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For a compact set $E \subset \mathbb{C}$, let

$$\hat{E} = \{z : \forall\, P \in \mathcal{P}(\mathbb{C}) \quad |P(z)| \le \|P\|_E\}. \tag{1}$$

Here and throughout, $\mathcal{P}(\mathbb{C}) = \bigcup_{n=0}^{\infty} \mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{C})$ denotes the space of all polynomials of one variable with complex coefficients and of degree not greater than n.

Let E be a non-polar compact set in the complex plane. From now on,

$$g_E(z) = \begin{cases} g_{\mathbb{C}_\infty \setminus \hat{E}}(z,\infty), & z \in \mathbb{C} \setminus \hat{E}, \\ 0, & z \in \hat{E}. \end{cases}$$

The Green functions have been extensively used in complex analysis, in potential theory and in partial differential equations. Among other applications, the Green functions have been particularly helpful in investigations of polynomial approximation, of 'small' sets, of conformal mappings and of some properties of analytic functions. For a comprehensive treatment and for references to the extensive literature on the subject one may refer to the books [Le2], [Ne], [Po], [Ra], [Kl].

There is a very close connection between the Green function and the maximal growth of polynomials outside a given compact set. Namely, for a non-polar compact set E, the Green function g_E (with zero on $\hat{E}$) coincides with the logarithm of the Leja-Siciak extremal function Φ_E, i.e. for $z \in \mathbb{C}$

$$g_E(z) = \log \Phi_E(z), \tag{2}$$

$$\Phi_E(z) = \sup\{|P(z)|^{1/\deg P} : P \in \mathcal{P}(\mathbb{C}), \deg P \ge 1, \|P\|_E \le 1\} \tag{3}$$

(see e.g. [Le1], [Si1]). If we take in (1) and (3) the space $\mathcal{P}(\mathbb{C}^N)$ of all polynomials of N variables with complex coefficients instead of $\mathcal{P}(\mathbb{C})$, formula (2) remains valid also for the pluricomplex Green function of $E \subset \mathbb{C}^N$ (see [Kl, Ch. 5] for more details).

A compact set $E \subset \mathbb{C}$ (or $\mathbb{C}^N$) is said to admit a *Hölder continuity property with exponent* $\alpha \in (0,1]$ if there exists a constant $A > 0$ such that

$$g_E(z) \le A[\operatorname{dist}(z,E)]^\alpha \text{ as } \operatorname{dist}(z,E) \le 1. \tag{4}$$

In this case, we will also say that g_E has the *Hölder continuity property* with $\alpha \in (0,1]$ and we will write $E \in HCP(\alpha)$. If $\alpha = 1$, we say that g_E is *Lipschitz continuous*. It is worth noticing that, by an argument due to Błocki, condition (4) is equivalent to the Hölder continuity of g_E in the whole complex plane ([Si4, Prop. 3.5]).

For a compact set $E \subset \mathbb{K}^N$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), we will consider *a Markov-type inequality*

$$\big\||\operatorname{grad} P|\big\|_E \le M_n \|P\|_E \tag{5}$$

where $\|f\|_E = \max\{|f(z)| : z \in E\}$ for $f : E \to \mathbb{C}$ and the constant $M_n > 0$ is independent of $P \in \mathcal{P}_n(\mathbb{K}^N)$.

This type of polynomial inequalities has been widely investigated in the last century (see e.g. [Jo-Wa], [Goe], [Li], [Ba], [Ba-Pl], [Bo-Er], [Si3], [To], [Gon], [Sk], [Kr-Re], [Ra-Sch], [Fr], [Ca-Le]). The most important applications of property (5) have been found in polynomial approximation, constructive theory and extension theory of functions. The classical papers here are [Pl1], [Pa-Pl], [Bo-Mi].

Let

$$M_n(E) := \sup\left\{ \frac{\||\text{grad}\,P|\|_E}{\|P\|_E} : P \in \mathcal{P}_n,\ P|_E \not\equiv 0 \right\} \quad \text{for } n = 1, 2, \ldots$$

It is possible to give exact values of the sequence $\{M_n(E)\}_{n=1,2,\ldots}$ only for a few compact sets E. For the unit disc $E = \{|z| \le 1\} \subset \mathbb{C}$ we have $M_n(E) = n$, $g_E(z) = \log|z|$ and $E \in HCP(1)$. If $E = [-1,1] \subset \mathbb{C}$ then $M_n(E) = n^2$ and $g_E(z) = \log|z + \sqrt{z^2-1}|$ with the branch of the square root chosen so that $t + \sqrt{t^2-1} > 1$ for $t > 1$. Thus $E \in HCP(\frac{1}{2})$. Now take $E = \{|z| \le 1\} \cup \{2\} \subset \mathbb{C}$. We can check that $M_n(E) \ge \frac{1}{6}2^n$, because for the polynomial $P(z) = z^{n-1}(z-2)$ we have $P'(2) = 2^{n-1}$ and $\|P\|_E = 3$. It is easily seen that g_E is not continuous.

A natural question arises whether there is a relationship between the smoothness of the Green function g_E and the growth of the sequence $\{M_n(E)\}_{n=1,2,\ldots}$. The paper deals with recent results giving some answers to this question. Most of the cited theorems deal with the one-dimensional case.

The paper is organized as follows. We start with Cantor-type sets, because this case was thoroughly investigated and the answers for questions about the smoothness of the Green functions are well known. Therefore, these sets will furnish some explicit examples. The third section deals with the existence of the Green function and the fourth one is connected with the continuity of g_E. The fifth section deals with the Hölder and Lipschitz continuity of the Green function. In the final section of the paper, we summarize presented results and we list some open problems.

2. Cantor-type sets. For a given sequence $(l_k)_{k=0,1,2,\ldots}$ of positive numbers such that

$$l_0 = 1 \quad \text{and} \quad l_k < \frac{1}{2} l_{k-1},\ k = 1, 2, \ldots$$

we construct a Cantor-type set as follows. Let $\{F_k\}_{k=0,1,2,\ldots}$ be a family of subsets of [0,1] such that every set F_k consists of 2^k intervals $I_{k,1}, \ldots, I_{k,2^k}$ (each of them has length l_k), F_0 =[0,1] and F_{k+1} is obtained by deleting the open middle subinterval of length $l_k - 2l_{k+1}$ from each interval $I_{k,n}$, $n = 1, \ldots, 2^k$. The set

$$E = \bigcap_{k=0}^{\infty} F_k$$

is a Cantor-type set associated with the sequence $(l_k)_{k=0,1,2,\ldots}$.

We can explicitly characterize all of the sets described above that have a positive logarithmic capacity or a continuous Green function, due to Carleson and Pleśniak.

THEOREM 2.1 (see [Ca]). *Let E be the Cantor-type set associated with the sequence $(l_k)_{k=0,1,2,\ldots}$. Then E is non-polar if and only if*

$$\sum_{k=0}^{\infty} \frac{\log l_k^{-1}}{2^k} < \infty.$$

THEOREM 2.2 (see [Pl1]). *A Cantor-type set has a continuous Green function if and only if it is non-polar.*

Evidently, the assertion of Theorem 2.2 is not true for an arbitrary compact set. The simplest example of this is furnished by $F = \{|z| \le 1\} \cup \{2\} \subset \mathbb{C}$. We can also construct a more interesting set K with a non-continuous Green function and with the following property: for any $z_0 \in K$ and $r > 0$ $\operatorname{cap}\big(K \cap \{\|z - z_0\| \le r\}\big) > 0$. By Wiener's criterion, an example of such K is easy to obtain. It is sufficient to take $K = \{0\} \cup \bigcup_{n=1}^{\infty} C_n$, $C_n = \{\frac{1}{2^n} e^{it} : t \in [0, 2^{-n^2}]\} \subset \mathbb{C}$.

In the class of the Cantor-type sets we can also characterize all of them that have a Hölder continuous Green function.

THEOREM 2.3 (see [To], cf. [BC1]). *If E is the Cantor-type set associated with a sequence $(l_k)_{k=0,1,2,\ldots}$ then the following conditions are equivalent*:

(i) *the Green function g_E is Hölder continuous,*
(ii) $\limsup\limits_{n\to\infty} \frac{\log M_n(E)}{\log n} < \infty$,
(iii) $\limsup\limits_{n\to\infty} \frac{\log l_k^{-1}}{k} < \infty$.

Let E be a Cantor-type set associated with (l_k). From the theorems given above it follows that

- if $l_k = \frac{1}{k}$ then $E \in HCP$,
- if $l_k = \frac{1}{(k+2)!}$ then $E \notin HCP$ but the Green function g_E is continuous (see [Pl1]).

It is worth noticing that for the Cantor-type set E associated with (l_k) we have the estimate $M_{2^n}(E) \ge \frac{1}{4 l_n}$ for $n = 0, 1, 2 \ldots$ (see [To]).

3. Existence of the Green function. Let E be a compact subset of the complex plane. According to the definition of the Green function given in the first section, we can see that g_E exists if and only if $\operatorname{cap} E > 0$. Therefore, in this section we consider a connection between the growth of $\{M_n(E)\}_{n=1,2,\ldots}$ and the non-polarity of the set E.

PROPOSITION 3.1. *If $E \subset \mathbb{C}$ is not polar, then*

$$\limsup_{n\to\infty} \frac{\log M_n(E)}{n} \le \log \frac{\operatorname{diam} E}{\operatorname{cap} E} < \infty,$$

where $\operatorname{diam} E = \max\{|z_1 - z_2| : z_1, z_2 \in E\}$.

Proof. Let $\{a_k^{(n)}\}_{k=0,\ldots,n}$ be a Fekete n-tuple for the set E, i.e.

$$\prod_{0\le i<j\le n} |a_i^{(n)} - a_j^{(n)}| = \max\Big\{ \prod_{0\le i<j\le n} |z_i - z_j| : z_0, \ldots, z_n \in E \Big\}.$$

Put

$$L_k^{(n)}(z) = \prod_{l=0, l\ne k}^{n} \frac{z - a_l^{(n)}}{a_k^{(n)} - a_l^{(n)}}.$$

By the Lagrange interpolation formula, for a fixed polynomial P of degree n and for any $z \in \mathbb{C}$

$$|P'(z)| \le \sum_{k=0}^{n} \big|P(a_k^{(n)})\big| \, \Big|\frac{d}{dz} L_k^{(n)}(z)\Big|.$$

Now for every $z \in E$,

$$|P'(z)| \le \|P\|_E \sum_{k=0}^{n} \sum_{j=0, j\neq k}^{n} \frac{\prod_{l=0, l\neq k, l\neq j}^{n} |z - a_l^{(n)}|}{\prod_{l=0, l\neq k}^{n} |a_k^{(n)} - a_l^{(n)}|}$$

$$\le \|P\|_E \sum_{k=0}^{n} \frac{n(\operatorname{diam} E)^{n-1}}{\prod_{l=0, l\neq k}^{n} |a_k^{(n)} - a_l^{(n)}|}.$$

It is known that $\|L_k^{(n)}\|_E = 1$ and $\|Q\|_E \ge (\operatorname{cap} E)^n$ for all monic polynomials Q of degree n (see [Ra, Th. 5.5.4]). Therefore $\prod_{l=0,l\neq k}^{n} |a_k^{(n)} - a_l^{(n)}| \ge (\operatorname{cap} E)^n$ and

$$M_n(E) \le \frac{n(n+1)}{\operatorname{diam} E}\Big(\frac{\operatorname{diam} E}{\operatorname{cap} E}\Big)^n,$$

which is the desired conclusion. ∎

REMARK 3.2. The assertion

$$\limsup_{n\to\infty} \frac{\log M_n(E)}{n} < \infty$$

of the above proposition is asymptotically optimal, because for any sequence $\alpha = \{\alpha_n\}_n$ growing more slowly to infinity than n, one can find a Cantor-type set $E_\alpha \subset \mathbb{R}$ such that $\operatorname{cap} E_\alpha > 0$ and $\limsup \frac{\log M_n(E_\alpha)}{\alpha_n} = \infty$ (see [To]).

THEOREM 3.3 (see [BC2]). *If E is a compact subset of $\mathbb{C}$ and the series*

$$\sum_{n=1}^{\infty} \frac{\log M_n(E)}{n^2} \quad \textit{is convergent}, \tag{6}$$

then $\operatorname{cap} E > 0$, and consequently, the Green function g_E exists.
Moreover,

$$\operatorname{cap} E \ge \exp\Big[-2\sum_{n=2}^{\infty} \frac{\log M_n(E)}{(n+1)(n+2)}\Big](\operatorname{diam} E)^{1/3}.$$

The assumption of the theorem is fulfilled if we have e.g. $M_n(E) \le \exp(\frac{nM}{\log^2 n})$ with some $M > 0$.

Note that the existence of the Green function does not imply the convergence of the series $\sum \frac{\log M_n(E)}{n^2}$. It is sufficient to consider $E = \{z \in \mathbb{C} : |z| \le 1\} \cup \{2\}$, because in this case we have $M_n(E) \ge \frac{1}{6}2^n$, hence $\sum \frac{\log M_n(E)}{n^2} \ge \sum \frac{\log(2^n/6)}{n^2} = +\infty$.

4. Continuity of the Green function. We start with a result obtained by Totik [To]. He has proved that the continuity of the Green function of a compact set $E \subset \mathbb{C}$ implies that

$$\lim_{n\to\infty} \frac{\log M_n(E)}{n} = 0.$$

The proof given in [To] for the one-dimensional case can be easily adapted to $E \subset \mathbb{C}^N$.

THEOREM 4.1 (cf. [To]). *If $E \subset \mathbb{C}^N$ is a compact set and its pluricomplex Green's function is continuous, then*

$$\lim_{n\to\infty} \frac{\log M_n(E)}{n} = 0. \tag{7}$$

Proof. Since the pluricomplex Green function is continuous, it follows that for any $\varepsilon > 0$ there exists $r > 0$ such that if $\operatorname{dist}(z, E) \le r$ then

$$\Big(\frac{|P(z)|}{\|P\|_E}\Big)^{1/n} \le e^{\varepsilon}$$

for all polynomials $P \in \mathcal{P}_n(\mathbb{C}^N)$.

Fix $z_0 \in E$. By Cauchy's inequality,

$$|\operatorname{grad} P(z_0)| \le \frac{c}{r}\|P\|_{\{\|z-z_0\|\le r\}}$$

with some $c > 0$. Hence

$$|\operatorname{grad} P(z_0)| \le \frac{c}{r} e^{n\varepsilon}\|P\|_E$$

and $M_n(E) \le ce^{n\varepsilon}/r$, which is the desired conclusion. ∎

REMARK 4.2. We follow [To] in noting that condition (7) is asymptotically optimal in the sense described in Remark 3.2. It is worth reminding the reader that a Cantor-type set is not polar if and only if its Green's function is continuous (see Section 2).

A sufficient condition for the continuity of the Green function g_E is known only in the case of $E \subset \mathbb{R}$.

THEOREM 4.3 (see [BC-E1]). *If $E \subset \mathbb{R}$ is a compact set and*

$$\limsup_{n\to\infty} \frac{\log M_n(E)}{\log n} < \infty \tag{8}$$

then the Green function g_E is continuous.

By Remarks 3.2 and 4.2, formula (8) is not a necessary condition for the continuity of g_E (see also [Pl1]).

We can prove Theorem 4.3 also in the complex plane but we need to assume that the set E satisfies a local Markov inequality.

DEFINITION 4.4. Let M be a positive number and $m, s \ge 1$. A compact subset E of $\mathbb{C}$ is said to admit a *local Markov property* if for every $z_0 \in E$ and $r \in (0, 1]$ the inequality

$$|P'(z_0)| \le \frac{Mn^s}{r^m}\|P\|_{E\cap\{|z-z_0|\le r\}}$$

is fulfilled for all polynomials $P \in \mathcal{P}_n(\mathbb{C})$.

THEOREM 4.5 (see [BC-E1]). *If a compact set $E \subset \mathbb{C}$ has the local Markov property then the Green function g_E is continuous.*

It is easily seen that the local Markov property implies the assumption of Theorem 4.3 (it suffices to take $r = 1$). But the converse is true only in the real case.

THEOREM 4.6 (see [Bo-Mi]). *A compact set $E \subset \mathbb{R}$ has the local Markov property if and only if condition* (8) *is fulfilled.*

Bos and Milman have proved an analogue of this theorem in $\mathbb{R}^N$. However, the equivalence of Theorem 4.6 is not true in the complex case [BC-E2]. Therefore, Theorem 4.3 is proven only for subsets of $\mathbb{R}$ and not of $\mathbb{C}$.

5. Hölder and Lipschitz continuity of the Green function. A necessary condition for the Hölder (and Lipschitz) continuity of the (pluricomplex) Green function has been known for over forty years (see [Si2] and the references given there).

THEOREM 5.1 (see e.g. [Si2]). *If $E \subset \mathbb{C}^N$ is a compact set and $E \in HCP(\alpha)$, then*

$$\limsup_{n\to\infty} \frac{\log M_n(E)}{\log n} \le \frac{1}{\alpha} < \infty.$$

Moreover,

$$\sup_n \frac{M_n(E)}{n^{1/\alpha}} < \infty.$$

COROLLARY 5.2. *If $E \subset \mathbb{C}^N$ is a compact set and $E \in HCP(1)$, then*

$$\lim_{n\to\infty} \frac{\log M_n(E)}{\log n} = 1.$$

Proof. Fix $w = (w_1, \dots, w_N)$, $z = (z_1, \dots, z_N) \in E$ and $k \in \{1, \dots, N\}$ such that

$$|w_k - z_k| = \max\Big\{\max_{l=1,\dots,N} |u_l - v_l| : u, v \in E\Big\} > 0.$$

Put $P(x) = (x_k - z_k)^n$ for $x = (x_1, \dots, x_N) \in \mathbb{C}^N$. We have

$$\|\operatorname{grad} P\|_E = n|w_k - z_k|^{n-1} = \frac{n}{|w_k - z_k|}\|P\|_E.$$

Consequently, $M_n(E) \ge \frac{n}{\operatorname{diam} E}$ and combining this with Theorem 5.1 yields the desired conclusion. ∎

A sufficient condition for the Hölder continuity (with $\alpha < 1$) of the (pluricomplex) Green function seems to be a difficult problem. A condition for the Lipschitz continuity of g_E ($E \subset \mathbb{C}$) has been described in [To-To]. The proof given there can easily be adapted to the case of several variables.

THEOREM 5.3 (cf. [To-To]). *Let E be a compact subset of $\mathbb{C}^N$. If*

$$\sup_n \frac{M_n(E)}{n} < \infty$$

then $E \in HCP(1)$.

Proof. For a fixed $z \in \mathbb{C} \setminus E$ find $z_0 \in E$ such that $\operatorname{dist}(z, E) = |z - z_0| \le 1$. By Taylor's formula, for any $P \in \mathcal{P}_n(\mathbb{C}^N)$

$$|P(z)| \le \sum_{|\beta|\le n} \frac{1}{\beta!}\,|D^\beta P(z_0)|\,|(z - z_0)^\beta|.$$

Since

$$\sum_{|\beta|=l} \frac{1}{\beta!} \le \frac{N^l}{l!},$$

we have

$$|P(z)| \le \sum_{l=0}^{n} \frac{N^l}{l!} M^l n^l \|P\|_E |z-z_0|^l \le e^{NM|z-z_0|n} \|P\|_E$$

where $M = \sup \frac{M_n(E)}{n}$. We conclude from (2) that $E \in HCP(1)$ as claimed. ■

COROLLARY 5.4. *If $E \subset \mathbb{C}^N$ is a compact set then $E \in HCP(1)$ if and only if*

$$\sup_n \frac{M_n(E)}{n} < \infty.$$

6. Summary and open problems. It seems to be of interest to look at a summary of the necessary conditions for the smoothness of the Green functions.

Let E be a compact set.

- If g_E exists (i.e. $\operatorname{cap} E > 0$) then $\displaystyle\limsup_{n\to\infty} \frac{\log M_n(E)}{n} < \infty.$
- If g_E is continuous then $\displaystyle\lim_{n\to\infty} \frac{\log M_n(E)}{n} = 0.$
- If g_E is Hölder continuous then $\displaystyle\limsup_{n\to\infty} \frac{\log M_n(E)}{\log n} < \infty.$
- If g_E is Lipschitz continuous then $\displaystyle\lim_{n\to\infty} \frac{\log M_n(E)}{\log n} = 1.$

We close this paper by offering some questions and problems for further research. The first three have been posed for a long time (see [Pl2], [Si3]) but they still remain open.

1. What condition for $\{M_n(E)\}_n$ is sufficient for the non-polarity of E in the space of several complex variables?
2. What assumption is sufficient for the continuity of the Green functions in the case of one or several complex variables?
3. Is there an equivalence between the conditions: $E \in HCP$ and $\limsup \frac{\log M_n(E)}{\log n} < \infty$? Any answer in the space of one or several variables should be interesting.
4. Can assumption (6) in Theorem 3.3 be replaced by a weaker condition for the sequence $\{M_n(E)\}_n$?
5. The same question as above but with assumption (8) in Theorem 4.3.

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