FUNCTION SPACES IX
BANACH CENTER PUBLICATIONS, VOLUME 92
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

A STRONGLY EXTREME POINT NEED NOT BE A DENTING POINT IN ORLICZ SPACES EQUIPPED WITH THE ORLICZ NORM

ADAM BOHONOS
School of Mathematics, West Pomeranian University of Technology
Al. Piastów 48/49, 70-310 Szczecin, Poland
E-mail: abohonos@zut.edu.pl

RYSZARD PŁUCIENNIK*
Institute of Mathematics, Poznań University of Technology
Piotrowo 3A, 60-965 Poznań, Poland
E-mail: ryszard.pluciennik@put.poznan.pl

Abstract. There are necessary conditions for a point x from the unit sphere to be a denting point of the unit ball of Orlicz spaces equipped with the Orlicz norm generated by arbitrary Orlicz functions. In contrast to results in [12, 17, 16], we present also examples of Orlicz spaces in which strongly extreme points of the unit ball are not denting points.

1. Introduction. Let $(X, \|\cdot\|_X)$ be a real Banach space and $B(X)$ $(S(X))$ be the closed unit ball (the unit sphere) of X. Let us denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and the set of reals, respectively. Before starting with our results, we need to recall some notions.

A point $x \in S(X)$ is called

a) an *extreme point* of the unit ball $B(X)$ (write $x \in \delta_e B(X)$) if for every $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$.

b) a *strongly extreme point* of the unit ball $B(X)$ (write $x \in \delta_{se} B(X)$) if for every sequences (y_n), (z_n) in $B(X)$ we have $\|y_n - x\|_X \to 0$ whenever $\|y_n + z_n - 2x\|_X \to 0$.

2010 *Mathematics Subject Classification*: 46E30, 46B20, 46B42.
Key words and phrases: extreme point, strongly extreme point, denting point, Orlicz space, Orlicz norm, interpolation space.
*Supported by State Committee for Scientific Research, Poland, Grant No. N N210 362236.
The paper is in final form and no version of it will be published elsewhere.

c) a *denting point of* $B(X)$ (write $x \in \delta_d B(X)$) if $x \notin \overline{co}\{B(X) \setminus [x + \varepsilon B(X)^0]\}$ for each $\varepsilon > 0$.

It is well known (see for example [11]) that

$$\delta_d B(X) \subset \delta_{se} B(X) \subset \delta_e B(X).$$

A Banach space X is said to have the *property* **R** (resp. **MLUR**, **G**) if $S(X) = \delta_e B(X)$ (resp. $S(X) = \delta_{se} B(X)$, $S(X) = \delta_d B(X)$).

The notion of denting point plays an important role because it is connected with the Radon-Nikodým Property (RNP), being one of the most important properties of Banach space. Namely, H. B. Maynard [15] proved that a Banach space X has the RNP if and only if every non-empty bounded closed set K in X has at least one denting point. It is also known that a Banach space X has the RNP if and only if for any equivalent norm in X the respective unit ball $B(X)$ has a denting point (see, e.g. [3, p. 30]).

R. Płuciennik, T. F. Wang and Y. L. Zhang [16] presented complete characterization of denting points of the unit ball in Orlicz spaces (for both Luxemburg and Orlicz norms) generated by an N-function. Moreover, they proved that the sets of denting points and strongly extreme points in Orlicz spaces coincide. Their results have been extended by B. L. Lin and Z. R. Shi [12] to Orlicz spaces generated by general Orlicz functions and equipped with the Luxemburg norm and by Z. R. Shi [17] to Orlicz spaces generated by essentially wider class of Orlicz functions than N-functions and equipped with the Orlicz norm.

The aim of this paper is to give the necessary conditions for a point x from the unit sphere to be a denting point of the unit ball of Orlicz spaces equipped with the Orlicz norm generated by arbitrary Orlicz functions (that is, Orlicz functions which vanish outside zero and which attain infinite values to the right of some point $u > 0$ are not excluded) and equipped with the Orlicz norm. In contrast to results in [12, 17, 16], we present also examples of Orlicz spaces in which strongly extreme points of the unit ball are not denting points. Moreover, we give examples of Orlicz spaces for which their unit balls have no denting points. It is important because such spaces lack Radon-Nikodým Property. As we will see below, the fact that the degenerated Orlicz functions are not excluded in our considerations is of great interest. Namely, the classical spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$ which are important in the interpolation theory as well as the spaces $L^p \cap L^\infty$ $(1 < p < \infty)$ become special cases of Orlicz spaces investigated by us.

Let (T, Σ, μ) be a σ-finite and complete measure space. By $L^0 = L^0(T)$ we denote the set of all μ-equivalence classes of real valued measurable functions defined on T.

To define Orlicz spaces, we start with so called Orlicz function. A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an *Orlicz function* if it is even, convex, left continuous on whole of $\mathbb{R}_+$, $\Phi(0) = 0$ and Φ is not identically equal to zero. For any Orlicz function Φ we set

$$a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\},$$
$$b(\Phi) = \sup\{u > 0 : \Phi(u) < \infty\}.$$

Since the quotient $\Phi(u)/u$ is nondecreasing on $\mathbb{R}_+$ for any Orlicz function Φ, the limit (finite or infinite) $A_\Phi = \lim\limits_{u \to \infty} (\Phi(u)/u)$ always exists. If we assume additionally that

$0 < \Phi(u) < \infty$ for any $u > 0$, $\lim_{u\to 0+}(\Phi(u)/u) = 0$ and $A_\Phi = \infty$, then the Orlicz function Φ is called an *N-function*. In the case $A_\Phi < \infty$, define

$$R(u) = A_\Phi\,|u| - \Phi(u).$$

We say that an Orlicz function Φ satisfies the Δ_2-*condition* for all $u \in \mathbb{R}$ (at infinity) [at zero] if there are positive constants K and u_0 with $0 < \Phi(u_0) < \infty$ such that $\Phi(2u) \le K\Phi(u)$ holds for all $u \in \mathbb{R}$ (for every $|u| \ge u_0$) [for every $|u| \le u_0$]. We denote these conditions by $\Phi \in \Delta_2$ ($\Phi \in \Delta_2(\infty)$) [$\Phi \in \Delta_2(0)$], respectively. We have that $\Phi \in \Delta_2$ if and only if $\Phi \in \Delta_2(0)$ and $\Phi \in \Delta_2(\infty)$.

For any Orlicz function Φ we can define on $L^0(T)$ a convex modular

$$I_\Phi(x) = \int_T \Phi(x(t))\,d\mu.$$

Then the set

$$L_\Phi = \{x \in L^0(T) : I_\Phi(cx) < \infty \text{ for some } c > 0 \text{ depending on } x\}$$

is called an *Orlicz space*. This space is usually equipped with the *Luxemburg norm*

$$\|x\|_\Phi = \inf\Big\{\varepsilon > 0 : I_\Phi\Big(\frac{x}{\varepsilon}\Big) \le 1\Big\}$$

or with the equivalent one

$$\|x\|_\Phi^0 = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

called the *Orlicz norm* in the *Amemiya form*. The Orlicz space L_Φ equipped with the Luxemburg (Orlicz) norm will be denoted by L_Φ (L_Φ^0, respectively). The set of all $k > 0$ at which the infimum in the Amemiya formula for $\|x\|_\Phi^0$ is attained will be denoted by $K(x)$. It is known (see [4, Theorem 5]) that the following conditions are equivalent:

(a) *There exists* $x \in L_\Phi^0 \setminus \{0\}$ *such that* $K(x) = \varnothing$;

(b) $R(u)$ *is upper bounded.*

We say that w is a *point of strict convexity* of Φ (we write $w \in SC(\Phi)$) if for every $u, v \in \mathbb{R}$ such that $u \ne v$ and $w = \frac{u+v}{2}$ we have

$$\Phi\Big(\frac{u+v}{2}\Big) < \frac{\Phi(u) + \Phi(v)}{2}\,.$$

The following results given in [5] are the crucial point in our consideration.

THEOREM 1.1. *Assume Φ is an arbitrary Orlicz function. A point $x \in S(L_\Phi^0)$ is an extreme point of $B(L_\Phi^0)$ if and only if the following conditions are satisfied*:

(a) *The set $K(x)$ is a singleton, that is $K(x) = \{k\}$;*

(b) *$kx(t) \in SC(\Phi)$ for μ-a.e. $t \in T$.*

THEOREM 1.2. *Let Φ be an arbitrary Orlicz function. A point $x \in S(L_\Phi^0)$ is a strongly extreme point of $B(L_\Phi^0)$ if and only if x is an extreme point of $B(L_\Phi^0)$ and either $\Phi(b(\Phi)) < \infty$ and x is of the form $k|x(t)| = b(\Phi)$ for μ-a.e. $t \in T$ or $\Phi \in \Delta_2(\infty)$ and at least one of the conditions holds*:

(i) $\mu(T) < \infty$;

(ii) $a(\Phi) > 0$;

(iii) $\Phi \in \Delta_2(0)$.

2. Results. The criteria for denting points of the unit ball in Orlicz spaces generated by some classes of Orlicz functions were given in [12, 17, 16]. In principle, Z. R. Shi in [17] assumed that $K(x) \neq \varnothing$ for any $x \in L^0_\Phi$, i.e. $R(u) = \infty$ for any $u \in \mathbb{R}_+$. Combining results obtained by R. Płuciennik, T. F. Wang and Y. L. Zhang in [16] with results (concerning only denting points) by Z. R. Shi in [17], we see that if the Orlicz function Φ takes only finite values, $\mu(T) < \infty$ and $R(u) = \infty$ for any $u \in \mathbb{R}_+$, then the following conditions are equivalent:

a) *x is a denting point of $B(L^0_\Phi)$;*

b) *x is a strongly extreme point of $B(L^0_\Phi)$;*

c) *$\Phi \in \Delta_2(\infty)$ and $kx(t) \in SC(\Phi)$ for μ-a.e. $t \in T$ and $K(x) = \{k\}$.*

This result is not true in general, i.e. for Orlicz spaces generated by arbitrary Orlicz functions. First we will prove the following.

Lemma 2.1. *Let Φ be an Orlicz function. If $b(\Phi) < \infty$, then the set of denting points of $B(L^0_\Phi)$ is empty.*

Proof. If $b(\Phi) < \infty$ and $\Phi(b(\Phi)) = \infty$, then $\Phi \notin \Delta_2(\infty)$, so, by Theorem 1.2, the unit ball $B(L^0_\Phi)$ has no strongly extreme points. Hence $B(L^0_\Phi)$ cannot have any denting points.

Assume now that $b(\Phi) < \infty$ and $\Phi(b(\Phi)) < \infty$. Then, by Theorem 1.2, the only strongly extreme points are of the form $|x| = \frac{b(\Phi)}{k_0}\chi_T$, where k_0 is such that $\big\|\frac{b(\Phi)}{k_0}\chi_T\big\|^0_\Phi = 1$. Since every denting point of the unit ball of Banach space is a strongly extreme, the only possible denting points are of the form $|x| = \frac{b(\Phi)}{k_0}\chi_T$ also.

Suppose first that $\mu(T) < \infty$. Take a sequence of pairwise disjoint measurable sets such that

$$T = \bigcup_{n=1}^{\infty} T_n \quad \text{and} \quad \mu(T_n) = 2^{-n}\mu(T)$$

for any $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define $x_n = x\chi_{T\setminus T_n}$. Then

$$\|x_n\|^0_\Phi \leq \|x\|^0_\Phi = 1$$

for any $n \in \mathbb{N}$. Consequently, $x_n \in B(L^0_\Phi)$ for any $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}\|x - x_n\|^0_\Phi = \Big\|\frac{b(\Phi)}{k_0}\chi_{T_n}\Big\|^0_\Phi &= \inf_{k>0}\Big\{\frac{1}{k}\Big(1 + \int_{T_n}\Phi\Big(\frac{kb(\Phi)}{k_0}\Big)\,d\mu\Big)\Big\} \\ &= \inf_{0<k\leq k_0}\Big\{\frac{1}{k}\Big(1 + \int_{T_n}\Phi\Big(\frac{kb(\Phi)}{k_0}\Big)\,d\mu\Big)\Big\} > \inf_{0<k\leq k_0}\frac{1}{k} = \frac{1}{k_0}\end{aligned}$$

and

$$\begin{aligned}\Big\|x - \frac{1}{n}\sum_{i=1}^{n} x_n\Big\|^0_\Phi &= \Big\|x - \frac{n-1}{n}\,x\chi_{\cup_{i=1}^n T_i} - x\chi_{\cup_{i=n+1}^\infty T_i}\Big\|^0_\Phi \\ &= \Big\|\frac{1}{n}\,x\chi_{\cup_{i=1}^n T_i}\Big\|^0_\Phi \leq \frac{1}{n}\,\|x\|^0_\Phi = \frac{1}{n}\end{aligned}$$

for any $n \in \mathbb{N}$. Hence $x \in \overline{co}\big(B(L^0_\Phi) \setminus \{x + \varepsilon B(L^0_\Phi)\}\big)$ for any $\varepsilon \in (0, 1/k_0)$, so x cannot be a denting point of $B(L^0_\Phi)$.

Now, let $\mu(T) = \infty$. Then it is easy to see that $|x| = \frac{b(\Phi)}{k_0}\chi_T$ implies that $x \in L^0_\Phi$ if and only if $a(\Phi) > 0$. To find k_0, we calculate the Orlicz norm of x. Since for any

$u > a(\Phi)$, $I_\Phi(u\chi_T) = \infty$, we have

$$1 = \left\| \frac{b(\Phi)}{k_0}\chi_T \right\|_\Phi^0 = \inf_{k>0} \left\{ \frac{1}{k}\Big(1 + I_\Phi\Big(\frac{kb(\Phi)}{k_0}\chi_T\Big)\Big) \right\}$$
$$= \inf_{0<k\le k_0 \frac{a(\Phi)}{b(\Phi)}} \left\{ \frac{1}{k}\Big(1 + I_\Phi\Big(\frac{kb(\Phi)}{k_0}\chi_T\Big)\Big) \right\} = \inf_{0<k\le k_0 \frac{a(\Phi)}{b(\Phi)}} \frac{1}{k} = \frac{b(\Phi)}{k_0 a(\Phi)}.$$

Hence $k_0 = \frac{b(\Phi)}{a(\Phi)}$ and consequently $|x| = a(\Phi)\chi_T$. Let (S_n) be a sequence of pairwise disjoint subsets of T such that $T = \bigcup_{n=1}^\infty S_n$ and $\mu(S_n) = 1$ for any $n \in \mathbb{N}$. Then, by the inequality $\|x_n\|_\Phi^0 \le \|x\|_\Phi^0 = 1$ $(n = 1, 2, \ldots)$, we have that $x_n \in B(L_\Phi^0)$ for any $n \in \mathbb{N}$. Further,

$$\|x - x_n\|_\Phi^0 = \|a(\Phi)\chi_{S_n}\|_\Phi^0 = \inf_{0<k<\frac{b(\Phi)}{a(\Phi)}} \left\{ \frac{1}{k}(1 + I_\Phi(ka(\Phi)\chi_{S_n})) \right\} \ge \frac{a(\Phi)}{b(\Phi)} > 0$$

and

$$\left\| x - \frac{1}{n}\sum_{i=1}^n x_n \right\|_\Phi^0 = \left\| \frac{1}{n} x\chi_{\cup_{i=1}^n S_i} \right\|_\Phi^0 \le \frac{1}{n}\|x\|_\Phi^0 = \frac{1}{n}$$

for any $n \in \mathbb{N}$. Therefore, the point x such that $|x| = a(\Phi)\chi_T$ is not a denting point, which finishes the proof. ∎

Lemma 2.2. *Let Φ be an Orlicz function and $\mu(T) = \infty$. If $a(\Phi) > 0$, then the set of denting points of $B(L_\Phi^0)$ is empty.*

Proof. Suppose that $a(\Phi) > 0$, $\mu(T) = \infty$ and $\delta_d B(L_\Phi^0) \ne \varnothing$. Let x_0 be a denting point of the unit ball $B(L_\Phi^0)$. Since x_0 is also an extreme point, by Theorem 1.1(a), there is the only one $k_0 \ge 1$ such that

$$\|x_0\|_\Phi^0 = \frac{1}{k_0}(1 + I_\Phi(k_0 x_0)).$$

Moreover, by Theorem 1.1(b), $k_0 x_0(t) \in SC(\Phi)$ for μ-a.e. $t \in T$. Obviously, $a(\Phi)$ is the smallest positive number in $SC(\Phi)$. Hence $k_0|x_0(t)| \ge a(\Phi)$ for μ-a.e. $t \in T$ and consequently $\operatorname*{infess}_{t\in T} |x_0(t)| \ge \frac{a(\Phi)}{k_0}$. Take the sequence (A_n) of pairwise disjoint measurable sets such that $T = \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) = \infty$ for any $n \in \mathbb{N}$. Define $x_n = x_0\chi_{T\setminus A_n}$ for any $n \in \mathbb{N}$. Then $\|x_n\|_\Phi^0 \le \|x_0\|_\Phi^0 = 1$ for any $n \in \mathbb{N}$. Moreover,

$$\|x_0 - x_n\|_\Phi^0 = \|x_0\chi_{A_n}\|_\Phi^0 \ge \left\| \frac{a(\Phi)}{k_0}\chi_{A_n} \right\|_\Phi^0$$
$$= \inf_{0<k<k_0} \left\{ \frac{1}{k}\Big(1 + I_\Phi\Big(k\frac{a(\Phi)}{k_0}\chi_{A_n}\Big)\Big) \right\} = \frac{1}{k_0} > 0$$

and

$$\left\| x_0 - \frac{1}{n}\sum_{i=1}^n x_n \right\|_\Phi^0 = \left\| \frac{1}{n} x_0\chi_{\cup_{i=1}^n A_i} \right\|_\Phi^0 \le \frac{1}{n}\|x_0\|_\Phi^0 = \frac{1}{n},$$

whence $x_0 \in \overline{co}(B(L_\Phi^0) \setminus \{x + \varepsilon B(L_\Phi^0)\})$ for any $\varepsilon \in (0, 1/k_0)$, so x_0 cannot be a denting point of $B(L_\Phi^0)$. ∎

THEOREM 2.3. *Assume Φ is an Orlicz function and $x \in S(L^0_\Phi)$. If x is a denting point of $B(L^0_\Phi)$, then the following conditions are satisfied*:

(a) *the set $K(x)$ is a singleton*;

(b) $kx(t) \in SC(\Phi)$ *for μ-a.e. $t \in T$*;

(c) $\Phi \in \Delta_2(\infty)$ *and at least one of the conditions holds*:

(i) $\mu(T) < \infty$;

(ii) $\Phi \in \Delta_2(0)$.

Proof. Let x be a denting point of $B(L^0_\Phi)$. Since every denting point is an extreme point, by Theorem 1.1, conditions (a) and (b) are necessary. Moreover, x is also a strongly extreme point, so at least one of conditions in Theorem 1.2(c) is satisfied. By Lemmas 2.1 and 2.2, the cases: $b(\Phi) < \infty$, $a(\Phi) > 0$ with $\mu(T) = \infty$ can be excluded. Hence, by eliminating from Theorem 1.2 the superfluous conditions, we get the assertion. ■

3. Examples. Now we will present some examples of Orlicz spaces L^0_Φ in which

$$\delta_{se}B(L^0_\Phi) \neq \delta_d B(L^0_\Phi)$$

EXAMPLE 1. Define

$$\Phi_\infty(u) = \begin{cases} 0 & \text{for } u \in [-1,1] \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that L^∞ is the Orlicz space $L^0_{\Phi_\infty}$ with the equality of the norms. By Corollary 4 from [5], the only strongly extreme points of $B(L^\infty)$ are functions $x \in L^0(\mu)$ such that $|x(t)| = 1$ for μ-a.e. $t \in T$. But, by Lemma 2.1, none of them is a denting point because $b(\Phi_\infty) = 1 < \infty$.

EXAMPLE 2. Consider the classical interpolation space $L^1 + L^\infty$ equipped with the norm

$$\|x\|_{L^1+L^\infty} = \inf\{\|y\|_1 + \|z\|_\infty : y+z = x,\ y \in L^1,\ z \in L^\infty\}$$

(see [1], [10]). It is known (see [6]) that the space $L^1 + L^\infty$ is the Orlicz space generated by the Orlicz function $\Phi_{\infty,1}$ defined by the formula $\Phi_{\infty,1}(u) = \max\{0, |u| - 1\}$. Moreover, $\|\cdot\|_{L^1+L^\infty} = \|\cdot\|^0_{\Phi_{\infty,1}}$. By Corollary 5 from [5], x is a strongly extreme point of $B(L^1+L^\infty)$ if and only if $\mu(T) > 1$ and $|x(t)| = 1$ for μ-a.e. $t \in T$. By Theorem 1 from [2], any strongly extreme point of $B(L^1 + L^\infty)$ is a denting point of $B(L^1 + L^\infty)$ only in the case when $1 < \mu(T) < \infty$. If $\mu(T) = \infty$, then, by Lemma 2.2, the unit ball $B(L^1 + L^\infty)$ has no denting points.

EXAMPLE 3. Let $\Phi_{1,\infty}(u) = \max\{|u|, \Phi_\infty(u)\}$. Then the Orlicz space $L^0_{\Phi_{1,\infty}}$ is the space $L^1 \cap L^\infty$ equipped with the norm $\|\cdot\|_{L^1\cap L^\infty} = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty}$. By Corollary 6 from [5], the set of strongly extreme points is nonempty if and only if $\mu(T) < \infty$. Moreover, the only strongly extreme points of $B(L^1 \cap L^\infty)$ are of the form $|x| = (1 + \mu(T))^{-1}\chi_T$. Since $b(\Phi_{1,\infty}) = 1 < \infty$, by Lemma 2.2, the set of denting points of $B(L^1 \cap L^\infty)$ is empty. We also will show directly that points of the form $|x| = (1 + \mu(T))^{-1}\chi_T$ are not denting points. Really, taking into account the sequence of pairwise disjoint measurable sets (T_n)

such that $\mu(T_n) = \frac{\mu(T)}{2^n}$ for any $n \in \mathbb{N}$, define $x_n = x\chi_{T\setminus T_n}$. Then

$$\|x_n\|^0_{\Phi_{1,\infty}} = \|x_n\|_{L^1\cap L^\infty} = \|x_n\|_{L^1} + \|x_n\|_{L^\infty} = \frac{\mu(T)-\mu(T_n)}{1+\mu(T)} + \frac{1}{1+\mu(T)}$$
$$= 1 - \frac{\mu(T_n)}{1+\mu(T)} = 1 - \frac{\mu(T)}{2^n(1+\mu(T))} \le 1,$$

so $x_n \in B(L^1 \cap L^\infty)$ for any $n \in \mathbb{N}$. Moreover,

$$\|x - x_n\|^0_{\Phi_{1,\infty}} = \|x\chi_{T_n}\|^0_{\Phi_{1,\infty}} = \frac{\mu(T_n)}{1+\mu(T)} + \frac{1}{1+\mu(T)} \ge \frac{1}{1+\mu(T)},$$

but

$$\Big\|x - \frac{1}{n}\sum_{i=1}^n x_n\Big\|^0_{\Phi_{1,\infty}} = \Big\|x - \frac{n-1}{n}x\chi_{\cup_{i=1}^n T_i} - x\chi_{\cup_{i=n+1}^\infty T_i}\Big\|^0_{\Phi_{1,\infty}} = \frac{1}{n}\,\|x\chi_{\cup_{i=1}^n T_i}\|^0_{\Phi_{1,\infty}}$$
$$= \frac{1}{n}\Big(\frac{\mu(T)}{1+\mu(T)}\Big(1-\frac{1}{2^{n-1}}\Big) + \frac{1}{1+\mu(T)}\Big)$$
$$= \frac{1}{n}\Big(1 - \frac{\mu(T)}{2^{n-1}(1+\mu(T))}\Big) < \frac{1}{n}.$$

Therefore x is not a denting point.

EXAMPLE 4. Let $\Phi_{p,\infty}(u) = \max\{|u|^p, \Phi_\infty(u)\}$, $1 < p < \infty$. It is proved in [8] that the Orlicz space $L^0_{\Phi_{p,\infty}}$ is the space $L^p \cap L^\infty$ equipped with the norm

$$\|x\|_{L^p\cap L^\infty} = \begin{cases} \beta(x)^{p-1}\,\|x\|_{L^p} + \|x\|_{L^\infty} & \text{if } \beta(x) \le (q/p)^{1/p} \\ p^{1/p}q^{1/q}\,\|x\|_{L^p} & \text{if } \beta(x) > (q/p)^{1/p}, \end{cases}$$

where $\beta(x) := \|x\|_{L^p} / \|x\|_{L^\infty}$ for $x \ne 0$ and $1/p + 1/q = 1$. By Corollary 7 from [5], the space $L^p \cap L^\infty$ is strictly convex. Moreover, if $\mu(T) \le q/p$, then a point $x \in S(L^p \cap L^\infty)$ is strongly extreme if and only if $|x| = (1+\mu(T))^{-1}\chi_T$. In the case when $\mu(T) > q/p$, the unit ball $B(L^p \cap L^\infty)$ has no strongly extreme points. Since $b(\Phi_{p,\infty}) = 1 < \infty$, by Lemma 2.2, the unit ball $B(L^p \cap L^\infty)$ has no denting points.

By Theorem 1 in [16], it is easy to conclude that L^0_Φ has property **G** whenever Φ is an N-function, $SC(\Phi) = \mathbb{R}$ and $\Phi \in \Delta_2$. Now, if we omit the assumption that Φ is an N-function, then L^0_Φ need not have property **G**. The following example shows this fact.

EXAMPLE 5. Define $\Phi(u) = \frac{u^2}{|u|+1}$. By the inequalities

$$\frac{1}{2}\,u^2 \le \frac{u^2}{|u|+1} \le u^2 \quad \text{for} \quad |u| \le 1$$

and

$$\frac{1}{2}\,|u| \le \frac{u^2}{|u|+1} \le |u| \quad \text{for} \quad |u| \ge 1,$$

we conclude that Φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$. Moreover, by the standard calculation, it is easy to prove that $SC(\Phi) = \mathbb{R}$. On the other hand, Φ is not an N-function because $\lim_{u\to\infty} \frac{\Phi(u)}{u} = 1$. Since

$$R(u) = |u| - \frac{u^2}{|u|+1} = \frac{|u|}{|u|+1} \le 1$$

for any $u \in \mathbb{R}$, there is $x \in L^0_\Phi \setminus \{0\}$ such that $K(x) = \varnothing$ (see [4, Theorem 5]). Define $x = \chi_A$ with $\mu(A) = 1$. Then

$$\|x\|^0_\Phi = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx)) = \inf_{k>0} \frac{1}{k}\Big(1 + \int_A \frac{k^2}{k+1}\, d\mu\Big)$$
$$= \inf_{k>0}\Big(\frac{1}{k} + \frac{k}{k+1}\Big) = \inf_{k>0} \frac{k^2+k+1}{k^2+k} = \lim_{k\to\infty} \frac{k^2+k+1}{k^2+k} = 1$$

and consequently $K(\chi_A) = \varnothing$. Hence, by Theorem 1 from [5], χ_A is not even an extreme point $B(L^0_\Phi)$. Therefore, L^0_Φ is not strictly convex, so it has not property **G**.

Open problem. Can Theorem 2.3 be reversed? The Authors feel sure that the answer is positive, but they still have no complete proof of it.

References

[1] C. Bennett, R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, 1988.

[2] A. Bohonos, R. Płuciennik, *Modulus of dentability in $L^1 + L^\infty$*, in: Function Spaces VIII, Banach Center Publ. 79, Polish Acad. Sci. Inst. Math., Warsaw, 2008, 39–51.

[3] R. D. Bourgin, *Geometric Aspects of Convex Set with the Radon-Nikodým Property*, Lecture Notes in Math. 993, Springer, Berlin, 1983.

[4] S. T. Chen, Y. A. Cui, H. Hudzik, *Isometric copies of l^1 and l^∞ in Orlicz spaces equipped with the Orlicz norm*, Proc. Amer. Math. Soc. 132 (2004), 473–480.

[5] Y. A. Cui, H. Hudzik, R. Płuciennik, *Extreme points and strongly extreme points in Orlicz spaces equipped with the Orlicz norm*, Z. Anal. Anwendungen 22 (2003), 789–817.

[6] H. Hudzik, *On the distance from the subspace of order continuous elements in $L^1 + L^\infty$*, Funct. Approx. Comment. Math. 25 (1997), 157–163.

[7] H. Hudzik, A. Kamińska, M. Mastyło, *Local geometry of $L^1 \cap L^\infty$ and $L^1 + L^\infty$*, Arch. Math. (Basel) 68 (1997), 159–168.

[8] H. Hudzik, L. Maligranda, *Amemiya norm equals Orlicz norm in general*, Indag. Math. (N.S.) 11 (2000), 573–585.

[9] L. V. Kantorovich, G. P. Akilov, *Functional Analysis*, Nauka, Moscow, 1984 (in Russian).

[10] S. G. Kreĭn, Yu. I. Petunin, E. M. Semenov, *Interpolation of Linear Operators*, Transl. Math. Monogr. 54, Amer. Math. Soc., Providence, 1982.

[11] B. L. Lin, P. K. Lin, S. L. Troyanski, *Characterization of denting points*, Proc. Amer. Math. Soc. 102 (1988), 526–528.

[12] B. L. Lin, Z. R. Shi, *Denting points and drop properties in Orlicz spaces*, J. Math. Anal. Appl. 201 (1996), 252–273.

[13] P. K. Lin, H. Sun, *Extremity in Köthe-Bochner function spaces*, J. Math. Anal. Appl. 218 (1998), 136–154.

[14] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces II*, Ergeb. Math. Grenzgeb. 97, Springer, Berlin, 1979.

[15] H. B. Maynard, *A geometrical characterization of Banach spaces having the Radon-Nikodým property*, Trans. Amer. Math. Soc. 185 (1973), 493–500.

[16] R. Płuciennik, T. F. Wang, Y. L. Zhang, *H-points and denting points in Orlicz spaces*, Comment. Math. Prace Mat. 33 (1993), 135–151.

[17] Z. R. Shi, *Denting points and drop properties in Orlicz spaces equipped with Orlicz norm*, Acta Sci. Math. (Szeged) 63 (1997), 513–532.

[18] W. Wnuk, *Banach Lattices with Order Continuous Norms*, PWN, Warszawa, 1999.