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WEAK STAR CONVERGENCE OF MARTINGALES IN A DUAL SPACE

C. CASTAING

Départment de Mathématiques, Université Montpellier II
34095 Montpellier Cedex 5, France
E-mail: charles.castaing@gmail.com

F. EZZAKI

Laboratoire modélisation et calcul scientifique
Département de Mathématiques, Faculté des Sciences et Techniques
BP 2202, Fès, Maroc
E-mail: fatimaezzaki@yahoo.fr

M. LAVIE

Laboratoire de Mathématiques appliquées, Université de Pau et des Pays de L'Adour
BP 1155, 64013 Pau cedex, France
E-mail: marc.lavie@univ-pau.fr

M. SAADOUNE

Département de Mathématiques, Université Ibnou Zohr
Lot. Addalha, B.P. 8106, Agadir, Maroc
E-mail: mohammed.saadoune@gmail.com

Abstract. In this paper we present various weak star Kuratowski convergence results for multivalued martingales, supermartingales and multivalued mils in the dual of a separable Banach space. We establish several integral representation formulas for convex weak star compact valued multifunctions defined on a Köthe space and derive several existence results of conditional expectation for multivalued Gelfand-integrable multifunctions. Similar convergence results for Gelfand-integrable martingales in the dual space are provided. We also present a new version of Mosco convergence result for unbounded closed convex integrable supermartingales in a sepa-

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rable Banach spaces having the Radon-Nikodym property. New application to the law of large numbers is also presented.

1. Introduction. In this paper we present new results of weak star Kuratowski (w^*K for short) convergence for Gelfand-integrable multivalued martingales, supermartingales and mils taking convex weak star compact values in the dual of a separable Banach space. By contrast with similar results in separable Banach space dealing with Mosco convergence, see [1, 4, 5, 15, 16, 17, 7, 20, 21, 22, 23, 27] and the references therein, the study of w^*K-convergence for the weak star closed convex valued multifunctions considered here is unusual because the dual space is no longer strongly separable. Our purpose is to present various w^*K-convergence results for Gelfand-integrable multivalued martingales in the dual of a separable Banach space by introducing new tools based on the conditional expectation for Gelfand-integrable multifunctions, multivalued Biting lemma and multivalued Dunford-Pettis theorems. The paper is organized as follows. In Section 3 we state the w^*K-convergence for convex weak star compact valued martingales. In Section 4 we state the w^*K-convergence for convex weak star compact valued mils. In Section 5 the w^*K-convergence for unbounded weak star closed convex valued for supermartingales is presented as well as the existence of Gelfand-integrable regular martingale selections. In Section 6 we establish several integral representation formulas for convex weak star compact valued multifunctions defined on Köthe space and provide several existence results of conditional expectation for convex weak star compact valued Gelfand-integrable multifunctions. In Section 7 new versions of Levy's theorem for convex weak star compact valued Gelfand-integrable multifunctions with application to the w^*K-convergence for convex weak star compact valued Gelfand-integrable martingales are provided. We also present in Section 8 a new version of Mosco convergence result for unbounded closed convex valued integrable supermartingales in a separable Banach space having the Radon-Nikodym property. In Section 9 we present an application to the law of large numbers in the space of Gelfand-integrable and mean bounded functions.

2. Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_n)_{n\in\mathbf{N}}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\bigcup_{n\geq 1}^{\infty} \mathcal{F}_n$. Let E be a separable Banach space, E^* the topological dual of E, $\overline{B}_E$ (resp. $\overline{B}_{E^*}$) the closed unit ball of E (resp. E^*), $D = (x_p)_{p\in\mathbf{N}}$ a dense sequence in $\overline{B}_E$. We denote by E^*_s (resp. E^*_b) the topological dual E^* endowed with the topology $\sigma(E^*, E)$ of pointwise convergence, alias w^* topology (resp. the topology associated with the dual norm $\|.\|_{E^*_b}$); and by $E^*_{m^*}$ the topological dual E^* endowed with the topology $m^* = \sigma(E^*, H)$, where H is the linear space of E generated by D, that is the Hausdorff locally convex topology defined by the sequence of semi-norms

$$P_k(x^*) = \max\{|\langle x^*, x_p\rangle| : p \leq k\}, \quad x^* \in E^*, k \in \mathbf{N}.$$

Recall that the topology m^* is metrizable, for instance, by the metric

$$d_{E^*_{m^*}}(x_1^*, x_2^*) := \sum_{p=1}^{p=\infty} \frac{1}{2^p} |\langle x_p, x_1^*\rangle - \langle x_p, x_2^*\rangle|, \quad x_1^*, x_2^* \in E^*.$$

We assume from now that $d_{E^*_{m^*}}$ is held fixed. Further, we have $m^* \subset w^* \subset s^*$. When E is infinite-dimensional these inclusions are strict. On the other hand, the restrictions of m^* and w^* to any bounded subset of E^* coincide and the Borel tribes $\mathcal{B}(E^*_s)$ and $\mathcal{B}(E^*_{m^*})$ associated with E^*_s and $E^*_{m^*}$ are equal, but the consideration of the Borel tribe $\mathcal{B}(E^*_b)$ associated with the topology of E^*_b is irrelevant here. Noting that E^* is the countable union of closed balls, we deduce that the space E^*_s is Suslin, as well as the metrizable topological space $E^*_{m^*}$. A $2^{E^*_s}$-valued multifunction (alias mapping for short) $X : \Omega \Rightarrow E^*_s$ is $\mathcal{F}$-measurable if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Given a $\mathcal{F}$-measurable mapping $X : \Omega \Rightarrow E^*_s$ and a Borel set $G \in \mathcal{B}(E^*_s)$, the set

$$X^-G = \{\omega \in \Omega : X(\omega) \cap G \neq \emptyset\}$$

is $\mathcal{F}$-measurable, that is $X^-G \in \mathcal{F}$. In view of the completeness hypothesis on the probability space, this is a consequence of the Projection Theorem (see e.g. Theorem III.23 of [13]) and of the equality

$$X^-G = \mathrm{proj}_\Omega \{Gr(X) \cap (\Omega \times G)\}.$$

In particular, if $X : \Omega \Rightarrow E^*_s$ is $\mathcal{F}$-measurable, the *domain* of X, defined by

$$\mathrm{dom}\, X = \{\omega \in \Omega : X(\omega) \neq \emptyset\}$$

is $\mathcal{F}$-measurable, because $\mathrm{dom}\, X = X^-E^*_s$. Further if $u : \Omega \to E^*_s$ is a scalarly $\mathcal{F}$-measurable mapping, that is, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable, then the function $f : (\omega, x^*) \mapsto \|x^* - u(\omega)\|_{E^*_b}$ is $\mathcal{F} \otimes \mathcal{B}(E^*_s)$-measurable, and for every fixed $\omega \in \Omega$, $f(\omega, .)$ is lower semicontinuous on E^*_s, for short, f is a normal integrand, indeed, we have

$$\|x^* - u(\omega)\|_{E^*_b} = \sup_{j\in\mathbf{N}} \langle e_j, x^* - u(\omega)\rangle$$

here $D_1 = (e_j)_{j\geq 1}$ is a dense sequence in the closed unit ball of E. As each function $(\omega, x^*) \mapsto \langle e_j, x^* - u(\omega)\rangle$ is $\mathcal{F} \otimes \mathcal{B}(E^*_s)$-measurable and continuous on E^*_s for each $\omega \in \Omega$, it follows that f is a normal integrand. Consequently, the graph of u belongs to $\mathcal{F} \otimes \mathcal{B}(E^*_s)$. Let us mention that the function distance $d_{E^*_b}(x^*, y^*) = \|x^* - y^*\|_{E^*_b}$ is lower semicontinuous on $E^*_s \times E^*_s$, being the supremum of w^*-continuous functions. If X is a $\mathcal{F}$-measurable mapping, the distance function $\omega \mapsto d_{E^*_b}(x^*, X(\omega))$ is $\mathcal{F}$-measurable, by using the lower semicontinuity of the function $d_{E^*_b}(x^*, .)$ on E^*_s and measurable projection theorem ([13], Theorem III.23) and recalling that E^*_s is a Suslin space. A mapping $u : \Omega \Rightarrow E^*_s$ is said to be scalarly integrable, alias Gelfand integrable, if, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable and integrable. We denoted by $G^1_{E^*}[E](\Omega, \mathcal{F}, P)$ ($G^1_{E^*}[E](\mathcal{F})$ for short) the space of all $\mathcal{F}$-measurable and scalarly integrable mappings $u : \Omega \Rightarrow E^*_s$. Here $L^1_{E^*}[E](\mathcal{F})$ is the subspace of $G^1_{E^*}[E](\mathcal{F})$ of all $\mathcal{F}$-measurable mappings u such that the function $|u| : \omega \mapsto \|u(\omega)\|_{E^*_b}$ is integrable. The measurability of $|u|$ follows easily from the above considerations. For any $2^{E^*_s}$-valued mapping $X : \Omega \Rightarrow E^*_s$, we denote by $\mathcal{S}^1_{Ge}(X)(\mathcal{F})$ (resp. $\mathcal{S}^1_X(\mathcal{F})$) the set of all $G^1_{E^*}[E](\mathcal{F})$-selections (resp. $L^1_{E^*}[E](\mathcal{F})$-selections) of X. We denote by Gelfand-$\int_A f\, dP$ (or, for short, G-$\int_A f\, dP$) the Gelfand integral of a Gelfand integrable mapping $f : \Omega \to E^*$ over a set $A \in \mathcal{F}$. The Aumann-

Gelfand integral, *AG-integral* for short, of X over a set $A \in \mathcal{F}$ is defined by

$$AG\text{-}\int_A X\,dP := \Big\{G\text{-}\int_A f\,dP : f \in \mathcal{S}^1_{Ge}(X)(\mathcal{F})\Big\}.$$

Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of $\mathcal{F}$-measurable w^*-closed convex mappings, the sequential weak* upper limit $w^*\text{-}ls\,X_n$ of $(X_n)_{n\in\mathbf{N}}$ is defined by

$$w^*\text{-}ls\,X_n = \{x^* \in E^* : x^* = \sigma(E^*,E)\text{-}\lim_{j\to\infty} x^*_j;\ x^*_j \in X_{n_j}\}.$$

Similarly the sequential weak* lower limit $w^*\text{-}li\,X_n$ of $(X_n)_{n\in\mathbf{N}}$ is defined by

$$w^*\text{-}li\,X_n = \{x^* \in E^* : x^* = \sigma(E^*,E)\text{-}\lim_{n\to\infty} x^*_n;\ x^*_n \in X_n\}.$$

The sequence $(X_n)_{n\in\mathbf{N}}$ weak star Kuratowski (w^*K for short) converges to a $\mathcal{F}$-measurable w^*-closed convex valued mapping $X_\infty : \Omega \rightrightarrows E^*_s$ if

$$w^*\text{-}ls\,X_n \subset X_\infty \subset w^*\text{-}li\,X_n$$

almost surely (a.s.). Briefly

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad \text{a.s.}$$

By $cwk(E^*_s)$ we denote the set of all nonempty convex $\sigma(E^*,E)$-compact subsets of E^*_s. A mapping $X : \Omega \to cwk(E^*_s)$ is scalarly $\mathcal{F}$-measurable if the function $\omega \to \delta^*(x, X(\omega))$ is $\mathcal{F}$-measurable for every $x \in E$. Let us recall that any scalarly $\mathcal{F}$-measurable $cwk(E^*_s)$-valued mapping is $\mathcal{F}$-measurable. Indeed, let $(e_k)_{k\in\mathbf{N}}$ be a sequence in E which separates the points of E^*, then we have $x \in X(\omega)$ iff $\langle e_k, x\rangle \le \delta^*(e_k, X(\omega))$ for all $k \in \mathbf{N}$. Further, we denote by $\mathcal{G}^1_{cwk(E^*_s)}(\Omega,\mathcal{F},P)$ (for short, $\mathcal{G}^1_{cwk(E^*_s)}(\mathcal{F})$) the space of all $\mathcal{F}$-measurable and *scalarly integrable* $cwk(E^*_s)$-valued mappings $X : \Omega \to cwk(E^*_s)$, that is, for every $x \in E$, the function $\omega \to \delta^*(x, X(\omega))$ is integrable. By $\mathcal{L}^1_{cwk(E^*_s)}(\Omega,\mathcal{F},P)$ (for short, $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$) we denote the subspace of $\mathcal{G}^1_{cwk(E^*_s)}(\mathcal{F})$ of all *integrably bounded* multifunctions X such that the function $|X| : \omega \to |X(\omega)|$ is integrable, here $|X(\omega)| := \sup_{y^*\in X(\omega)} \|y^*\|_{E^*_b}$, by the above consideration, it is easy to see that $|X|$ is $\mathcal{F}$-measurable. Similarly, $\mathcal{L}^\infty_{cwk(E^*_s)}(\Omega,\mathcal{F},P)$ (for short, $\mathcal{L}^\infty_{cwk(E^*_s)}(\mathcal{F})$) is the space of all $\mathcal{F}$-measurable and scalarly integrable $cwk(E^*_s)$-valued mapping $X : \Omega \to cwk(E^*_s)$, such that $|X|$ belongs to $L^\infty(\Omega,\mathcal{F},P)$. Let $\mathcal{H}^*_{E^*_b}$ be the Hausdorff distance associated with the dual norm $\|.\|_{E^*_b}$ on bounded closed convex subsets in E^*, and X, Y be two convex weak* compact valued measurable mappings, then $\mathcal{H}^*_{E^*_b}(X,Y)$ is measurable because

$$\mathcal{H}_{E^*_b} = \sup_{j\in\mathbf{N}} [\delta^*(e_j, X) - \delta^*(e_j, Y)],$$

where $(e_j)_{j\in\mathbf{N}}$ is a dense sequence in $\overline{B}_E$. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is *bounded* (resp. *uniformly integrable*) if $(|X_n|)_{n\in\mathbf{N}}$ is bounded (resp. uniformly integrable) in $\mathcal{L}^1_{\mathbf{R}}(\Omega,\mathcal{F},P)$. We refer to [19] for the weak star convergence of closed bounded convex sets in a dual space.

3. Martingales in $L^1_{E^*}[E](\mathcal{F})$ and $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. For the convenience of the reader we recall and summarize the existence and uniqueness of the conditional expectation in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. See [26, Theorem 3].

THEOREM 3.1. *Given* $\Gamma \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ *and a sub-*σ*-algebra* $\mathcal{B}$ *of* $\mathcal{F}$*, there exists a unique (for equality a.s.) mapping* $\Sigma := E^{\mathcal{B}}\Gamma \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{B})$*, that is the conditional expectation of* Γ *with respect to* $\mathcal{B}$*, which enjoys the following properties*:

a) $\int_\Omega \delta^*(v, \Sigma)\, dP = \int_\Omega \delta^*(v, \Gamma)\, dP$ *for all* $v \in L^\infty_E(\mathcal{B})$.
b) $\Sigma \subset E^{\mathcal{B}}|\Gamma|\overline{B}_{E^*}$ *a.s.*
c) $\mathcal{S}^1_\Sigma(\mathcal{B})$ *is sequentially* $\sigma(L^1_{E^*}[E](\mathcal{B}), L^\infty_E(\mathcal{B}))$ *compact* (*here* $\mathcal{S}^1_\Sigma(\mathcal{B})$ *denotes the set of all* $L^1_{E^*}[E](\mathcal{B})$ *selections of* Σ) *and satisfies the inclusion*
$$E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F}) \subset \mathcal{S}^1_\Sigma(\mathcal{B}).$$
d) *Furthermore one has*
$$\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) = \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B}))$$
for all $v \in L^\infty_E(\mathcal{B})$.
e) $E^{\mathcal{B}}$ *is increasing*: $\Gamma_1 \subset \Gamma_2$ *a.s. implies* $E^{\mathcal{B}}\Gamma_1 \subset E^{\mathcal{B}}\Gamma_2$ *a.s.*

For more information for the conditional expectation of multifunctions, we refer to [12, 22, 26].

DEFINITION 3.2. An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is

— a *martingale* if $X_n = E^{\mathcal{F}_n}X_{n+1}$ for all $n \in \mathbf{N}$,
— a *submartingale* if $X_n \subset E^{\mathcal{F}_n}X_{n+1}$ for all $n \in \mathbf{N}$,
— a *supermartingale* if $E^{\mathcal{F}_n}X_{n+1} \subset X_n$ for all $n \in \mathbf{N}$.

Here $E^{\mathcal{F}_n}X_{n+1} \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F}_n)$ denotes the convex weakly compact valued conditional expectation of X_{n+1} as defined in Theorem 3.1.

We begin with a simple convergence result for martingales in $L^1_{E^*}[E](\mathcal{F})$ which is a starting point of our study.

PROPOSITION 3.3. *Let* $(X_n)_{n\in\mathbf{N}}$ *be a bounded martingale in* $L^1_{E^*}[E](\mathcal{F})$*. Then there is* $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ *which enjoys the following properties*

(a) $(X_n)_{n\in\mathbf{N}}$ *weak* converges a.s. to* X_∞.
(b) $\lim_{n\to\infty} \|X_n - y^*\|_{E^*_b} = \|X_\infty - y^*\|_{E^*_b}$ *a.s., for each* $y^* \in E^*$*, here the negligible set depends on* $y^* \in E^*$.

Proof. It is clear that $(|X_n|)_{n\in\mathbf{N}}$ is a bounded submartingale in $L^1_{\mathbf{R}}(\mathcal{F})$. Hence $(|X_n|)_{n\in\mathbf{N}}$ converges a.s. So $\sup_{n\in\mathbf{N}} |X_n|(\omega) < \infty$ a.s. Now, since, for each $x \in E$, $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ is a bounded martingale in $L^1_{\mathbf{R}}(\mathcal{F})$, it converges a.s. to a function $m_x \in L^1_{\mathbf{R}}(\mathcal{F})$. Using [12, Theorem 6.1(4)] provides an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ with $\lim_{p\to\infty} P(A_p) = 1$, a function $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ and a subsequence $(X'_n)_{n\in\mathbf{N}}$ such that
$$\lim_{n\to\infty} \int_{A_p} \langle h, X'_n\rangle\, dP = \int_{A_p} \langle h, X_\infty\rangle\, dP$$
for all $h \in L^\infty_E(\mathcal{F})$. So by identifying the limit, we get $m_x = \langle x, X_\infty\rangle$ a.s., so (a) follows using the separability of E and the pointwise boundedness of $(X_n)_{n\in\mathbf{N}}$. It remains to prove (b). We have
$$\|X_n - y^*\|_{E^*_b} = \sup_{j\in\mathbf{N}} |\langle e_j, X_n - y^*\rangle|$$

here $D_1 = (e_j)_{j\in\mathbf{N}}$ denotes a dense sequence in the closed unit ball of E. As, for each $j \in \mathbf{N}$, $(|\langle e_j, X_n - y^*\rangle|)_{n\in\mathbf{N}}$ are real-valued submartingales which converge a.s. to $|\langle e_j, X_\infty - y^*\rangle|$, and $(|X_n|)_{n\in\mathbf{N}}$ is $L^1_{\mathbf{R}}(\mathcal{F})$-bounded we can apply Lemma V.2.9 in [24]. So we have

$$\begin{aligned}\lim_{n\to\infty} \|X_n - y^*\|_{E_b^*} = \lim_{n\to\infty} \sup_{j\in\mathbf{N}} |\langle e_j, X_n - y^*\rangle| &= \sup_{j\in\mathbf{N}} \lim_{n\to\infty} |\langle e_j, X_n - y^*\rangle| \\ &= \sup_{j\in\mathbf{N}} |\langle e_j, X_\infty - y^*\rangle| = \|X_\infty - y^*\|_{E_b^*}\end{aligned}$$

almost surely. ∎

Now we proceed to a multivalued version of the preceding result dealing with submartingales in $\mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$. For this purpose we begin with the regular martingale $E^{\mathcal{F}_n}X$ where $X \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$.

THEOREM 3.4. *Let* $X \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$. *Then the following hold*:

(a) $\lim_{n\to\infty} \delta^*(x, E^{\mathcal{F}_n}X) = \delta^*(x, X)$ *a.s.* $\forall x \in \overline{B}_E$.
(b) $w^*\text{-}ls\, E^{\mathcal{F}_n}X \subset X \subset w^*\text{-}li\, E^{\mathcal{F}_n}X$ *a.s.*

Proof. (a) On account of Theorem 3.1, $E^{\mathcal{F}_n}X \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$ with

$$E^{\mathcal{F}_n}X \subset E^{\mathcal{F}_n}|X|\overline{B}_{E^*}. \tag{3.4.1}$$

Further by applying Levy's theorem to $(E^{\mathcal{F}_n}\delta^*(x, X))_{n\in\mathbf{N}}$ we have

$$\lim_{n\to\infty} E^{\mathcal{F}_n}\delta^*(x, X) = \delta^*(x, X)$$

a.s. for each $x \in E$. Since $E^{\mathcal{F}_n}\delta^*(x, X) = \delta^*(x, E^{\mathcal{F}_n}X)$ and E is separable, using (3.4.1), it is not difficult to check that

$$\lim_{n\to\infty} \delta^*(x, E^{\mathcal{F}_n}X) = \delta^*(x, X) \quad \text{a.s.} \quad \forall x \in \overline{B}_E. \tag{3.4.2}$$

(b) Let us check the inclusion

$$X(\omega) \subset w^*\text{-}li\, E^{\mathcal{F}_n}X(\omega) \quad \text{a.s.}$$

Let $f \in \mathcal{S}^1_X$. By applying (a) to the single-valued mapping $f \in L^1_{E^*}[E](\mathcal{F})$ (see 3.4.2) we deduce that $\lim_{n\to\infty} E^{\mathcal{F}_n}f(\omega) = f(\omega)$ a.s. with respect to the $\sigma(E^*, E)$ topology. Since $E^{\mathcal{F}_n}f(\omega) \in E^{\mathcal{F}_n}X(\omega)$, it follows that $f(\omega) \in w^*\text{-}li\, E^{\mathcal{F}_n}X(\omega)$ a.s. Taking a Castaing representation of X (see [13, Theorem III.37]) we deduce that $X(\omega) \subset w^*\text{-}li\, E^{\mathcal{F}_n}X(\omega)$ a.s. The inclusion $w^*\text{-}ls\, E^{\mathcal{F}_n}X \subset X$ a.s. follows again from (a). Indeed, let $\omega \in \Omega$ be fixed but arbitrary for which the equality

$$\lim_{n\to\infty} \delta^*(x, E^{\mathcal{F}_n}X(\omega)) = \delta^*(x, X(\omega))$$

holds for all $x \in E$, and let $x^* \in w\text{-}ls\, E^{\mathcal{F}_n}X(\omega)$. There is a sequence $(x_k^*)_{k\in\mathbf{N}}$ in E^* with $x_k^* \in E^{\mathcal{F}_{n_k}}X(\omega)$ such that $(x_k^*)_{k\in\mathbf{N}}$ weakly* converges to x^*. Then, for each $x \in E$, we have

$$\langle x, x^*\rangle = \lim_{k\to\infty} \langle x, x_k^*\rangle \le \lim_{k\to\infty} \delta^*(x, E^{\mathcal{F}_{n_k}}X(\omega)) = \delta^*(x, X(\omega)).$$

According to Proposition III.35 in [13], we deduce that $x^* \in X(\omega)$. ∎

Now we proceed to the convergence of martingales in $\mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$.

THEOREM 3.5. *Let $(X_n)_{n\in\mathbf{N}}$ be a bounded martingale in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Then there is $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ which enjoys the following properties*

(a) $\lim_{n\to\infty} \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n}X_\infty) = 0$ *a.s.*

Consequently

(b) $\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty)$ *a.s.* $\forall x \in \overline{B}_E$.

Proof. (a) Let $D_1 = (e_j)_{j\in\mathbf{N}}$ denote a dense sequence in $\overline{B}_E$. As $(X_n)_{n\in\mathbf{N}}$ is a bounded martingale in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, for each $j \in \mathbf{N}$, $(\delta^*(e_j, X_n))_{n\in\mathbf{N}}$ is a bounded real-valued martingale in $L^1_{\mathbf{R}}(\mathcal{F})$. So for each $j \in \mathbf{N}$, $(\delta^*(e_j, X_n))_{n\in\mathbf{N}}$ converges a.s. to an integrable function m_j in $L^1_{\mathbf{R}}(\mathcal{F})$. Applying [12, Theorem 6.1(4)] gives $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(e_j, X_n) = m_j = \delta^*(e_j, X_\infty) \quad \text{a.s.}$$

and then in view of Theorem 3.4

$$\lim_{n\to\infty} [\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n}X_\infty)] = 0 \quad \text{a.s.}$$

Furthermore, for each $j \in \mathbf{N}$, $(\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n}X_\infty))_{n\in\mathbf{N}}$ are real-valued L^1-bounded martingales thus, invoking Lemma V.2.9 in [24], we see that

$$\begin{aligned}\mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n}X_\infty) &= \sup_{x\in\overline{B}_E} \left|\delta^*(x, X_n) - \delta^*(x, E^{\mathcal{F}_n}X_\infty)\right| \\ &= \sup_{j\geq 1}\left|\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n}X_\infty)\right| \to \sup_{j\geq 1}\lim_{n\to\infty}\left|\delta^*(e_j, X_n) - \delta^*(e_j, E^{\mathcal{F}_n}X_\infty)\right| = 0\end{aligned}$$

almost surely, which proves (a). Hence we deduce that

$$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty), \quad \text{a.s.} \quad \forall x \in \overline{B}_E.$$

Indeed, for each $x \in \overline{B}_E$ let us write

$$\begin{aligned}[\delta^*(x, X_n) - \delta^*(x, X_\infty)] = [\delta^*(x, X_n) - \delta^*(x, E^{\mathcal{F}_n}X_\infty)]& \\ + [\delta^*(x, E^{\mathcal{F}_n}X_\infty) - \delta^*(x, X_\infty)]&.\end{aligned}$$

From (a), it is obvious that the first term $[\delta^*(x, X_n) - \delta^*(x, E^{\mathcal{F}_n}X_\infty]$ goes to 0 a.s. for all $x \in \overline{B}_E$ when n goes to ∞ and so is the second term

$$[\delta^*(x, E^{\mathcal{F}_n}X_\infty) - \delta^*(x, X_\infty)].$$

Indeed, since

$$[\delta^*(e_j, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_j, X_\infty)] \to 0 \quad \text{a.s.} \quad \forall j \in \mathbf{N}$$

when n goes to ∞ and

$$\sup_{n\in\mathbf{N}} |E^{\mathcal{F}_n}X_\infty(\omega)| < \infty \quad \text{a.s.} \quad \omega \in \Omega$$

it is straightforward (using a density argument) to check that

$$[\delta^*(x, E^{\mathcal{F}_n}X_\infty) - \delta^*(x, X_\infty)] \to 0 \quad \text{a.s.} \quad \forall x \in \overline{B}_E$$

when n goes to ∞. ∎

COROLLARY 3.6. *Let $(X_n)_{n\in\mathbf{N}}$ be a bounded martingale in $L^1_{E^*}[E](\mathcal{F})$. Then there exist $X_\infty \in L^1_{E^*}[E](\mathcal{F})$, a regular martingale $(Y_n)_{n\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ and a martingale $(Z_n)_{n\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ such that $X_n = Y_n + Z_n$ for all $n \in \mathbf{N}$ and such that $(Y_n)_{n\in\mathbf{N}}$ weak* converges a.s. to X_∞ and $(Z_n)_{n\in\mathbf{N}}$ norm converges to 0 a.s.*

Proof. By Proposition 3.3, there exists $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ such that $(X_n)_{n\in\mathbf{N}}$ weak* converges a.s. to X_∞. As $X_n = E^{\mathcal{F}_n}X_\infty + [X_n - E^{\mathcal{F}_n}X_\infty]$, the result follows from Theorems 3.4 and 3.5 by putting $Y_n = E^{\mathcal{F}_n}X_\infty$ and $Z_n = X_n - E^{\mathcal{F}_n}X_\infty$. ∎

Now we proceed to the w^*K convergence of bounded martingales in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. For this purpose we need the following result that is dual version of a similar result ([1], Proposition 3.1) in the primal space E.

LEMMA 3.7. *Let E be a separable Banach space. Let $(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ be two uniformly bounded sequences in $cwk(E^*_s)$ and $B_\infty \in cwk(E^*_s)$ satisfying:*

(i) $w^*K\text{-}\lim\limits_{n\to\infty} B_n := B_\infty$.

(ii) $\lim_{n\to\infty} \mathcal{H}^*_{E^*_b}(A_n, B_n) = 0$, *where $\mathcal{H}_{E^*_b}$ denotes the Hausdorff distance associated with the norm $\|.\|_{E^*_b}$ on the closed bounded convex subsets of E^*_b.*

Then

$$w^*K\text{-}\lim_{n\to\infty} A_n = B_\infty.$$

Proof. We may assume that the sets A_n, B_n and B_∞ are included in a convex $\sigma(E^*, E)$-compact subset L. Then on L the $\sigma(E^*, E)$ topology coincides with the metric topology given by $d_{E^*_{m^*}}$, and so for any sequence $(C_n)_{n\in\mathbf{N}}$ of convex $\sigma(E^*, E)$-compact subsets included in L, and for any $x^* \in L$, we have that $x^* \in w^*\text{-}li\, C_n$ iff $\lim_{n\to\infty} d_{E^*_{m^*}}(x^*, C_n) = 0$. Given $x^* \in w^*\text{-}li\, B_n$, there exists $x^*_n \in B_n$ such that $x^* = w^*\text{-}\lim\limits_{n\to\infty} x^*_n$. We have the estimate

$$d_{E^*_{m^*}}(x^*, A_n) \le d_{E^*_{m^*}}(x^*, x^*_n) + d_{E^*_{m^*}}(x^*_n, A_n) \le d_{E^*_{m^*}}(x^*, x^*_n) + \mathcal{H}^*_{E^*_b}(A_n, B_n).$$

Indeed, for every $z^* \in A_n$, we have

$$d_{E^*_{m^*}}(x^*_n, z^*) = \sum_{p=1}^{\infty} \frac{1}{2^p}\langle x^*_n - z^*, x_p\rangle \le \sum_{p=1}^{\infty} \frac{1}{2^p}\|x^*_n - z^*\|_{E^*_b} = \|x^*_n - z^*\|_{E^*_b}$$

so that

$$d_{E^*_{m^*}}(x^*_n, A_n) = \inf_{z^*\in A_n} d_{E^*_{m^*}}(x^*_n, z^*) \le \inf_{z^*\in A_n} \|x^*_n - z^*\|_{E^*_b}$$
$$= d_{E^*_b}(x^*_n, A_n) \le \mathcal{H}^*_{E^*_b}(A_n, B_n).$$

By (ii) we deduce that $\lim_{n\to\infty} d_{E^*_{m^*}}(x^*, A_n) = 0$. Hence

$$(*) \qquad w^*\text{-}li\, B_n \subset w^*\text{-}li\, A_n.$$

Now let $x^* \in w^*\text{-}ls\, A_n$. There exist a sequence $(x^*_k)_{k\in\mathbf{N}}$ weak* converging to x^* with $x^*_k \in A_{n_k}$ for all $k \in \mathbf{N}$. Let us pick $y^*_k \in B_{n_k}$ such that

$$d_{E^*_{m^*}}(x^*_k, y^*_k) = d_{E^*_{m^*}}(x^*_k, B_{n_k}) + \frac{1}{k} \le \mathcal{H}^*_{E^*_b}(A_{n_k}, B_{n_k}) + \frac{1}{k}$$

because $d_{E^*_{m^*}}(x^*_k, B_{n_k}) \le \mathcal{H}^*_{E^*_b}(A_{n_k}, B_{n_k})$ similarly. As $y^*_k = x^*_k + (y^*_k - x^*_k)$, $(y^*_k)_{k\in\mathbf{N}}$ weak* converges to x^*, therefore $x^* \in w^*\text{-}ls\, B_n$. Hence

$$(**) \qquad w^*\text{-}ls\, A_n \subset w^*\text{-}ls\, B_n.$$

Consequently, combining (i), (*), (**) we get

$$w^*K\text{-}\lim_{n\to\infty} A_n = B_\infty. \ \blacksquare$$

Now we present the weak star Kuratowski convergence for bounded martingales in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$.

THEOREM 3.8. *Let $(X_n)_{n\in\mathbf{N}}$ be a bounded martingale in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Then there is $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ which enjoys the following properties*

$$\text{(a)} \qquad \lim_{n\to\infty} |X_n| = |X_\infty| \quad a.s.$$

$$\text{(b)} \qquad \lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad a.s. \quad \forall x \in \overline{B}_E.$$

$$\text{(c)} \qquad w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Proof. Arguing as in the proof of Theorem 3.5 via Neveu lemma [24, Lemme 5.2.9], we conclude that

$$\text{(3.8.1)} \qquad \lim_{n\to\infty} |X_n| = |X_\infty| \quad \text{a.s.}$$

$$\text{(3.8.2)} \qquad \lim_{n\to\infty} \mathcal{H}_{E^*_b}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad \text{a.s.}$$

So (b) follows easily by repeating the arguments in the proof of Theorem 3.5. In view of Theorem 3.4 one has

$$\text{(3.8.3)} \qquad w^*K\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_\infty = X_\infty \quad \text{a.s.}$$

From (3.8.1), (3.8.2), (3.8.3) and Lemma 3.7 we conclude that $(X_n)_{n\in\mathbf{N}}$ w^*K-converges to X_∞ a.s. $\blacksquare$

REMARKS. The above results are not comparable with those given in [18, Proposition 1] dealing with norm convergence a.s. of strongly measurable vector-valued martingales taking values in a strongly separable subspace of a dual space. In the next section, we will present similar results for multivalued mils in E^*.

4. Multivalued mils in a dual space.

Before going further, let us introduce the definition of mils in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$.

DEFINITION 4.1. An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ is a *mil* if for every $\varepsilon > 0$, there exists p such that for $n \ge p$, we have

$$P\big(\sup_{n\ge q\ge p} \mathcal{H}^*_{E^*_b}(X_q, E^{\mathcal{F}_q} X_n) > \varepsilon\big) < \varepsilon$$

where $\mathcal{H}^*_{E^*_b}$ stands for the Hausdorff distance associated with the dual norm $\|.\|_{E^*_b}$ on $cwk(E^*_s)$.

It is obvious that if $(X_n)_{n\in\mathbf{N}}$ is a mil in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, then for every x in the unit ball $\overline{B}_E$ of E, the sequence $(\delta^*(x, X_n))_{n\in\mathbf{N}}$ is a mil in $\mathcal{L}^1_{\mathbf{R}}(\mathcal{F})$, since we have

$$\mathcal{H}^*_{E^*_b}(X_q, E^{\mathcal{F}_q}X_n) = \sup_{x\in\overline{B}_E} \big[\delta^*(x, X_q) - E^{\mathcal{F}_q}\delta^*(x, X_n)\big]$$

Further, if $(X_n)_{n\in\mathbf{N}}$ is single-valued, Definition 4.1 is reduced to

Definition 4.2. An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ is a *mil* if for every $\varepsilon > 0$, there exists p such that for $n \geq p$, we have

$$P\big(\sup_{n\geq q\geq p} \|X_q - E^{\mathcal{F}_q}X_n\|_{E^*_b} > \varepsilon\big) < \varepsilon.$$

Similarly if $(X_n)_{n\in\mathbf{N}}$ is a mil in $L^1_{E^*}[E](\mathcal{F})$, then for every x in the unit ball $\overline{B}_E$ of E, the sequence $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ is a mil in $L^1_{\mathbf{R}}(\mathcal{F})$, since we have

$$\|X_q - E^{\mathcal{F}_q}X_n\|_{E^*_b} = \sup_{x\in\overline{B}_E} [\langle x, X_q - E^{\mathcal{F}_q}X_n\rangle].$$

Now we are ready to state the convergence of $cwk(E^*_s)$-valued mils.

Theorem 4.3. *Let $(X_n)_{n\in\mathbf{N}}$ be a bounded mil in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ and $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that, for each $x \in \overline{B}_E$,*

$$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad a.s.$$

Then

(i) $$\lim_{n\to\infty} \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n}X_\infty) = 0 \quad a.s.$$

Consequently, we have

(ii) $$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad a.s. \quad \forall x \in \overline{B}_E.$$

Assume further that $\sup_{n\in\mathbf{N}} |X_n(\omega)| < \infty$ *a.s., then one has*

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Proof. We will proceed in three steps.

Step 1. Claim. $\lim_{n\to\infty} \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n}X_\infty) = 0$ a.s. If $(X_n)_{n\in\mathbf{N}}$ is a bounded mil in $L^1_{E^*}[E](\mathcal{F})$, then $\|X_n - E^{\mathcal{F}_n}X_\infty\|_{E^*_b}$ goes to 0 a.s. by repeating the techniques of Talagrand developed in [25, Theorem 6, p. 1193] because for each $x \in \overline{B}_E$, the real-valued L^1-bounded mil $(\langle x, X_n - E^{\mathcal{F}_n}X_\infty\rangle)_{n\in\mathbf{N}}$ converges to 0 a.s. In the multivalued case, the claim (i) is true by using Definition 4.1 and a careful adaptation of the mentioned techniques of Talagrand, namely $\lim_{n\to\infty} \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n}X_\infty) = 0$ a.s. (see [4, 5] for details).

Step 2. $\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty)$ a.s. $\forall x \in \overline{B}_E$.

Let $(e_j)_{j\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_E$ with respect to the topology of norm. For each $x \in \overline{B}_E$ let us write

$$[\delta^*(x, X_n) - \delta^*(x, X_\infty)] = [\delta^*(x, X_n) - \delta^*(x, E^{\mathcal{F}_n}X_\infty)] + [\delta^*(x, E^{\mathcal{F}_n}X_\infty) - \delta^*(x, X_\infty)].$$

From (i), it is obvious that the first term $[\delta^*(x, X_n) - \delta^*(x, E^{\mathcal{F}_n} X_\infty)]$ goes to 0 a.s. for all $x \in \overline{B}_E$ when n goes to ∞ and so is the second term

$$[\delta^*(x, E^{\mathcal{F}_n} X_\infty) - \delta^*(x, X_\infty)].$$

Indeed, by Levy's theorem it is obvious that for all $j \in \mathbf{N}$

$$[\delta^*(e_j, E^{\mathcal{F}_n} X_\infty) - \delta^*(e_j, X_\infty)] \to 0 \quad \text{a.s.}$$

when n goes to ∞. Since

$$\sup_{n \in \mathbf{N}} E^{\mathcal{F}_n} |X_\infty|(\omega) < \infty \quad \text{a.s.} \quad \omega \in \Omega$$

by using a density argument, it is easy to check that

$$[\delta^*(x, E^{\mathcal{F}_n} X_\infty) - \delta^*(x, X_\infty)] \to 0 \quad \text{a.s.} \quad \forall x \in \overline{B}_E$$

when n goes to ∞.

Step 3. From the pointwise boundedness condition $\sup_{n \in \mathbf{N}} |X_n(\omega)| < \infty$ a.s., the assertion

$$w^* K\text{-}\lim_{n \to \infty} X_n = X_\infty \quad a.s.$$

follows by applying (i)–(ii) and Lemma 3.7. ∎

Here are some corollaries.

Corollary 4.4. *Let $(X_n)_{n \in \mathbf{N}}$ be a bounded mil in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that $\sup_{n \in \mathbf{N}} |X_n(\omega)| < \infty$ a.s. Then there exists $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that*

$$\lim_{n \to \infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad a.s. \quad \forall x \in \overline{B}_E$$

$$\lim_{n \to \infty} \mathcal{H}^*_{E^*_b}(X_n, E^{\mathcal{F}_n} X_\infty) = 0 \quad a.s.$$

Consequently, we have

$$w^* K\text{-}\lim_{n \to \infty} X_n = X_\infty \quad a.s.$$

Proof. Let $D_1 = (e_j)_{j \in \mathbf{N}}$ denote a dense sequence in $\overline{B}_E$. As $(X_n)_{n \in \mathbf{N}}$ is a bounded mil in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, for each $j \in \mathbf{N}$, $(\delta^*(e_j, X_n))_{n \in \mathbf{N}}$ is a bounded mil in $L^1_{\mathbf{R}}(\mathcal{F})$. So for each $j \in \mathbf{N}$, $(\delta^*(e_j, X_n))_{n \in \mathbf{N}}$ converges a.s. to an integrable function m_j in $L^1_{\mathbf{R}}(\mathcal{F})$. Applying [12, Theorem 6.1(4)] gives $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that

$$\lim_{n \to \infty} \delta^*(e_j, X_n) = m_j = \delta^*(e_j, X_\infty) \quad \text{a.s.}$$

Since $(X_n)_{n \in \mathbf{N}}$ is pointwise bounded, we deduce that

$$\lim_{n \to \infty} \delta^*(x, X_n) = \delta^*(x, X_\infty) \quad \text{a.s.} \quad \forall x \in \overline{B}_E$$

by a density argument. Then the conclusion follows from Theorem 4.3. ∎

Corollary 4.5. *Let $(X_n)_{n \in \mathbf{N}}$ be a bounded mil in $L^1_{E^*}[E](\mathcal{F})$. Then there exist $X_\infty \in L^1_{E^*}[E](\mathcal{F})$, a regular martingale $(Y_n)_{n \in \mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ and a mil $(Z_n)_{n \in \mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$ such that $X_n = Y_n + Z_n$ for all $n \in \mathbf{N}$ and such that $(Y_n)_{n \in \mathbf{N}}$ weak* converges a.s. to X_∞ and $(Z_n)_{n \in \mathbf{N}}$ norm converges to 0 a.s.*

Proof. As $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ is a real-valued bounded mil in $L^1_{\mathbf{R}}$ for each $x \in \overline{B}_E$, $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ converges a.s. to an integrable function m_x. Using [10, Proposition 6.5.11(4)] provides an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ with $\lim_{p\to\infty} P(A_p) = 1$, a function $X_\infty \in L^1_{E^*}[E](\mathcal{F})$ and a subsequence (X'_n) such that for each $p \in \mathbf{N}$

$$\lim_{n\to\infty} \int_{A_p} \langle h, X'_n\rangle\, dP = \int_{A_p} \langle h, X_\infty\rangle\, dP$$

for all $h \in L^\infty_E(\mathcal{F})$. So by identifying the limit, we get $m_x = \langle x, X_\infty\rangle$ a.s. As $X_n = E^{\mathcal{F}_n} X_\infty + [X_n - E^{\mathcal{F}_n} X_\infty]$, the result follows from Theorem 3.4 and Theorem 4.3(i) by putting $Y_n = E^{\mathcal{F}_n} X_\infty$ and $Z_n = X_n - E^{\mathcal{F}_n} X_\infty$. ■

5. w^*K convergence of integrable supermartingale with unbounded weak*-closed convex values. Let us recall that, given a w^*-closed convex $\mathcal{F}$-measurable mapping Γ such that $S^1_\Gamma(\mathcal{F})$ is nonempty (Γ is integrable, for short), and a sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$, there is a unique w^*-closed convex $\mathcal{B}$-measurable and integrable multifunction, denoted by $E^{\mathcal{B}}\Gamma$ that is the conditional expectation of Γ satisfying, for every $u \in S^1_\Gamma(\mathcal{F})$,

1) $E^{\mathcal{B}}u(\omega) \in E^{\mathcal{B}}\Gamma(\omega)$ a.s.
2) $S^1_{E^{\mathcal{B}}\Gamma} = \sigma(L^1_{E^*}[E], L^\infty_E)\text{-}cl(\{E^{\mathcal{B}}f : f \in S^1_\Gamma\})$.
3) $\int \delta^*(u, \Gamma)\, dP = \int \delta^*(u, E^{\mathcal{B}}\Gamma)\, dP$.
4) $E^{\mathcal{B}}$ is increasing: $\Gamma_1 \subset \Gamma_2$ a.s. implies $E^{\mathcal{B}}\Gamma_1 \subset E^{\mathcal{B}}\Gamma_2$ a.s.

See [26, Theorem 3] or [13, Theorem VIII.34] for more details. Now we proceed to the w^*K convergence for w^*-closed convex integrable supermartingales. We begin with some useful lemmas.

LEMMA 5.1. *Assume that Γ is a w^*-closed convex $\mathcal{F}$-measurable and integrable multifunction and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. Then for any $x^* \in E^*$, we have*

$$d_{E^*_b}(x^*, E^{\mathcal{B}}\Gamma) \le E^{\mathcal{B}} d_{E^*_b}(x^*, \Gamma) \quad a.s.$$

Proof. We will use some arguments from [20, Lemma 4.3]. Nevertheless this needs a careful look. Recall that Γ is integrable iff $d_{E^*_b}(0, \Gamma)$ is integrable (see [8, Lemma 5.6]). Note that the function $\|.\|_{E^*_b}$ is inf-w^*-compact in the sense of convex analysis. By using the measurable choice theorem (see e.g. Theorem III.6 in [13]) there is a $\mathcal{F}$-measurable selection g of Γ such that

$$(*) \qquad d_{E^*_b}(x^*, \Gamma(\omega)) = \inf_{y^*\in\Gamma(\omega)} \|x^* - y^*\|_{E^*_b} = \|x^* - g(\omega)\|_{E^*_b} \quad \text{a.s.}$$

so that $g \in L^1_{E^*}[E](\mathcal{F})$. By Theorem 3.1, one can consider the conditional expectation $E^{\mathcal{B}}g$ which belongs to $L^1_{E^*}[E](\mathcal{B})$ and satisfies $E^{\mathcal{B}}g(\omega) \in E^{\mathcal{B}}\Gamma(\omega)$ a.s. Now taking the conditional expectation in the equality $(*)$ gives

$$(**) \qquad E^{\mathcal{B}}\|x^* - g\|_{E^*_b} = E^{\mathcal{B}} d_{E^*_b}(x^*, \Gamma).$$

By Theorem 3.1 and $(**)$ we deduce that

$$\|x^* - E^{\mathcal{B}}g\|_{E^*_b} = \|E^{\mathcal{B}}(x^* - g)\|_{E^*_b} \le E^{\mathcal{B}}\|x^* - g\|_{E^*_b} = E^{\mathcal{B}} d_{E^*_b}(x^*, \Gamma).$$

As $E^{\mathcal{B}}g \in E^{\mathcal{B}}\Gamma$ a.s., from the preceding estimate it follows that

$$d_{E^*_b}(x^*, E^{\mathcal{B}}\Gamma) \le E^{\mathcal{B}} d_{E^*_b}(x^*, \Gamma) \quad \text{a.s. } ■$$

Recall that the Banach space E is weakly compactly generated (WCG) if there exist a weakly compact subset of E whose linear span is dense in E.

LEMMA 5.2. *Let $(X_n)_{n\in\mathbf{N}}$ be a uniformly integrable supermartingale ($E^{\mathcal{F}_m}X_n \subset X_m$ for $m < n$) in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that $\sup_{n\in\mathbf{N}} |X_n(\omega)| < \infty$ for each $\omega \in \Omega$. Then there is $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that*

(a) *$\lim_{n\to\infty} \delta^*(x, X_n) = \delta^*(x, X_\infty)$ a.s. for all $x \in \overline{B}_E$.*
(b) *$E^{\mathcal{F}_m}X_\infty \subset X_m$ a.s. for all $m \in \mathbf{N}$.*
(c) *Consequently, if E is WCG, then*

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad \text{a.s.}$$

Proof. (a) Let $D_1 = (e_j)_{j\in\mathbf{N}}$ be a dense sequence in the closed unit ball of E. As $(X_n)_{n\in\mathbf{N}}$ is a uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$, for each $j \in \mathbf{N}$, $(\delta^*(e_j, X_n))_{n\in\mathbf{N}}$ is a L^1-bounded real-valued supermartingale. So it converges a.s. for every $j \in \mathbf{N}$ to a function in L^1. Applying [12, Theorem 6.1(4)] to the uniformly integrable $(X_n)_{n\in\mathbf{N}}$ provides a subsequence $(X'_n)_{n\in\mathbf{N}}$ and $X_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \int_A \delta^*(e_j, X'_n)\, dP = \int_A \delta^*(e_j, X_\infty)\, dP \quad \forall j \in \mathbf{N} \quad \forall A \in \mathcal{F}.$$

By identifying the limits we get

$$\lim_{n\to\infty} \delta^*(e_j, X_n) = \delta^*(e_j, X_\infty), \quad \text{a.s.} \quad \forall j \in \mathbf{N}.$$

So taking into account the condition $\sup_{n\in\mathbf{N}} |X_n(\omega)| < \infty$ for each $\omega \in \Omega$, (a) follows by a density argument. Let $m < n$ and $A \in \mathcal{F}_m$, by the supermartingale property

$$\int_A \delta^*(e_j, X_n)\, dP \le \int_A \delta^*(e_j, X_m)\, dP.$$

It follows that

$$\int_A \delta^*(e_j, X_\infty)\, dP \le \int_A \delta^*(e_j, X_m)\, dP.$$

Therefore by taking the conditional expectation of X_∞ (see Theorem 3.1) we get

$$\int_A \delta^*(e_j, E^{\mathcal{F}_m}X_\infty)\, dP = \int_A \delta^*(e_j, X_\infty)\, dP \le \int_A \delta^*(e_j, X_m)\, dP$$

thereby proving (b) (see e.g. Proposition III.35 in [13]).

(c) If E is WCG, on account of (a), the pointwise boundedness of $(X_n)_{n\in\mathbf{N}}$ and Theorem 4.1 in [19], we get

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad \text{a.s.} \ \blacksquare$$

THEOREM 5.3. *Assume that E is WCG. Let $(X_n)_{n\in\mathbf{N}}$ be a w^*-closed convex integrable supermartingale ($E^{\mathcal{F}_m}X_n \subset X_m$ for $m < n$) satisfying: $(X_n)_{n\in\mathbf{N}}$ admits a regular martingale selection $(f_n, \mathcal{F}_n)_{n\ge1}$ in $L^1_{E^*}[E](\mathcal{F})$. Then one can find a w^*-closed convex valued integrable multifunction X_∞ such that*

$$w^*\text{-}lsX_n \subset X_\infty \subset w^*cl[w^*\text{-}liX_n] \quad \text{a.s.}$$
$$E^{\mathcal{F}_n}X_\infty \subset X_n \quad \text{a.s.} \quad \forall n \in \mathbf{N}.$$

Proof. We will proceed in several steps.

Step 1. Here we will use a careful adaptation of a truncation technique developed in [15, Theorem 2.16]. By our assumption there is $f \in L^1_{E^*}[E](\mathcal{F})$ such that $f_n = E^{\mathcal{F}_n} f$ for all $n \in \mathbf{N}$. For each $k \in \mathbf{N}$, let us consider the multifunction

$$X_n^k = X_n \cap [f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_{E^*}].$$

We are going to check that $(X_n^k)_{n\in\mathbf{N}}$ is a uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E^*)}(\mathcal{F})$. Let $m < n$. As $X_n^k \subset X_n$, one has by the monotony of conditional expectation and supermartingale property

$$E^{\mathcal{F}_m} X_n^k \subset E^{\mathcal{F}_m} X_n \subset X_m.$$

Taking the conditional expectation in the inclusion

$$X_n^k \subset [f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_{E^*}]$$

yields

$$E^{\mathcal{F}_m} X_n^k \subset E^{\mathcal{F}_m}\left[f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_{E^*}\right] = f_m + E^{\mathcal{F}_m}(|f| + k)\overline{B}_{E^*}.$$

Hence we get $E^{\mathcal{F}_m} X_n^k \subset X_m^k$. Note that $|X_n^k| \leq h_n^k$ for all $n \in \mathbf{N}$, where $h_n^k := |f_n| + E^{\mathcal{F}_n}(|f| + k)$. Further the positive uniformly integrable submartingale $(h_n^k)_{n\in\mathbf{N}}$ converges a.s. to a positive integrable function h^k. Hence there exists a positive constant r^k depending on $\omega \in \Omega$ such that $h_n^k \leq r^k$ a.s. for all $n \in \mathbf{N}$. So $|X_n^k| \leq r^k$ a.s. for all $n \in \mathbf{N}$. Applying Lemma 5.2 to the uniformly integrable supermartingale $(X_n^k)_{n\in\mathbf{N}}$ provides $X_\infty^k \in \mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$ and a negligible set N_k such that

$$E^{\mathcal{F}_n} X_\infty^k(\omega) \subset X_n^k(\omega) \qquad \forall n \in \mathbf{N} \quad \forall \omega \in \Omega \setminus N_k, \tag{5.3.1}$$

$$w^*K\text{-}\lim_{n\to\infty} X_n^k(\omega) = X_\infty^k(\omega) \qquad \forall \omega \in \Omega \setminus N_k. \tag{5.3.2}$$

Step 2. Convergence and conclusion. By construction, we have $X_n^k \subset X_n^{k+1}$, so that $(X_\infty^k)_{k\in\mathbf{N}}$ is increasing. Let us set $N = \bigcup_{k=1}^\infty N_k$ and

$$X_\infty(\omega) = \begin{cases} w^*cl[\bigcup_{k=1}^\infty X_\infty^k(\omega)] & \text{if} \quad \omega \in \Omega \setminus N \\ 0 & \text{if} \quad \omega \in N \end{cases}$$

We need to check that

$$w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Let $x^* \in w^*\text{-}lsX_n(\omega)$ with $\omega \notin N$. There is a sequence $(x_j^*)_{j\in\mathbf{N}}$ weak* converging to x^* with $x_j^* \in X_{n_j}(\omega)$. Pick a large enough integer $p \in \mathbf{N}$ such that $\|x_j^*\| \leq p$ for all $j \geq 1$. By Jensen's inequality we have

$$\|x_j^* - f_{n_j}(\omega)\| = \|x_j^* - E^{\mathcal{F}_{n_j}}(f)(\omega)\| \leq p + E^{\mathcal{F}_{n_j}}(|f|)(\omega) = E^{\mathcal{F}_{n_j}}(|f| + p)(\omega),$$

in other words

$$x_j^* \in f_{n_j}(\omega) + E^{\mathcal{F}_{n_j}}(|f| + p)(\omega)\overline{B}_{E^*}$$

and so $x_j^* \in X_{n_j}^p(\omega)$, coming back to the definition of X_n^p. Hence using (5.3.2) and the definition of X_∞ we get

$$x^* \in w^*\text{-}lsX_{n_j}^p(\omega) \subset X_\infty^p(\omega) \subset X_\infty(\omega).$$

By Theorem 3 in [26] that we recalled in the beginning of this section, and by our construction we have

$$E^{\mathcal{F}_n} X_\infty(\omega) = w^* cl\Big[\bigcup_{k=1}^{\infty} E^{\mathcal{F}_n} X_\infty^k(\omega)\Big] \quad \text{a.s.}$$

Hence by (5.3.1)

$$E^{\mathcal{F}_n} X_\infty(\omega) \subset w^* cl\Big[\bigcup_{k=1}^{\infty} X_n^k(\omega)\Big] \subset X_n(\omega) \quad \text{a.s.} \tag{5.3.3}$$

Let $f \in S^1_{X_\infty}$. By (5.3.3) we have $E^{\mathcal{F}_n} f(\omega) \in X_n(\omega)$. Applying Theorem 3.4 (a) to the single-valued mapping f yields $w^*\text{-}\lim_n E^{\mathcal{F}_n} f = f$ a.s. Hence $f(\omega) \in w^*\text{-}li X_n(\omega)$ a.s. Taking a Castaing representation of X_∞ we get $X_\infty(\omega) \in w^* cl[w^*\text{-}li X_n(\omega)]$ a.s. ∎

Taking into account Theorem 5.3, we proceed to the existence of regular martingale selections in $L^1_{E^*}[E](\mathcal{F})$.

THEOREM 5.4. *Let $(X_n)_{n\in\mathbf{N}}$ be a w^*-closed convex integrable supermartingale ($E^{\mathcal{F}_m} X_n \subset X_m$ for $m < n$) such that $(d_{E_b^*}(0, X_n))_{n\in\mathbf{N}}$ is uniformly integrable in $L^1_{\mathbf{R}}(\mathcal{F})$. Then there is $Z \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that*

$$E^{\mathcal{F}_n} Z \subset X_n \quad a.s. \quad \forall n \in \mathbf{N}.$$

Hence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ admits a regular martingale selection $(f_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_{E^*}[E](\mathcal{F})$.

Proof. We will use some techniques from [24, Theorem IV-1-2] and [20, Theorem 4.4]. Set $V_n := d_{E_b^*}(0, X_n) + 1$ for each $n \in \mathbf{N}$. Then by using Lemma 5.1 and the supermartingale property

$$d_{E_b^*}(0, X_n) \le d_{E_b^*}(0, E^{\mathcal{F}_n} X_{n+1}) \le E^{\mathcal{F}_n} d_{E_b^*}(0, X_{n+1}) \quad \text{a.s.}$$

we see that $(V_n)_{n\in\mathbf{N}}$ is a positive uniformly integrable submartingale. So $(V_n)_{n\in\mathbf{N}}$ converges a.s. to a nonnegative integrable function $V_\infty \in L^1_{\mathbf{R}}(\mathcal{F})$. For each fixed $n \in \mathbf{N}$, the sequence $(E^{\mathcal{F}_n} V_p, p \ge n)$ is increasing, so, if $p \ge n$, let us set $M_n = \uparrow \lim_{p\to\infty} E^{\mathcal{F}_n} V_p$. Then $(M_n)_{n\in\mathbf{N}}$ is a positive integrable martingale such that $M_n \ge V_n$ for every $n \in \mathbf{N}$. But $(V_n)_{n\in\mathbf{N}}$ converges to V_∞ in $L^1_{\mathbf{R}}(\mathcal{F})$ so that $\lim_{p\to\infty} E^{\mathcal{F}_n} V_p = E^{\mathcal{F}_n} V_\infty$ in $L^1_{\mathbf{R}}(\mathcal{F})$. By identifying the limits, we get $M_n = E^{\mathcal{F}_n} V_\infty$ a.s. So we conclude that $(M_n)_{n\in\mathbf{N}}$ is a regular integrable martingale. Set

$$Y_n := X_n \cap M_n \overline{B}_{E^*}.$$

Now using the supermartingale property and the monotony of the conditional expectation, it is easy to check that $(Y_n)_{n\in\mathbf{N}}$ is a $cwk(E_s^*)$-valued uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E_s^*)}(\mathcal{F})$. Indeed, we have for each $n \in \mathbf{N}$

$$\begin{aligned} E^{\mathcal{F}_n} Y_{n+1} &= E^{\mathcal{F}_n}[X_{n+1} \cap M_{n+1} \overline{B}_{E^*}] \\ &\subset E^{\mathcal{F}_n} X_{n+1} \cap E^{\mathcal{F}_n}[M_{n+1} \overline{B}_{E^*}] \subset X_n \cap E^{\mathcal{F}_n}[M_{n+1} \overline{B}_{E^*}]. \end{aligned}$$

Let $(e_j)_{j\in\mathbf{N}}$ be a dense sequence in $\overline{B}_E$. Taking the conditional expectation of $M_{n+1}(.)\overline{B}_{E^*}$ gives

$$\begin{aligned}\delta^*(e_j, E^{\mathcal{F}_n}[M_{n+1}(.)\overline{B}_{E^*}]) &= E^{\mathcal{F}_n}[\delta^*(e_j, M_{n+1}(.)\overline{B}_{E^*})]\\ &= E^{\mathcal{F}_n}[\delta^*(e_j,\overline{B}_{E^*})]M_{n+1}(.) = \delta^*(e_j,\overline{B}_{E^*})E^{\mathcal{F}_n}M_{n+1}(.) = \delta^*(e_j,\overline{B}_{E^*})M_n(.)\end{aligned}$$

a.s. for all $j\in\mathbf{N}$. Hence $E^{\mathcal{F}_n}[M_{n+1}(.)\overline{B}_{E^*}] = M_n(.)\overline{B}_{E^*}$ a.s., thus proving that $E^{\mathcal{F}_n}Y_{n+1}\subset Y_n$ a.s. Now applying Lemma 5.2 to the uniformly integrable supermartingale $(Y_n)_{n\in\mathbf{N}}$ yields an $Z\in\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that $E^{\mathcal{F}_n}Z\subset Y_n\subset X_n$ a.s. for each $n\in\mathbf{N}$. ∎

The above consideration leads to a characterization of regular martingales in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$.

Proposition 5.5. *Let $(X_n)_{n\in\mathbf{N}}$ be a uniformly integrable martingale in $\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$. Then there is $X_\infty\in\mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ such that $X_m = E^{\mathcal{F}_m}X_\infty$ for all $m\in\mathbf{N}$.*

Proof. Since $(X_n)_{n\in\mathbf{N}}$ is a martingale, for every $m\in\mathbf{N}$ and for every $A\in\mathcal{F}_m$ we have

$$\lim_{n>m,n\to\infty} AG\text{-}\int_A X_n\,dP = AG\text{-}\int_A X_m\,dP$$

here the limit can be taken with respect to the Hausdorff distance $\mathcal{H}^*_{E^*_b}$. As $(|X_n|)_{n\in\mathbf{N}}$ is uniformly integrable, applying Theorem 6.1(4) in [12] provides a subsequence $(X'_n)_{n\in\mathbf{N}}$ and $X_\infty\in\mathcal{L}^1_{cwk(E^*)}(\mathcal{F})$ such that

$$\lim_{n\to\infty}\int_\Omega \delta^*(u,X'_n)\,dP = \int_\Omega \delta^*(u,X_\infty)\,dP \quad \forall u\in L^\infty_E(\mathcal{F}).$$

It follows that, for every $m\in\mathbf{N}$, for $A\in\mathcal{F}_m$

$$AG\text{-}\int_A X_\infty\,dP = AG\text{-}\int_A X_m\,dP.$$

In other words $X_m = E^{\mathcal{F}_m}X_\infty$. ∎

Corollary 5.6. *Let $(X_n)_{n\in\mathbf{N}}$ be a uniformly integrable martingale in $L^1_{E^*}[E](\mathcal{F})$. Then there is $X_\infty\in L^1_{E^*}[E](\mathcal{F})$ such that $X_m = E^{\mathcal{F}_m}X_\infty$ for all $m\in\mathbf{N}$.*

Proof. Since $(X_n)_{n\in\mathbf{N}}$ is a martingale, for every $m\in\mathbf{N}$ and for every for $A\in\mathcal{F}_m$ we have

$$\underset{n>m,n\to\infty}{w^*\text{-lim}}\ G\text{-}\int_A X_n\,dP = G\text{-}\int_A X_m\,dP$$

where the limit can be taken with respect to the weak star topology. As $(|X_n|)_{n\in\mathbf{N}}$ is uniformly integrable, applying Theorem 6.5.9 in [10], provides a subsequence $(X'_n)_{n\in\mathbf{N}}$ and $X_\infty\in L^1_{E^*}[E](\mathcal{F})$ such that

$$\lim_{n\to\infty}\int_\Omega\langle u,X'_n\rangle\,dP = \int_\Omega\langle u,X_\infty\rangle\,dP \quad \forall u\in L^\infty_E(\mathcal{F}).$$

It follows that, for every $m\in\mathbf{N}$ and for every $A\in\mathcal{F}_m$,

$$G\text{-}\int_A X_\infty\,dP = G\text{-}\int_A X_m\,dP.$$

In other words $X_m = E^{\mathcal{F}_m}X_\infty$. ∎

6. Conditional expectation in $\mathcal{G}^1_{cwk(E^*_s)}(\mathcal{F})$. Now the existence and uniqueness for the conditional expectation in $\mathcal{G}^1_{cwk(E^*_s)}(\mathcal{F})$ comes.

THEOREM 6.1. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and let X be a $cwk(E^*_s)$-valued Gelfand-integrable mapping such that $E^{\mathcal{B}}|X| \in [0,+\infty[$. Then there exists a unique $\mathcal{B}$-measurable, $cwk(E^*_s)$-valued Gelfand-integrable mapping, denoted by $Ge\text{-}E^{\mathcal{B}}X$ which enjoys the following property*: *For every $h \in L^\infty(\mathcal{B})$, one has*

$$AG\text{-}\int_\Omega hGe\text{-}E^{\mathcal{B}}X\,dP = AG\text{-}\int_\Omega hX\,dP.$$

$Ge\text{-}E^{\mathcal{B}}X$ is called the Gelfand conditional expectation of X.

Proof. Theorem 6.1 is a corollary of a general integral representation (see Theorem 6.3) given below. ∎

Let us mention a useful corollary.

COROLLARY 6.2. *Under the hypotheses and notation of Theorem* 6.1, *the following hold*:

1) *For every $h \in L^\infty(\mathcal{B})$ and for every $x \in E$ and for every $f \in \mathcal{S}^1_{Ge}(X)$, one has*

$$\langle h\otimes x, Ge\text{-}E^{\mathcal{B}}f\rangle := \int_\Omega \langle h\otimes x, f\rangle\,dP \le \int_\Omega \delta^*(h\otimes x, X)\,dP = \int_\Omega \delta^*(h\otimes x, Ge\text{-}E^{\mathcal{B}}X)\,dP$$

and hence $Ge\text{-}E^{\mathcal{B}}f(\omega) \in Ge\text{-}E^{\mathcal{B}}X(\omega)$ a.s.

2) *For every $h \in L^\infty(\mathcal{B})$ and for every $x \in E$, one has*

$$\begin{aligned}\delta^*(h\otimes x, Ge\text{-}E^{\mathcal{B}}(\mathcal{S}^1_{Ge}(X)) &= \sup\{\langle h\otimes x, Ge\text{-}E^{\mathcal{B}}f\rangle : f\in\mathcal{S}^1_{Ge}(X)\}\\ &= \sup\{\langle h\otimes x^*, f\rangle : f\in\mathcal{S}^1_{Ge}(X)\} = \delta^*(h\otimes x, \mathcal{S}^1_{Ge}(X))\\ &= \int_\Omega \delta^*(h\otimes x, X)\,dP = \int_\Omega \delta^*(h\otimes x, Ge\text{-}E^{\mathcal{B}}X)\,dP = \delta^*(h\otimes x, \mathcal{S}^1_{Ge}(Ge\text{-}E^{\mathcal{B}}X))\end{aligned}$$

3) *For all $B\in\mathcal{B}$, for all $x \in E$,*

$$\int_B \delta^*(x, X(\omega))\,dP = \int_B \delta^*(x, Ge\text{-}E^{\mathcal{B}}X(\omega))\,dP$$

hence $\delta^*(x, Ge\text{-}E^{\mathcal{B}}X(\omega)) = E^{\mathcal{B}}\delta^*(x, X(\omega))$ a.s.

Proof. 1) Equality $\langle h\otimes x, Ge\text{-}E^{\mathcal{B}}f\rangle = \int_\Omega \langle h\otimes x, f\rangle\,dP$ and equality

$$\int_\Omega \delta^*(h\otimes x, X)\,dP = \int_\Omega \delta^*(h\otimes x, Ge\text{-}E^{\mathcal{B}}X)\,dP$$

follow from Theorem 6.1. In particular, by taking the functions $1_A\otimes x_i$ where $A\in\mathcal{B}$ and $(x_i)_{i\in\mathbf{N}}$ is a dense sequence in E, we get

$$\int_A \langle x_i, Ge\text{-}E^{\mathcal{B}}f\rangle\,dP \le \int_A \delta^*(x_i, Ge\text{-}E^{\mathcal{B}}X)\,dP$$

and hence

$$\langle x_i, Ge\text{-}E^{\mathcal{B}}f\rangle \le \delta^*(x_i, Ge\text{-}E^{\mathcal{B}}X)$$

a.s. for all $i \in \mathbf{N}$. By Proposition III.35 in [13], we get

$$Ge\text{-}E^{\mathcal{B}} f(\omega) \in Ge\text{-}E^{\mathcal{B}} X(\omega) \quad \text{a.s.}$$

2) follows from the Strassen formula [13, Theorem V-14] applied to the Aumann-Gelfand integral of the $cwk(E_s^*)$-valued Gelfand-integrable X and $Ge\text{-}E^{\mathcal{B}}X$.

3) follows from the calculus of support functionals in Theorem 6.3. ■

Theorem 6.1 is a consequence of the following integral representation.

THEOREM 6.3. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. Let us consider a $cwk(E_s^*)$-valued mapping $M : L^\infty(\mathcal{B}) \to E_s^*$ satisfying the following conditions:*

(i) *For each $x \in E$, the scalar function $h \mapsto \delta^*(x, M(h))$ is continuous on bounded subset of $L^\infty(\mathcal{B})$ with respect to the convergence in probability.*
(ii) *$M(f+g) = M(f) + M(g)$ if $fg \geq 0$, for $f, g \in L^\infty(\mathcal{B})$.*
(iii) *There is a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*)}(\Omega, \mathcal{F}, P)$ and a $\mathcal{B}$-measurable partition $(B_n)_{n\in\mathbf{N}}$ of Ω satisfying*

$$M(1_{B_n} h) = AG\text{-}\int_{B_n} hX_n \, dP \quad \forall h \in L^\infty(\mathcal{B}) \quad \forall n \in \mathbf{N}.$$

Then there exists a unique $\mathcal{B}$-measurable, $cwk(E_s^*)$-valued Gelfand-integrable mapping Γ satisfying the following property:

$$M(h) = AG\text{-}\int_\Omega h\Gamma \, dP \quad \forall h \in L^\infty(\mathcal{B}).$$

Here $AG\text{-}\int_\Omega h\Gamma\, dP$ denotes the $cwk(E_s^*)$-valued Aumann-Gelfand integral of $h\Gamma$.

Proof. Let $n \in \mathbf{N}$. By (iii) and Theorem 3.1, there is a unique $cwk(E_s^*)$-valued $\mathcal{B}$-measurable and integrably bounded mapping $\Gamma_n := E^{\mathcal{B}} X_n$

$$M(h1_{B_n}) = AG\text{-}\int_\Omega h1_{B_n} X_n \, dP = AG \int_{B_n} h\Gamma_n \, dP \quad \forall h \in L^\infty(\mathcal{B}).$$

Let us define $\Gamma(\omega) = \Gamma_n(\omega)$ if $\omega \in B_n$. Then Γ is $\mathcal{B}$-measurable. By using (ii) it is not difficult to check that

$$M\Big(\sum_{n=l}^m 1_{B_n} h\Big) = AG\text{-}\int_\Omega \sum_{n=l}^m 1_{B_n} h\Gamma \, dP \tag{6.3.1}$$

for every $h \in L^\infty(\mathcal{B})$ and for every $m, l \in \mathbf{N}$. Let us consider an arbitrary $\mathcal{B}$-measurable selection g of Γ. Then we have

$$G\text{-}\int_\Omega \sum_{n=l}^m 1_{B_n} hg \, dP \in M\Big(\sum_{n=l}^m 1_{B_n} h\Big) \tag{6.3.2}$$

which implies

$$\delta^*\Big(-x, M\Big(\sum_{n=l}^m 1_{B_n} h\Big)\Big) \leq \int_\Omega \Big\langle x, \sum_{n=l}^m 1_{B_n} hg\Big\rangle \, dP \leq \delta^*\Big(x, M\Big(\sum_{n=l}^m 1_{B_n} h\Big)\Big).$$

for every $x \in E$. By (i) the mapping $h \mapsto M(h)$ is scalarly continuous on bounded subsets of $L^\infty(\mathcal{B})$ with respect to the convergence in probability, from the above estimate we see that the sequence $(\langle x, \sum_{n=1}^m 1_{B_n} g\rangle)_{m\in\mathbf{N}}$ is a $\sigma(L^1(\mathcal{B}), L^\infty(\mathcal{B}))$ Cauchy sequence. But the pointwise limit of this sequence is $\langle x, g\rangle$, therefore by classical property of L^1 space we have

$$\lim_{m\to\infty}\sum_{n=1}^{m} 1_{B_n}\langle x, g\rangle = \langle x, g\rangle \tag{6.3.3}$$

for the norm topology of L^1. By (6.3.3) we conclude that g is Gelfand-integrable with $G\text{-}\int_\Omega hg\,dP \in M(h)$ by passing to the limit when m goes to ∞ in (6.3.2). Now we prove that Γ is Gelfand-integrable. Let $x \in E$. By the measurable implicit Theorem III.38 in [13], there is $\mathcal{B}$-measurable selection σ of Γ such that $\langle x, \sigma\rangle = \delta^*(x, \Gamma)$. We conclude that the $cwk(E^*)$-valued $\mathcal{B}$-measurable mapping Γ is Gelfand-integrable. Let us denote by $\mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})$ the set of all Gelfand-integrable selections of Γ. Then $\mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})$ is nonempty convex $\sigma(G^1_{E^*}[E](\mathcal{B}), L^\infty(\mathcal{B})\otimes E)$ compact, by applying Theorem V-14 in [13]. We finish the proof by showing that

$$M(h) = AG\text{-}\int_\Omega h\Gamma\,dP = \Big\{G\text{-}\int_\Omega hg\,dP : g \in \mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})\Big\}$$

By invoking Theorem V-14 in [13] we see that the Aumann-Gelfand integral $AG\text{-}\int_\Omega h\Gamma\,dP$ is convex weak* compact. In order to prove the desired equality, we proceed as in the proof of Theorem V-17 in [13]. It is clear that

$$AG\text{-}\int_\Omega h\Gamma\,dP = \Big\{G\text{-}\int_\Omega hg\,dP : g \in \mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})\Big\} \subset M(h).$$

Assume that there is a $\zeta \in M(h) \setminus AG\text{-}\int_\Omega hY\,dP$. By the Hahn-Banach theorem, there is an $x \in E$ such that

$$\forall g \in \mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B}) \quad \int_\Omega \langle x, g\rangle h\,dP < \langle x, \zeta\rangle \le \delta^*(x, M(h)). \tag{6.3.4}$$

By the measurable implicit Theorem III.38 in [13], there is a $\mathcal{B}$-measurable and Gelfand-integrable selection $\widetilde{g}$ of Γ such that

$$\langle x, \widetilde{g}\rangle h = \delta^*(x, h\Gamma) \quad \forall\omega \in \Omega.$$

This implies by integrating on each B_n

$$\int_{B_n} \langle x, \widetilde{g}\rangle h\,dP = \int_{B_n} \delta^*(x, h\Gamma)\,dP = \delta^*(x, M(1_{B_n}h))$$

By (6.3.2) we have

$$M\Big(\sum_{n=1}^{m} 1_{B_n}h\Big) = AG\text{-}\int_\Omega \sum_{n=1}^{m} 1_{B_n}h\Gamma\,dP = \sum_{n=1}^{m} AG\text{-}\int_\Omega 1_{B_n}h\Gamma\,dP = \sum_{n=1}^{m} M(1_{B_n}h)$$

Hence we get

$$\delta^*\Big(x, M\Big(\sum_{n=1}^{m} 1_{B_n}h\Big)\Big) = \int_\Omega \sum_{n=1}^{m}\langle x, 1_{B_n}\widetilde{g}\rangle h\,dP$$

By passing to the limit when m goes to ∞, we get by Lebesgue's theorem

$$\delta^*(x, M(h)) = \int_\Omega \langle x, \widetilde{g}\rangle h\,dP$$

This contradicts the inequality

$$\int_\Omega \langle x, \widetilde{g}\rangle h\, dP < \delta^*(x, M(h))$$

in (6.3.4) and completes the proof. ■

The preceding result recovers Theorem 6.1. Indeed, it is enough to put $M(h) = AG\text{-}\int hX\, dP$, $h \in L^\infty(\mathcal{B})$. Using the assumption $E^{\mathcal{B}}|X| \in [0, +\infty[$ provides a $\mathcal{B}$-measurable partition $(B_n)_{n\in\mathbf{N}}$ of Ω and a sequence $(X_n)_{n\in\mathbf{N}} := (1_{B_n}X)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E^*_s)}(\Omega, \mathcal{F}, P)$ such that

$$M(1_{B_n}h) = AG\text{-}\int_{B_n} hX_n\, dP \quad \forall h \in L^\infty(\mathcal{B}) \quad \forall n \in \mathbf{N}.$$

In the same vein we provide another integral representation theorem for a $cwk(E^*)$-valued mapping $M : \Lambda^* \to cwk(E^*)$ involving a pair of Köthe spaces (Λ^*, Λ) in the same style as Theorem V-17 in [13]. Recall that (Λ^*, Λ) is a pair of vector subspaces of L^1 placed in duality by the bilinear form

$$\langle f, g\rangle = \int_\Omega fg\, dP, \quad f \in \Lambda^*, \quad g \in \Lambda.$$

THEOREM 6.4. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. Let us consider a $cwk(E^*_s)$-valued mapping $M : \Lambda^*(\mathcal{B}) \to E^*_s$ satisfying the following conditions*:

(i) *For each $x \in E$, the scalar function $f \mapsto \delta^*(x, M(f))$ is continuous on $\Lambda^*(\mathcal{B})$ with respect to the topology of $L^1(\mathcal{B})$.*
(ii) *$M(f+g) = M(f) + M(g)$ if $fg \geq 0$, for $f, g \in \Lambda^*(\mathcal{B})$.*
(iii) *There is a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^\infty_{cwk(E^*)}(\Omega, \mathcal{F}, P)$ and a $\mathcal{B}$-measurable partition $(B_n)_{n\in\mathbf{N}}$ of Ω satisfying*

$$M(1_{B_n}f) = AG\text{-}\int_{B_n} fX_n\, dP \quad \forall f \in \Lambda^*(\mathcal{B}) \quad \forall n \in \mathbf{N}.$$

Then there exists a unique $\mathcal{B}$-measurable, $cwk(E^*_s)$-valued Gelfand-integrable mapping Γ satisfying the following properties:

(a) $$\delta^*(x, \Gamma) \in \Lambda(\mathcal{B}) \quad \forall x \in E.$$

(b) $$M(f) = AG\text{-}\int_\Omega f\Gamma\, dP \quad \forall f \in \Lambda^*(\mathcal{B}).$$

Here $AG\text{-}\int_\Omega f\Gamma\, dP$ denotes the $cwk(E^*_s)$-valued Aumann-Gelfand integral of $f\Gamma$.

Proof. Let $n \in \mathbf{N}$. By (iii) and virtue of Theorem 3.1, there is a unique $cwk(E^*_s)$-valued $\mathcal{B}$-measurable and bounded mapping $\Gamma_n := E^{\mathcal{B}}X_n$ satisfying

$$AG\text{-}\int_{B_n} f\Gamma_n\, dP = AG\text{-}\int_{B_n} fX_n\, dP$$

for every $f \in L^\infty(\mathcal{B})$. Coming back to (iii) we have

$$M(f1_{B_n}) = AG\text{-}\int_\Omega f1_{B_n}X_n\, dP = AG\text{-}\int_{B_n} f\Gamma_n\, dP \quad \forall f \in L^\infty(\mathcal{B}).$$

Since the set of simple functions is dense in L^1, by using (i) it is not difficult to see that

$$M(f1_{B_n}) = AG\text{-}\int_\Omega f1_{B_n} X_n\,dP = AG\text{-}\int_{B_n} f\Gamma_n\,dP \quad \forall f \in \Lambda^*(\mathcal{B}). \tag{6.4.1}$$

Let us define $\Gamma(\omega) = \Gamma_n(\omega)$ if $\omega \in B_n$. Then Γ is $\mathcal{B}$-measurable. Accordingly, from (ii) and (6.4.1)

$$M\Big(\sum_{n=l}^{m} 1_{B_n} f\Big) = AG\text{-}\int_\Omega \sum_{n=l}^{m} 1_{B_n} f\Gamma\,dP \tag{6.4.2}$$

for every $f \in \Lambda^*(\mathcal{B})$, and for every $m, l \in \mathbf{N}$. Let $\mathcal{S}_\Gamma$ be the set of all $\mathcal{B}$-measurable selections of Γ. Let g be a $\mathcal{B}$-measurable selection of Γ and $h \in L^\infty(\mathcal{B})$. Then we have

$$G\text{-}\int_\Omega \sum_{n=l}^{m} 1_{B_n} hfg\,dP \in M\Big(\sum_{n=l}^{m} 1_{B_n} hf\Big) \tag{6.4.3}$$

which implies

$$\delta^*\Big(-x, M\Big(\sum_{n=l}^{m} 1_{B_n} hf\Big)\Big) \le \int_\Omega \Big\langle x, \sum_{n=l}^{m} 1_{B_n} fg\Big\rangle h\,dP \le \delta^*\Big(x, M\Big(\sum_{n=l}^{m} 1_{B_n} hf\Big)\Big)$$

for every $x \in E$. By (i) and the preceding estimate we see that the sequence

$$\Big(\Big\langle x, \sum_{n=1}^{m} 1_{B_n} fg\Big\rangle\Big)_{m\in\mathbf{N}}$$

is a $\sigma(L^1(\mathcal{B}), L^\infty(\mathcal{B}))$ Cauchy sequence. But the pointwise limit of this sequence is $\langle x, fg\rangle$, therefore by the classical property of L^1 space we have

$$\lim_{m\to\infty} \sum_{n=1}^{m} 1_{B_n} \langle x, fg\rangle = \langle x, fg\rangle \tag{6.4.4}$$

for the norm topology of L^1. By (6.4.4) we conclude that fg is Gelfand-integrable with $Ge\text{-}\int_\Omega fg\,dP \in M(f)$ by passing to the limit when m goes to ∞ in (6.4.3). This implies that $\langle x, g\rangle \in \Lambda$ for all $x \in E$. Now we prove that Γ is Gelfand-integrable. Let $x \in E$. By the measurable implicit Theorem III.38 in [13], there is $\mathcal{B}$-measurable selection σ of Γ such that $\langle x, \sigma\rangle = \delta^*(x, \Gamma)$. We conclude that the $cwk(E^*)$-valued $\mathcal{B}$-measurable mapping Γ is Gelfand-integrable and also $\delta^*(x, \Gamma) \in \Lambda$ for all $x \in E$. Let us denote by $\mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})$ the set of all Gelfand-integrable selections of Γ. Then $\mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})$ is nonempty convex $\sigma(G^1_{E^*}[E](\mathcal{B}), L^\infty(\mathcal{B}) \otimes E)$ compact, by applying Theorem V-14 in [13]. We finish the proof as in Theorem 6.3 or Theorem V-17 in [13] by showing that

$$M(f) = AG\text{-}\int_\Omega f\Gamma\,dP = \{G\text{-}\int_\Omega fg\,dP : g \in \mathcal{S}^1_{Ge}(\Gamma)(\mathcal{B})\}. \ \blacksquare$$

7. Levy's theorem for convex weak* compact valued Gelfand-integrable mapping. We begin with a version of Levy's theorem for a Gelfand integrable mapping.

THEOREM 7.1. *Assume that E is a separable Banach space and X is a Gelfand-integrable E^*-valued mapping such that*

$$E^{\mathcal{F}_n}|X| \in [0, +\infty[\quad \textit{for each } n \in \mathbf{N}. \tag{$\mathcal{C}$}$$

Then we have

$$w^*\text{-}\lim_{n\to\infty} Ge\text{-}E^{\mathcal{F}_n} X = X \quad a.s.$$

where $Ge\text{-}E^{\mathcal{F}_n} X$ *denotes the Gelfand conditional expectation of* X.

Proof. By condition $(\mathcal{C})$ and Theorem 6.1, $Ge\text{-}E^{\mathcal{F}_n} X$ is $\mathcal{F}_n$-measurable and Gelfand-integrable satisfying

$$G\text{-}\int_\Omega h Ge\text{-}E^{\mathcal{F}_n} X \, dP = G\text{-}\int_\Omega hX \, dP \tag{7.1.1}$$

for every $h \in L^\infty(\mathcal{F}_n)$. Applying $(\mathcal{C})$ for $n = 1$ provides a $\mathcal{F}_1$-measurable partition $(B_k)_{k\in\mathbf{N}}$ of Ω such that $X_k := X 1_{B_k} \in L^1_{E^*}[E](\mathcal{F})$ for each $k \in \mathbf{N}$. By virtue of Theorem 3.2, we have

$$X_k = w^*\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_k \quad \text{a.s.} \tag{7.1.2}$$

for each $k \in \mathbf{N}$. We claim that

$$X = w^*\text{-}\lim_{n\to\infty} Ge\text{-}E^{\mathcal{F}_n} X \quad \text{a.s.}$$

As $B_k \in \mathcal{F}_1 \subset \mathcal{F}_n$ for every $k \in \mathbf{N}$ and for every $n \in \mathbf{N}$, it is obvious by using (7.1.1) that

$$E^{\mathcal{F}_n} X 1_{B_k} = 1_{B_k} Ge\text{-}E^{\mathcal{F}_n} X. \tag{7.1.3}$$

By (7.1.2) and (7.1.3) we have

$$X = \sum_{k=1}^\infty X_k = \sum_{k=1}^\infty w^*\text{-}\lim_{n\to\infty} E^{\mathcal{F}_n} X_k = \sum_{k=1}^\infty 1_{B_k}\, w^*\text{-}\lim_{n\to\infty} Ge\text{-}E^{\mathcal{F}_n} X = w^*\text{-}\lim_{n\to\infty} Ge\text{-}E^{\mathcal{F}_n} X.$$

a.s., thereby proving the claim and completing the proof. ∎

Now here is a version of Levy's theorem for $cwk(E^*_s)$-valued Gelfand-integrable mapping. A similar version for convex weakly compact valued Pettis-integrable mappings is available in [1, Theorem 5.1].

THEOREM 7.2. *Let* X *be a* $cwk(E^*_s)$*-valued Gelfand-integrable mapping such that* $(\mathcal{C})$ *holds. Then we have*

$$w^*K\text{-}\lim_{n\to\infty} Ge\text{-}E^{\mathcal{F}_n} X = X \quad a.s.$$

where $Ge\text{-}E^{\mathcal{F}_n} X$ *denotes the* $cwk(E^*_s)$*-valued Gelfand conditional expectation of* X.

Proof. 1) By condition $(\mathcal{C})$ and Theorem 6.1, $Ge\text{-}E^{\mathcal{F}_n} X$ is $\mathcal{F}_n$-measurable and $cwk(E^*_s)$-valued Gelfand-integrable satisfying

$$AG\text{-}\int_\Omega h Ge\text{-}E^{\mathcal{F}_n} X \, dP = AG\text{-}\int_\Omega hX \, dP \tag{7.2.1}$$

for every $h \in L^\infty(\mathcal{F}_n)$. Applying $(\mathcal{C})$ for $n = 1$ provides a $\mathcal{F}_1$-measurable partition $(B_k)_{k\in\mathbf{N}}$ of Ω such that $X_k := X 1_{B_k} \in \mathcal{L}^1_{cwk(E^*_s)}(\mathcal{F})$ for each $k \in \mathbf{N}$. From equality (7.2.1) it follows easily that $E^{\mathcal{F}_n} 1_{B_k} X = 1_{B_k} Ge\text{-}E^{\mathcal{F}_n} X$. So

$$1_{B_k} \left| Ge\text{-}E^{\mathcal{F}_n} X \right| = \left| E^{\mathcal{F}_n} 1_{B_k} X \right| \le E^{\mathcal{F}_n} 1_{B_k} |X| \quad \text{a.s.}$$

Hence

$$\sup_{n\in\mathbf{N}} |E^{\mathcal{F}_n}X| < \infty \quad \text{a.s. on each } B_k,$$

which implies that $w^*\text{-}li\, Ge\text{-}E^{\mathcal{F}_n}X(\omega)$ is convex and w^*-compact a.s.

2) Since X is Gelfand-integrable, the measurable selections of X are Gelfand-integrable. So, by [13, Theorem III.22], there is a sequence $(f_n)_{n\in\mathbf{N}}$ in $G^1_{E^*_s}(\mathcal{F})$ such that

$$X(\omega) = w^*\text{-}cl\{f_n(\omega)\} \tag{7.2.2}$$

for every $\omega \in \Omega$.

Claim 1. $X \subset w^*\text{-}li\, Ge\text{-}E^{\mathcal{F}_n}X$ a.s. Let $f \in S^1_{Ge}(\mathcal{F})(X)$. By Theorem 6.1 and Part 3) of Corollary 6.2, $Ge\text{-}E^{\mathcal{F}_n}f$ is $\mathcal{F}_n$-measurable and Gelfand-integrable and satisfies

$$Ge\text{-}E^{\mathcal{F}_n}f(\omega) \in Ge\text{-}E^{\mathcal{F}_n}X(\omega) \quad \text{a.s.}$$

Furthermore, by Theorem 7.1, $w^*\text{-}\lim_{n\to\infty} Ge\text{-}E^{\mathcal{F}_n}f = f$ a.s. So we conclude that $f \in w^*\text{-}li\, Ge\text{-}E^{\mathcal{F}_n}X$ a.s. Since this is true for any $f \in S^1_{Ge}(\mathcal{F})(X)$, by invoking (7.2.2) we see that Claim 1 is true.

Claim 2. $w^*\text{-}ls\, Ge\text{-}E^{\mathcal{F}_n}X \subset X$ a.s. Let $(h_j)_{j\in\mathbf{N}}$ be a dense sequence in E. Then Part 3) of Corollary 6.2 implies

$$\delta^*(h_j, Ge\text{-}E^{\mathcal{F}_n}X(\omega)) = E^{\mathcal{F}_n}\delta^*(h_j, X(\omega)) \quad \text{a.s.} \quad \forall j \in \mathbf{N} \quad \forall n \in \mathbf{N}.$$

By Levy's theorem, we have

$$\lim_{n\to\infty} E^{\mathcal{F}_n}\delta^*(h_j, X(\omega)) = \delta^*(h_j, X(\omega)) \quad \text{a.s.} \quad \forall j \in \mathbf{N}.$$

Let $\omega \in \Omega$ be such that the preceding relations are satisfied. Let $x^* \in w^*\text{-}ls\, Ge\text{-}E^{\mathcal{F}_n}X(\omega)$. Then $x^*_k \to x^*$ in E^*_s for some $x^*_k \in Ge\text{-}E^{\mathcal{F}_{n_k}}(X)(\omega)$ and hence, for each $j \in \mathbf{N}$,

$$\begin{aligned}\langle h_j, x^*\rangle = \lim_{k\to\infty}\langle h_j, x^*_k\rangle &\le \limsup_{k\to\infty} \delta^*(h_j, Ge\text{-}E^{\mathcal{F}_{n_k}}(X)(\omega)) \\ &= \limsup_{k\to\infty} E^{\mathcal{F}_{n_k}}\delta^*(h_j, X(\omega)) = \lim_{k\to\infty} E^{\mathcal{F}_{n_k}}\delta^*(h_j, X(\omega)) = \delta^*(h_j, X(\omega))\end{aligned}$$

So $x^* \in X(\omega)$ because X is convex weakly* compact valued ([13], Proposition III.35) and hence Claim 2 follows. ∎

The following is a w^*K-convergence result for $cwk(E^*_s)$-valued Gelfand-integrable martingales.

THEOREM 7.3. *Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be an adapted sequence of $cwk(E^*)$-valued Gelfand-integrable multifunctions satisfying*:

(i) $E^{\mathcal{F}_q}|X_n| < \infty$ *for each* $n \in \mathbf{N}$ *and each* $1 \le q < n$.
(ii) $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ *is a $cwk(E^*)$-valued martingale, that is,* $Ge\text{-}E^{\mathcal{F}_n}X_{n+1} = X_n$ *for all* $n \in \mathbf{N}$.
(iii) $\sup_{n\ge1}\sup_{x\in\overline{B}_E}\int_\Omega |\delta^*(x, X_n)|\,dP < \infty$.
(iv) *There is a partition $(A_n)_{n\in\mathbf{N}}$ in $\bigcup_{n=1}^\infty \mathcal{F}_n$ such that for each $m \in \mathbf{N}$, $(X_n|_{A_m})_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E^*_s)}(A_m)$.*

Then there is a $cwk(E^*_s)$-valued Gelfand-integrable multifunction X_∞ such that $w^*K\text{-}\lim_{n\to\infty} X_n = X_\infty$ *a.s.*

Proof. By (i) and Theorem 6.1, for $1 \leq q < n$ the Gelfand conditional expectations $Ge\text{-}E^{\mathcal{F}_q} X_n$ exist and belong to $\mathcal{G}^1_{cwk(E^*_s)}(\mathcal{F})$. Accordingly the $cwk(E^*_s)$-valued martingale given in (ii) exists. Now for each $m \in \mathbf{N}$, let $n(m) \in \mathbf{N}$ be such that $A_m \in \mathcal{F}_{n(m)}$. Then, by (iv) $(X_n|_{A_m}, \mathcal{F}_n|_{A_m})_{n \geq n(m)}$ is a $cwk(E^*_s)$-valued bounded martingale in $\mathcal{L}^1_{cwk(E^*_s)}(A_m)$. Hence for each $m \in \mathbf{N}$, using Theorem 3.8, we can find $X^m_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(A_m)$ such that

$$w^*K\text{-}\lim_{n\to\infty, n\geq n(m)} X_n = X^m_\infty \quad \text{a.s.} \quad \omega \in A_m$$

Put $X_\infty = \sum_{m=1}^\infty X^m_\infty 1_{A_m}$. Then obviously $(X_n)_{n\in\mathbf{N}}$ w^*K-converges to X_∞. It is easy to check that X_∞ is scalarly $\mathcal{F}$-measurable. By (iii) it follows that, for every $x \in \overline{B}_E$, $(\delta^*(x, X_n))_{n\in\mathbf{N}}$ is a real-valued L^1-bounded martingale, so it converges a.s. to a function in L^1. Hence X_∞ is Gelfand-integrable. ■

8. Mosco convergence results for closed convex valued supermartingales. We present a new version of Mosco convergence results for closed convex valued supermartingales in a separable Banach space E having the Radon-Nikodym property (RNP). For this purpose, in the remainder of this section, the conditional expectation is taken in the sense of Hiai-Umegaki [22]. If $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{F}$, $F : \Omega \rightrightarrows E$ is an integrable $\mathcal{F}$-measurable multifunction, Hiai and Umegaki [22] showed the existence of a $\mathcal{B}$-measurable and integrable multifunction G such that

$$S^1_G(\mathcal{B}) = cl\{E^{\mathcal{B}} f : f \in S^1_F(\mathcal{F})\}$$

where $S^1_F(\mathcal{F})$ (resp. $S^1_G(\mathcal{B})$) is the set of all $\mathcal{F}$- (resp $\mathcal{B}$-) measurable and integrable selections of F (resp. G), here the closure is taken in $L^1_E(\mathcal{F})$. G is the multivalued conditional expectation of F relative to $\mathcal{B}$. The conditional expectation $G := E^{\mathcal{B}} F$ of Hiai and Umegaki can be defined as the essential supremum of $\{E^{\mathcal{B}} f : f \in S^1_F(\mathcal{F})\}$. For more information on the Hiai-Umegaki conditional expectation, see [22]. A closed convex valued integrable sequence $(X_n)_{n\in\mathbf{N}}$ is a supermartingale if X_n is $\mathcal{F}_n$-measurable for each $n \in \mathbf{N}$ and $E^{\mathcal{F}_n} X_{n+1} \subset X_n$ for each $n \in \mathbf{N}$. Recall that an adapted sequence $(f_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a regular integrable martingale if there is $f \in L^1_E(\mathcal{F})$ such that $f_n = E^{\mathcal{F}_n} f$ for each $n \in \mathbf{N}$. A nonempty closed convex set A in E is *weak ball-compact* if for any $x \in A$, for any $r > 0$, the set $A \cap [x + r\overline{B}_E]$ is $\sigma(E, E^*)$-compact.

We need a preliminary lemma.

LEMMA 8.1. *Assume that E is a separable Banach space having the RNP and the strong dual E^*_b of E is separable. Let $(X_n)_{n\in\mathbf{N}}$ be a uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying*:

(i) $\sup_{n\in\mathbf{N}} |X_n(\omega)| < \infty$ *for each* $\omega \in \Omega$.
(ii) *For each $A \in \mathcal{F}$, the set $\bigcup_{n=1}^\infty \int_A X_n \, dP$ is relatively weakly compact.*

Then there is $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the following properties:

(a) $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty)$ *a.s. for all* $x^* \in \overline{B}_{E^*}$.
(b) $\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega))$ *a.s. for all* $x \in E$.
(c) $E^{\mathcal{F}_m} X_\infty(\omega) \subset X_m(\omega)$ *a.s. for all* $m \in \mathbf{N}$.

Proof. (a) Let $D_1^* = (e_j^*)_{j\in\mathbf{N}}$ be a dense sequence in $\overline{B}_{E^*}$ for the topology associated with the dual norm. As $(X_n)_{n\in\mathbf{N}}$ is a uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, for each $j \in \mathbf{N}$, $(\delta^*(e_j^*, X_n))_{n\in\mathbf{N}}$ is a bounded real-valued supermartingale in $L^1_{\mathbf{R}}(\mathcal{F})$. So it converges a.s. for every $j \in \mathbf{N}$ to a function m_j in L^1. Thanks to (ii) we may apply [11, Theorem 3.4(b)] to the uniformly integrable $cwk(E)$ sequence $(X_n)_{n\in\mathbf{N}}$. This provides a subsequence $(X'_n)_{n\in\mathbf{N}}$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that for each $v \in L^\infty_{E^*}(\mathcal{F})$

$$\lim_{n\to\infty} \int_\Omega \delta^*(v, X'_n)\, dP = \int_\Omega \delta^*(v, X_\infty)\, dP.$$

So by identifying the limits we get

$$\lim_{n\to\infty} \delta^*(e_j^*, X_n) = m_j = \delta^*(e_j^*, X_\infty) \quad \text{a.s.}$$

Therefore (a) follows by using (i) and a density argument. Applying Lemma V.2.9 in [24] to the family of real-valued L^1-bounded submartingales

$$(\langle e_j^*, x\rangle - \delta^*(e_j^*, X_n))_{n\in\mathbf{N}}$$

gives (b). Now let $m < n$ and $A \in \mathcal{F}_m$. By the supermartingale property we have

$$\int_A \delta^*(e_j^*, X_n)\, dP \le \int_A \delta^*(e_j^*, X_m)\, dP.$$

It follows that

$$\int_A \delta^*(e_j^*, X_\infty)\, dP \le \int_A \delta^*(e_j^*, X_m)\, dP.$$

Therefore by taking the conditional expectation of X_∞ we get

$$\int_A \delta^*(e_j^*, E^{\mathcal{F}_m} X_\infty)\, dP = \int_A \delta^*(e_j^*, X_\infty)\, dP \le \int_A \delta^*(e_j^*, X_m)\, dP$$

thereby proving (c) (see Proposition III.35 in [13]). ∎

The following result is an extension of a similar one due to Choukairi [15, Theorem 2.14] dealing with reflexive separable Banach space and is a variant of a result due to Hess [20, Theorem 5.12].

THEOREM 8.2. *Assume that E is a separable Banach space having the RNP and the strong dual E^*_b of E is separable. Let $(X_n)_{n\in\mathbf{N}}$ be a weak-ball compact closed convex valued integrable supermartingale ($E^{\mathcal{F}_m} X_n \subset X_m$ for $m < n$) satisfying:*

(i) *$(X_n)_{n\in\mathbf{N}}$ admits a regular martingale selection $(f_n)_{n\in\mathbf{N}} = (E^{\mathcal{F}_n} f)_{n\in\mathbf{N}}$ in $L^1_E(\mathcal{F})$.*
(ii) *For each $k \in \mathbf{N}$, and for each $A \in \mathcal{F}$,*

$$\bigcup_{n=1}^{\infty} \int_A X_n \cap [f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_E]\, dP$$

is relatively weakly compact.

Then one can find a closed convex valued integrable multifunction X_∞ such that

$$M\text{-}\lim_n X_n = X_\infty \quad a.s.$$

$$E^{\mathcal{F}_n} X_\infty \subset X_n \quad a.s. \quad \forall n \in \mathbf{N}.$$

Proof. We will proceed in several steps.

Step 1. By our assumption there is an $f \in L^1_E(\mathcal{F})$ such that $f_n = E^{\mathcal{F}_n} f$ for all $n \in \mathbf{N}$. For each $k \in \mathbf{N}$, let us consider the multifunction

$$X_n^k = X_n \cap [f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_E].$$

We are going to check that $(X_n^k)_{n\in\mathbf{N}}$ is a uniformly integrable supermartingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Let $m < n$. As $X_n^k \subset X_n$, by the supermartingale property and by monotony of conditional expectation one has

$$E^{\mathcal{F}_m} X_n^k \subset E^{\mathcal{F}_m} X_n \subset X_m.$$

Taking the conditional expectation $E^{\mathcal{F}_m}$ in the inclusion

$$X_n^k \subset f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_E$$

yields

$$E^{\mathcal{F}_m} X_n^k \subset f_m + E^{\mathcal{F}_m}(|f| + k)\overline{B}_E.$$

Hence we get $E^{\mathcal{F}_m} X_n^k \subset X_m^k$. Note that $|X_n^k| \leq h_n^k$ for all $n \in \mathbf{N}$, where $h_n^k := |f_n| + E^{\mathcal{F}_n}(|f| + k)$. Further the uniformly integrable submartingale $(h_n^k)_{n\in\mathbf{N}}$ converges a.s. to a positive integrable function h^k. Hence there exists a positive constant r^k depending on $\omega \in \Omega$ such that $h_n^k \leq r^k$ a.s. for all $n \in \mathbf{N}$. So $|X_n^k| \leq r^k$ a.s. for all $n \in \mathbf{N}$. As $(X_n)_{n\in\mathbf{N}}$ is weak-ball compact closed convex valued, by (ii) we may apply Lemma 8.1 (a), (b), (c) to the uniformly integrable $cwk(E)$-valued supermartingale $(X_n^k)_{n\in\mathbf{N}}$. That provides $X_\infty^k \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and a negligible set N_k such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n^k(\omega)) = \delta^*(x^*, X_\infty^k(\omega)) \qquad \forall x^* \in \overline{B}_{E^*} \quad \forall \omega \in \Omega \setminus N_k, \tag{8.2.1}$$

$$\lim_{n\to\infty} d(x, X_n^k(\omega)) = d(x, X_\infty^k(\omega)) \qquad \forall x \in E \quad \forall \omega \in \Omega \setminus N_k, \tag{8.2.2}$$

$$E^{\mathcal{F}_n} X_\infty^k(\omega) \subset X_n^k(\omega) \qquad \forall n \in \mathbf{N} \quad \forall \omega \in \Omega \setminus N_k, \tag{8.2.3}$$

so that by (8.2.1) and (8.2.2)

$$\underset{n\to\infty}{M\text{-lim}}\, X_n^k(\omega) = X_\infty^k(\omega) \quad \forall \omega \in \Omega \setminus N_k. \tag{8.2.4}$$

Step 2. Convergence and conclusion. By construction, we have $X_n^k \subset X_n^{k+1}$, so that $(X_\infty^k)_{k\in\mathbf{N}}$ is increasing. Let us set $N = \bigcup_{k=1}^\infty N_k$ and

$$X_\infty(\omega) = \begin{cases} cl\Big[\bigcup_{k=1}^\infty X_\infty^k(\omega)\Big] & \text{if} \quad \omega \in \Omega \setminus N \\ 0 & \text{if} \quad \omega \in N. \end{cases}$$

We need to check that

$$\underset{n\to\infty}{M\text{-lim}}\, X_n(\omega) = X_\infty(\omega) \quad \text{a.s.}$$

Let $x \in w\text{-}lsX_n(\omega)$ with $\omega \notin N$. There is a sequence $(x_j)_{j\in\mathbf{N}}$ weakly converging to x with $x_j \in X_{n_j}(\omega)$. Pick a large enough integer $p \in \mathbf{N}$ such that $\|x_j\| \leq p$ for all $j \in \mathbf{N}$. By Jensen's inequality we have

$$\|x_j - f_{n_j}(\omega)\| = \|x_j - E^{\mathcal{F}_{n_j}}(f)(\omega)\| \leq p + E^{\mathcal{F}_{n_j}}|f|(\omega) = E^{\mathcal{F}_{n_j}}(|f| + p)(\omega),$$

therefore $x_j \in f_{n_j}(\omega) + E^{\mathcal{F}_{n_j}}(|f|+p)(\omega)\overline{B}_E$, and so $x_j \in X^p_{n_j}(\omega)$. Hence using (8.2.4) and the definition of X_∞ we get

$$x \in w\text{-}lsX^p_{n_j}(\omega) \subset X^p_\infty(\omega) \subset X_\infty(\omega).$$

By Theorem 2.1 in [21] we have

$$E^{\mathcal{F}_n}X_\infty(\omega) = cl\Big[\bigcup_{k=1}^{\infty} E^{\mathcal{F}_n}X^k_\infty(\omega)\Big] \quad \text{a.s.}$$

Hence by (8.2.3)

$$E^{\mathcal{F}_n}X_\infty(\omega) \subset cl\Big[\bigcup_{k=1}^{\infty} X^k_n(\omega)\Big] \subset X_n(\omega) \quad \text{a.s.} \tag{8.2.5}$$

Let $f \in S^1_{X_\infty}$. By (8.2.5) $E^{\mathcal{F}_n}f(\omega) \in X_n(\omega)$. By Levy's theorem $\lim_{n\to\infty} E^{\mathcal{F}_n}f = f$ a.s. Hence $f(\omega) \in s\text{-}liX_n(\omega)$ a.s. Taking a Castaing representation of X_∞ we get $X_\infty(\omega) \in s\text{-}liX_n(\omega)$ a.s. ∎

9. Application to the law of large numbers. Here we need some specific notation and definitions. Let Γ be a element in $L^1_{E^*}[E](\mathcal{F})$. The law (or distribution) P_Γ of Γ is given by $P_\Gamma(B) = P(\Gamma^{-1}B)$ for each $B \in \mathcal{B}(E^*_s)$. Two elements Γ and Δ in $L^1_{E^*}[E](\mathcal{F})$ are said to be equidistributed (or to have the same distribution) if $P_\Gamma = P_\Delta$. Two $L^1_{E^*}[E](\mathcal{F})$-mappings Γ and Δ are said to be independent if

$$P_{(\Gamma,\Delta)} = P_\Gamma \otimes P_\Delta.$$

Given Γ in $L^1_{E^*}[E](\mathcal{F})$, $\mathcal{A}_\Gamma$ is the σ-algebra on Ω generated by Γ. For shortness we provide a simple application to the law of large numbers in $L^1_{E^*}[E](\mathcal{F})$, see [9] for a related result with different approach. An element $\Gamma \in L^1_{E^*}[E](\mathcal{F})$ is often called random vector or random element.

PROPOSITION 9.1. *Let $(X_n)_{n\ge1}$ be a sequence of independent random elements in $L^1_{E^*}[E](\mathcal{F})$. Let*

$$S_n := X_1 + X_2 + \ldots + X_n$$

and assume that for all n and for every $j \in \{1,2,\ldots,n\}$, (S_n, X_1) and (S_n, X_j) have the same distribution. Then

$$w^*\text{-}\lim_{n\to\infty} \frac{1}{n} S_n = \int_\Omega X_1\, dP \quad a.s.$$

Proof. We will sketch the proof, using several arguments developed in [6] involving the existence of conditional expectation in Theorem 3.1.

Fact 1. Using the arguments of Theorem 3.2 in [6] we have

$$\forall n\ \forall j \in \{1,2,\ldots,n\} \quad E^{\mathcal{A}_{S_n}}X_j = E^{\mathcal{A}_{S_n}}X_1 \quad \text{a.s.} \tag{9.1.1}$$

Fact 2. Let $(x_p)_{p\in\mathbf{N}}$ be a dense sequence in the closed unit ball of E. Since $\langle x_p, X_1\rangle$ and $\mathcal{A}_{S_n}$ are independent, by a well known result (see e.g. [2, Theorem 10.1.4]) we get

$$E(\langle x_p, X_1\rangle | S_n, X_{n+1}, X_{n+2}, \ldots) = E^{\mathcal{A}_{S_n}}\langle x_p, X_1\rangle \quad \text{a.s.} \tag{9.1.2}$$

By combining (9.1.1) and (9.1.2) we get for a.s. $\omega \in \Omega$

$$\frac{1}{n}\langle x_p, S_n\rangle = E\Big(\frac{1}{n}\sum_{j=1}^{n}\langle x_p, X_j\rangle|S_n\Big) = \frac{1}{n}\sum_{j=1}^{n}E(\langle x_p, X_j\rangle|S_n)$$
$$= E(\langle x_p, X_1\rangle|S_n) = E(\langle x_p, X_1\rangle|S_n, X_{n+1}, X_{n+2}, \dots).$$

That implies by using the properties of conditional expectation and the separability of E,

$$\frac{1}{n}S_n = E(X_1|S_n) \quad a.s.$$

Let

$$\mathcal{G}_n = \sigma(\mathcal{A}_{S_n}, \mathcal{A}_{S_{n+1}}, \dots) \quad \forall n \geq 1.$$

Then we have

$$E^{\mathcal{A}_{S_n}}X_1 = E^{\mathcal{G}_n}X_1 \quad \text{a.s.}$$

Hence

$$w^*\text{-}\lim_{n\to\infty}\frac{1}{n}S_n = w^*\text{-}\lim_{n\to\infty}E^{\mathcal{G}_n}X_1 = E^{\mathcal{G}_\infty}X_1 \quad \text{a.s.}$$

where $\mathcal{G}_\infty := \bigcap_{n=1}^{\infty}\mathcal{G}_n$. We finish the proof by proving

Fact 3. $E^{\mathcal{G}_\infty}X_1 = \int_\Omega X_1\,dP$ and conclusion.

Observe that for each fixed integer $m \geq 1$,

$$w^*\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{j=m}^{n}X_j = E^{\mathcal{G}_\infty}X_1 \quad \text{a.s.}$$

So that $E^{\mathcal{G}_\infty}X_1$ is $\sigma(\mathcal{A}_{S_j}, j \geq m)$-measurable. By invoking the independence of $(\mathcal{A}_{S_n})_{n\geq 1}$ and the Kolmogorov's Zero-One Law, we conclude that $E^{\mathcal{G}_\infty}X_1 = \int_\Omega X_1\,dP$. ∎

References

[1] F. Akhiat, C. Castaing, F. Ezzaki, *Some various convergence results for multivalued martingales*, Adv. Math. Econ. 13 (2010), 1–33.

[2] H. Bauer, *Probability Theory and Elements of Measure Theory*, Holt, Rinehart and Winston, Inc., 1972.

[3] C. Castaing, *Méthode de compacité et de décomposition, applications: minimisation, convergence des martingales, lemme de Fatou multivoque*, Ann. Mat. Pura Appl. (4) 164 (1993), 51–75.

[4] C. Castaing, F. Ezzaki, *Some convergence results for multivalued martingales in the limit*, Sém. Anal. Convexe 20 (1990), exposé no. 1.

[5] C. Castaing, F. Ezzaki, *Décomposition et convergence des martingales à la limite à valeurs convexes faiblement compactes*, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), 695–698.

[6] C. Castaing, F. Ezzaki, *SLLN for convex random sets and random lower semicontinuous integrands*, Atti Sem. Mat. Fis. Univ. Modena 45 (1997), 527–553.

[7] C. Castaing, F. Ezzaki, C. Hess, *Convergence of conditional expectation for unbounded closed convex random sets*, Studia Math. 124 (1997), 133–148.

[8] C. Castaing, C. Hess, M. Saadoune, *Tightness conditions and integrability of the sequential weak upper limit of a sequence of multifunctions*, Adv. Math. Econ. 11 (2008), 11–44.

[9] C. Castaing, P. Raynaud de Fitte, $\mathfrak{S}$-*Uniform scalar integrability and strong laws of large numbers for Pettis-integrable functions with values in a separable locally convex space*, J. Theoret. Probab. 13 (2000), 93–134.

[10] C. Castaing, P. Raynaud de Fitte, M. Valadier, *Young Measures on Topological Spaces. With Applications in Control Theory and Probability Theory*, Math. Appl. 571, Kluwer Acad. Publ., Dordrecht, 2004.

[11] C. Castaing, M. Saadoune, *Dunford-Pettis-types theorem and convergences in set-valued integration*, J. Nonlinear Convex Anal. 1 (1999), 37–71.

[12] C. Castaing, M. Saadoune, *Convergences in a dual space with applications to Fatou Lemma*, Adv. Math. Econ. 12 (2009), 23–69.

[13] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, Berlin, 1977.

[14] A. Choukairi, *Sur la convergence d'espérances conditionnelles de multiapplications*, Séminaire d'Analyse Convexe, Montpellier, 1987, exposé no. 10.

[15] A. Choukairi-Dini, *M-convergence et régularité des martingales multivoques: épi-martingales*, Sém. Anal. Convexe, Montpellier, 1987, exposé no. 5; J. Multivariate Anal. 33 (1990), 49–71.

[16] A. Choukairi-Dini, *M-convergence des martingales. Applications aux amarts uniformes et aux intégrandes*, Sém. Anal. Convexe 19 (1989), exposé no. 7.

[17] A. Choukairi-Dini, *M-convergence des martingales (asymptotiques) multivoques, Épi-martingales*, C. R. Acad. Sci. Paris Sér. I Math. 309 (1989), 889–892.

[18] W. J. Davis, N. Ghoussoub, W. B. Johnson, S. Kwapień, B. Maurey, *Weak convergence of vector-valued martingales*, in: Probability in Banach Spaces 6 (Sandbjerg 1986), Birkhäuser, Boston, Ma, 1990, 41–50.

[19] S. Fitzpatrick, A. S. Lewis, *Weak-star convergence of convex sets*, J. Convex Anal. 13 (2006), 711–719.

[20] C. Hess, *On multivalued martingales whose values may be unbounded: martingale selectors and Mosco convergence*, J. Multivariate Anal. 39 (1991), 175–201.

[21] F. Hiai, *Convergence of conditional expectations and strong laws of large numbers for multivalued random variables*, Trans. Amer. Math. Soc. 291 (1985), 613–627.

[22] F. Hiai, H. Umegaki, *Integrals, conditional expectations and martingales of multivalued functions*, J. Multivariate Anal. 7 (1977), 149–182.

[23] U. Mosco, *Convergence of convex sets and solutions of variational inequalities*, Advanced in Math. 3 (1969), 510–585.

[24] J. Neveu, *Martingales à temps discret*, Masson et Cie, Paris, 1972.

[25] M. Talagrand, *Some structure results for martingales in the limit and pramarts*, Ann. Probab. 13 (1985), 1192–1203.

[26] M. Valadier, *On conditional expectation of random sets*, Ann. Mat. Pura Appl. (4) 126 (1980), 81–91.

[27] R. M. Wang, *Essential (convex) closure of a family of random sets and its applications*, J. Math. Anal. Appl. 262 (2001), 667–687.