ABSOLUTELY CONTINUOUS LINEAR OPERATORS ON KÖTHE-BOCHNER SPACES

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Abstract. Let $E$ be a Banach function space over a finite and atomless measure space $(\Omega, \Sigma, \mu)$ and let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces. A linear operator $T$ acting from the Köthe-Bochner space $E(X)$ to $Y$ is said to be absolutely continuous if $\|T(1_{A_n}, f)\|_Y \to 0$ whenever $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$. In this paper we examine absolutely continuous operators from $E(X)$ to $Y$. Moreover, we establish relationships between different classes of linear operators from $E(X)$ to $Y$.

1. Introduction and notation. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and let $B_X$ stand for the closed unit ball in $X$. Let $X^*$ and $Y^*$ stand for the Banach duals of $X$ and $Y$ respectively. Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural and real numbers.

Now we establish terminology concerning function spaces (see [AB], [KA], [Z]). Throughout the paper we assume that $(\Omega, \Sigma, \mu)$ is a finite and atomless measure space. By $1_A$ we will denote the characteristic function of a set $A \in \Sigma$. By $L^0$ we denote the corresponding space of $\mu$-equivalence classes of $\Sigma$-measurable real valued functions defined on $\Omega$. Let $(E, \|\cdot\|_E)$ be a Köthe function space in $L^0$, that is, $E$ is an ideal of $L^0$ with $\text{supp } E = \Omega$ and $\|\cdot\|_E$ is a Riesz norm.

Now we recall terminology and basic concepts from the theory of vector-valued function spaces (see [CM], [DU], [L]). By $L^0(\mu, X)$ we denote the space of $\mu$-equivalence classes of all strongly $\Sigma$-measurable functions $f : \Omega \to X$. For $f \in L^0(\mu, X)$ let us set $\tilde{f}(\omega) := \|f(\omega)\|_X$ for $\omega \in \Omega$. The linear space $E(X) = \{f \in L^0(\mu, X) : \tilde{f} \in E\}$ provided with the norm $\|f\|_{E(X)} := \|\tilde{f}\|_E$ is a Banach space and is called a Köthe-Bochner space.

Orlicz [O] and Orlicz and Wnuk [OW] defined and studied absolutely continuous operators acting from Banach function spaces $E$ to a Banach space $Y$. In this paper we...
extend the results of [OW] to the vector-valued setting, i.e., we study linear operators from \( E(X) \) to \( Y \).

For each \( u \in E^+ \) the set \( I_u = \{ f \in E(X) : \tilde{f} \leq u \} \) will be called an order interval in \( E(X) \).

Nowak ([N1], [N2]) studied order-bounded and order-weakly compact operators acting from \( E(X) \) to \( Y \). Recall that a linear operator \( T : E(X) \to Y \) is said to be order-weakly compact (resp. order-bounded) whenever for each \( u \in E^+ \) the set \( T(I_u) \) is relatively-weakly compact (resp. norm-bounded) in \( Y \). We will need the following result (see [N1, Theorem 2.3]).

**Proposition 1.1.** A linear operator \( T : E(X) \to Y \) is order-bounded if and only if \( T \) is \((\| \cdot \|_E, \| \cdot \|_Y)\)-continuous.

Moreover, if \( \| \cdot \|_E \) is order continuous and \( X \) is a reflexive Banach space, then every \((\| \cdot \|_E, \| \cdot \|_Y)\)-continuous linear operator \( T : E(X) \to Y \) is order-weakly compact (see [N2, Theorem 3.6]).

2. Absolutely continuous operators on Köthe-Bochner spaces

**Definition 2.1.** A linear operator \( T : E(X) \to Y \) will be called absolutely continuous whenever for every \( f \in E(X), T(\mathbb{1}_{A_n} f) \to 0 \) as \( \mu(A_n) \to 0, (A_n) \subset \Sigma \).

**Proposition 2.2.** If \( T : E(X) \to Y \) is absolutely continuous, then it maps order bounded sequences in \( E(X) \) with pairwise disjoint terms into null sequences.

**Proof.** Let \( u \in E^+ \) and let \( (f_n) \) be a sequence in \( I_u \) with pairwise disjoint terms, i.e., \( \text{supp} f_n \cap \text{supp} f_m = \emptyset \) if \( n \neq m \). Let \( f : \Omega \to X \) be the function defined by

\[
f(\omega) = \begin{cases} 
  f_n(\omega) & \text{if } \omega \in \text{supp} f_n, \ n = 1, 2, \ldots \\
  0 & \text{if } \omega \in \Omega \setminus \bigcup_{n=1}^{\infty} \text{supp} f_n.
\end{cases}
\]

Thus \( \tilde{f} \leq u \), so \( f \in I_u \).

Since \( f_n = \mathbb{1}_{\text{supp} f_n} f \) and \( \sum_{n=1}^{\infty} \mu(\text{supp} f_n) = \mu(\bigcup_{n=1}^{\infty} \text{supp} f_n) \leq \mu(\Omega) < \infty \), we get \( \mu(\text{supp} f_n) \to 0 \) and it follows that \( T(f_n) = T(\mathbb{1}_{\text{supp} f_n} f) \to 0 \) in \( Y \). \( \blacksquare \)

**Lemma 2.3.** If an operator \( T : E(X) \to Y \) is absolutely continuous, then for each \( f \in E(X) \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \| T(\mathbb{1}_A g) \|_Y \leq \varepsilon \) for every \( A \in \Sigma \) with \( \mu(A) \leq \delta \) and \( g \in I_{\tilde{f}} \).

**Proof.** Assume that there exist \( f \in E(X), \varepsilon > 0 \) and sequences \((A_n)\) in \( \Sigma \), \((g_n)\) in \( I_{\tilde{f}} \) such that \( \sum_{n=1}^{\infty} \mu(A_n) < \infty \) and \( \| T(\mathbb{1}_{A_n} g_n) \|_Y > \varepsilon \).

Since \( T \) is absolutely continuous, there exists \( k_1 \in \mathbb{N} \) such that \( \| T(\mathbb{1}_{\bigcup_{i=k_1}^{\infty} A_i}, g_1) \|_Y \leq \frac{\varepsilon}{2} \). Then, we can find \( k_2 > k_1 \) with \( \| T(\mathbb{1}_{\bigcup_{i=k_1}^{k_2 - 1} A_i}, g_{k_1}) \|_Y \leq \frac{\varepsilon}{2} \). Following this way we are able
to find a sequence \((k_n)\) such that \(\| \mathcal{T}(1_{\bigcup_{n=k_n+1}^\infty A_n}, g_{k_n}) \|_Y \leq \frac{\varepsilon}{2} \) for \(n \in \mathbb{N}\). Thus we have

\[
\varepsilon < \| \mathcal{T}(A_{k_n}, g_{k_n}) \|_Y = \| \mathcal{T}(1_{A_{k_n}} \cup \bigcup_{i=n+1}^\infty A_{k_i}, g_{k_n}) \|_Y + \| \mathcal{T}(1_{A_{k_n}} \setminus \bigcup_{i=n+1}^\infty A_{k_i}, g_{k_n}) \|_Y \leq \| \mathcal{T}(1_{A_{k_n}} \setminus \bigcup_{i=n+1}^\infty A_{k_i}, g_{k_n}) \|_Y \leq \| \mathcal{T}(1_{A_{k_n}} \setminus \bigcup_{i=n+1}^\infty A_{k_i}, g_{k_n}) \|_Y + \frac{\varepsilon}{2}.
\]

So

\[
(*) \quad \| \mathcal{T}(1_{A_{k_n}} \setminus \bigcup_{i=n+1}^\infty A_{k_i}, g_{k_n}) \|_Y > \frac{\varepsilon}{2} \quad \text{for} \quad n = 1, 2, \ldots.
\]

Let us put \(B_n = A_{k_n} \setminus \bigcup_{i=n+1}^\infty A_{k_i}\) and \(h_n = \mathcal{T}_n g_{k_n}\) for \(n \in \mathbb{N}\). Then the sets \(B_1, B_2, \ldots\) are pairwise disjoint and \(h_n \leq g_{k_n} \leq f\) for \(n \in \mathbb{N}\). By Proposition 2.2 we get \(\| \mathcal{T}(h_n) \|_Y \to 0\), which contradicts \((*)\).

**Theorem 2.4.** If a linear operator \(\mathcal{T} : E(X) \to Y\) is absolutely continuous, then it is \((\| \cdot \|_E(X), \| \cdot \|_Y)\)-continuous.

**Proof.** By Proposition 1.1 it is enough to show that \(\mathcal{T}(I_u)\) is bounded in \(Y\) for every \(u \in E^+\). If \(\mathcal{T}(I_u)\) were not bounded, then we would find a sequence \((f_n)\) in \(I_u\) such that \(\| \mathcal{T}(f_n) \|_Y \to \infty\). Thus we have \(\| \mathcal{T}(f_n) \|_Y > 2\) for some \(n_1 \in \mathbb{N}\). Using the absolute continuity of \(\mathcal{T}\) we can find \(k \in \mathbb{N}\) such that \(\| \mathcal{T}(1_{A_i} f_n) \|_Y < 1\) for every \(A \in \Sigma\) with \(\mu(A) \leq \frac{\mu(\Omega)}{k}\). Since the measure \(\mu\) is atomless, there exist pairwise disjoint sets \(A_1, \ldots, A_k\) in \(\Sigma\) such that \(\Omega = \bigcup_{i=1}^k A_i\) and \(\mu(A_i) = \frac{\mu(\Omega)}{k}\) for \(i = 1, \ldots, k\). Thus we have \(\| \mathcal{T}(1_{A_i} f_n) \|_Y < 1\) for \(i = 1, \ldots, k\) and \(\sup_n \| \mathcal{T}(1_{A_j} f_n) \|_Y = \infty\) for some \(j \in \{1, \ldots, k\}\). Putting \(g_1 = 1_{\Omega \setminus A_j} f_n\), we obtain \(\| \mathcal{T}(g_j) \|_Y > 1\). Moreover, we can find a natural number \(n_2 > n_1\) such that \(\| \mathcal{T}(1_{A_j} f_{n_2}) \|_Y > 2\). Using the absolute continuity of \(\mathcal{T}\) again, we can find \(m \in \mathbb{N}\) such that \(\| \mathcal{T}(1_{A_j} f_{n_2}) \|_Y < 1\) for every \(A \in \Sigma\) with \(\mu(A) \leq \frac{\mu(A)}{m}\).

Obviously, there exist pairwise disjoint sets \(I_{B_1}, \ldots, I_{B_m}\) in \(\Sigma\) such that \(A_j = \bigcup_{i=1}^m I_{B_i}\) and \(\mu(B_i) = \frac{\mu(A_j)}{m}\) for \(i = 1, \ldots, m\). Then we have \(\| \mathcal{T}(1_{B_i} f_{n_2}) \|_Y < 1\) for \(i = 1, \ldots, m\) and \(\sup_n \| \mathcal{T}(1_{B_j} f_{n_2}) \|_Y = \infty\) for some \(j \in \{1, \ldots, m\}\). Let us put \(g_2 = 1_{A_j \setminus B_i} f_{n_2}\). Note that \(\| \mathcal{T}(g_2) \|_Y > 1\) and \(\tilde{g}_1 \land \tilde{g}_2 = 0\).

By induction we can define a sequence \((g_n)\) of pairwise disjoint functions in \(I_u\) with \(\| \mathcal{T}(g_n) \|_Y > 1\) for \(n = 1, 2, \ldots\).

The last inequality contradicts Proposition 2.2 and it proves that \(\mathcal{T}\) is continuous.

**Definition 2.5.**

(i) A linear operator \(\mathcal{T} : E(X) \to Y\) is said to be \(\sigma\)-smooth if \(\tilde{f_n} \underset{(\alpha)}{\to} 0\) in \(E\) implies \(\| \mathcal{T}(f_n) \|_Y \to 0\).

(ii) A linear operator \(\mathcal{T} : E(X) \to Y\) is said to be smooth if \(\tilde{f_\alpha} \underset{(\alpha)}{\to} 0\) in \(E\) implies \(\| \mathcal{T}(f_\alpha) \|_Y \to Y\).

**Theorem 2.6.** Assume that \(L^\infty \subset E\). For a linear operator \(\mathcal{T} : E(X) \to Y\) the following statements are equivalent:

(i) \(\mathcal{T}\) is absolutely continuous.

(ii) \(\mathcal{T}\) is \(\sigma\)-smooth.
Proof. (i)⇒(ii) Assume that $T$ is absolutely continuous. Choose a sequence $(f_n)$ in $E(X)$ with $\tilde{f}_n \overset{(o)}{\longrightarrow} 0$ in $E$. Then there exists a decreasing sequence $(u_n)$ in $E^+$ such that $\tilde{f}_n \leq u_n \downarrow 0$ in $E$. Fix $\varepsilon > 0$. By Lemma 2.3 for $u_1$ there exists $\delta > 0$ such that $\sup_n \|T(1_A f_n)\|_Y \leq \frac{\varepsilon}{2}$ whenever $A \in \Sigma$ with $\mu(A) \leq \delta$. According to the Egoroff theorem there exists a set $A_0 \in \Sigma$ with $\mu(\Omega \setminus A_0) \leq \delta$ such that $1_{A_0} f_n(\omega) \longrightarrow 0$ uniformly on $\Omega$. It follows that $1_{A_0} f_n \longrightarrow 0$ in $(E(X), \| \cdot \|)$. Thus $\|T(1_{A_0} f_n)\|_Y \longrightarrow 0$ because by Theorem 2.4 $T$ is continuous. Hence

$$\|T(f_n)\|_Y \leq \|T(1_{A_0} f_n)\|_Y + \|T(1_{\Omega \setminus A_0} f_n)\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for sufficiently large $n$'s, i.e., $\|T(f_n)\|_Y \longrightarrow 0$.

(ii)⇒(i) It is sufficient to show that for every $f \in E(X)$ $\|T(1_{A_n} f)\|_Y \longrightarrow 0$ for every $(A_n)$ in $\Sigma$ with $\mu(A_n) \longrightarrow 0$. Let us take $f \in E(X)$ and a sequence $(A_n)$ in $\Sigma$ such that $\mu(A_n) \longrightarrow 0$. Obviously $1_{A_n} \tilde{f} \overset{(o)}{\longrightarrow} 0$ and it follows that $\|T(1_{A_n} f)\|_Y \longrightarrow 0$ because $T$ is $\sigma$-smooth. ■

Now we briefly recall terminology concerning locally solid topologies and the duality of $E(X)$ (see [FN], [N2]). A subset $H$ of $E(X)$ is said to be solid whenever $\tilde{f}_1 \leq \tilde{f}_2$ and $f_1 \in E(X)$, $f_2 \in H$ implies $f_1 \in H$. A linear topology $\tau$ on $E(X)$ is said to be locally solid if it has a local base at zero consisting of solid sets. A locally solid topology $\tau$ on $E(X)$ is said to be a Lebesgue topology whenever for any net $(f_\alpha)$ in $E(X)$, $\tilde{f}_\alpha \overset{(o)}{\longrightarrow} 0$ in $E$ implies $f_\alpha \longrightarrow 0$ for $\tau$.

A linear functional $F$ on $E(X)$ is said to be order continuous whenever $\tilde{f}_\alpha \overset{(o)}{\longrightarrow} 0$ in $E$ implies $F(f_\alpha) \longrightarrow 0$. The set consisting of all order continuous linear functionals on $E(X)$ will be denoted by $E(X)^\sim$ and called the order continuous dual of $E(X)$. Then $E(X)^* = E(X)^\sim$ if and only if the norm $\| \cdot \|_E$ is order continuous.

The following theorem will be of importance (see [N3, Theorem 4.1]).

**Theorem 2.7.** Assume that $X^*$ has the Radon-Nikodym property. Then the Mackey topology $\tau(E(X), E(X)^\sim)$ is a locally convex-solid Lebesgue topology on $E(X)$.

Now we are ready to state the following corollary.

**Corollary 2.8.** Assume that $L^\infty \subset E$ and $X^*$ has the Radon-Nikodym property. Then for a linear operator $T : E(X) \longrightarrow Y$ the following statements are equivalent:

(i) $y^* \circ T \in E(X)^\sim$ for every $y^* \in Y^*$.
(ii) $T$ is $(\sigma(E(X), E(X)^\sim), \sigma(Y, Y^*))$-continuous.
(iii) $T$ is $(\tau(E(X), E(X)^\sim), \| \cdot \|_Y)$-continuous.
(iv) $T$ is smooth.
(v) $T$ is $\sigma$-smooth.
(vi) $T$ is absolutely continuous.

**Proof.** (i)⇔(ii) See [AB, Theorem 9.26].

(ii)⇔(iii) See [AB, Example 11, p. 149].

(iii)⇒(iv) It follows from Theorem 2.7.

(iv)⇒(v) It is obvious.

(v)⇒(vi) See Theorem 2.6.

(v)⇒(i) It is obvious. ■
References


