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ABSOLUTELY CONTINUOUS LINEAR OPERATORS ON KÖTHE-BOCHNER SPACES

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Abstract. Let E be a Banach function space over a finite and atomless measure space (Ω, Σ, μ) and let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces. A linear operator T acting from the Köthe-Bochner space $E(X)$ to Y is said to be absolutely continuous if $\|T(\mathbb{1}_{A_n} f)\|_Y \to 0$ whenever $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$. In this paper we examine absolutely continuous operators from $E(X)$ to Y. Moreover, we establish relationships between different classes of linear operators from $E(X)$ to Y.

1. Introduction and notation. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces and let B_X stand for the closed unit ball in X. Let X^* and Y^* stand for the Banach duals of X and Y respectively. Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural and real numbers.

Now we establish terminology concerning function spaces (see [AB], [KA], [Z]). Throughout the paper we assume that (Ω, Σ, μ) is a finite and atomless measure space. By $\mathbb{1}_A$ we will denote the characteristic function of a set $A \in \Sigma$. By L^0 we denote the corresponding space of μ-equivalence classes of Σ-measurable real valued functions defined on Ω. Let $(E, \|\cdot\|_E)$ be a Köthe function space in L^0, that is, E is an ideal of L^0 with $\operatorname{supp} E = \Omega$ and $\|\cdot\|_E$ is a Riesz norm.

Now we recall terminology and basic concepts from the theory of vector-valued function spaces (see [CM], [DU], [L]). By $L^0(\mu, X)$ we denote the space of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \longrightarrow X$. For $f \in L^0(\mu, X)$ let us set $\widetilde{f}(\omega) := \|f(\omega)\|_X$ for $\omega \in \Omega$. The linear space $E(X) = \{f \in L^0(\mu, X) : \widetilde{f} \in E\}$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is a Banach space and is called a *Köthe-Bochner space*.

Orlicz [O] and Orlicz and Wnuk [OW] defined and studied absolutely continuous operators acting from Banach function spaces E to a Banach space Y. In this paper we

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extend the results of [OW] to the vector-valued setting, i.e., we study linear operators from $E(X)$ to Y.

For each $u \in E^+$ the set $I_u = \{f \in E(X) : \widetilde{f} \le u\}$ will be called an order interval in $E(X)$.

Nowak ([N1], [N2]) studied order-bounded and order-weakly compact operators acting from $E(X)$ to Y. Recall that a linear operator $T : E(X) \to Y$ is said to be order-weakly compact (resp. order-bounded) whenever for each $u \in E^+$ the set $T(I_u)$ is relatively-weakly compact (resp. norm-bounded) in Y. We will need the following result (see [N1, Theorem 2.3]).

PROPOSITION 1.1. *A linear operator $T : E(X) \longrightarrow Y$ is order-bounded if and only if T is $(\|\cdot\|_E, \|\cdot\|_Y)$-continuous.*

Moreover, if $\|\cdot\|_E$ is order continuous and X is a reflexive Banach space, then every $(\|\cdot\|_E, \|\cdot\|_Y)$-continuous linear operator $T : E(X) \longrightarrow Y$ is order-weakly compact (see [N2, Theorem 3.6]).

2. Absolutely continuous operators on Köthe-Bochner spaces

DEFINITION 2.1. A linear operator $T : E(X) \longrightarrow Y$ will be called *absolutely continuous* whenever for every $f \in E(X)$, $T(\mathbb{1}_{A_n} f) \longrightarrow 0$ as $\mu(A_n) \longrightarrow 0$, $(A_n) \subset \Sigma$.

PROPOSITION 2.2. *If $T : E(X) \longrightarrow Y$ is absolutely continuous, then it maps order bounded sequences in $E(X)$ with pairwise disjoint terms into null sequences.*

Proof. Let $u \in E^+$ and let (f_n) be a sequence in I_u with pairwise disjoint terms, i.e., $\operatorname{supp} f_n \cap \operatorname{supp} f_m = \emptyset$ if $n \ne m$. Let $f : \Omega \longrightarrow X$ be the function defined by

$$f(\omega) = \begin{cases} f_n(\omega) & \text{if } \omega \in \operatorname{supp} f_n,\ n = 1, 2, \dots \\ 0 & \text{if } \omega \in \Omega \setminus \bigcup_{n=1}^{\infty} \operatorname{supp} f_n\,. \end{cases}$$

Thus $\widetilde{f} \le u$, so $f \in I_u$.

Since $f_n = \mathbb{1}_{\operatorname{supp} f_n} f$ and $\sum_{n=1}^{\infty} \mu(\operatorname{supp} f_n) = \mu(\bigcup_{n=1}^{\infty} \operatorname{supp} f_n) \le \mu(\Omega) < \infty$, we get $\mu(\operatorname{supp} f_n) \longrightarrow 0$ and it follows that $T(f_n) = T(\mathbb{1}_{\operatorname{supp} f_n} f) \longrightarrow 0$ in Y. ∎

LEMMA 2.3. *If an operator $T : E(X) \longrightarrow Y$ is absolutely continuous, then for each $f \in E(X)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(\mathbb{1}_A\, g)\|_Y \le \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \le \delta$ and $g \in I_{\widetilde{f}}$.*

Proof. Assume that there exist $f \in E(X)$, $\varepsilon > 0$ and sequences (A_n) in Σ, (g_n) in $I_{\widetilde{f}}$ such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ and $\|T(\mathbb{1}_{A_n} g_n)\|_Y > \varepsilon$.

Since T is absolutely continuous, there exists $k_1 \in \mathbb{N}$ such that $\|T(\mathbb{1}_{\bigcup_{i=k_1}^{\infty} A_i}\, g_1)\|_Y \le \frac{\varepsilon}{2}$. Then, we can find $k_2 > k_1$ with $\|T(\mathbb{1}_{\bigcup_{i=k_2}^{\infty} A_i}\, g_{k_1})\|_Y \le \frac{\varepsilon}{2}$. Following this way we are able to find a sequence (k_n) such that $\|T(\mathbb{1}_{\bigcup_{i=k_{n+1}}^{\infty} A_i} g_{k_n})\|_Y \le \frac{\varepsilon}{2}$ for $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}\varepsilon < \|T(\mathbb{1}_{A_{k_n}} g_{k_n})\|_Y &= \|T(\mathbb{1}_{A_{k_n} \setminus \bigcup_{i=n+1}^{\infty} A_{k_i}} g_{k_n} + \mathbb{1}_{\bigcup_{i=n+1}^{\infty} A_{k_i}} g_{k_n})\|_Y \\ &\le \|T(\mathbb{1}_{A_{k_n} \setminus \bigcup_{i=n+1}^{\infty} A_{k_i}} g_{k_n})\|_Y + \|T(\mathbb{1}_{\bigcup_{i=n+1}^{\infty} A_{k_i}} g_{k_n})\|_Y \\ &\le \|T(\mathbb{1}_{A_{k_n} \setminus \bigcup_{i=n+1}^{\infty} A_{k_i}} g_{k_n})\|_Y + \frac{\varepsilon}{2},\end{aligned}$$

so

$$(*) \qquad \|T(\mathbb{1}_{A_{k_n} \setminus \bigcup_{i=n+1}^{\infty} A_{k_i}} g_{k_n})\|_Y > \frac{\varepsilon}{2} \quad \text{for} \quad n = 1, 2, \ldots .$$

Let us put $B_n = A_{k_n} \setminus \bigcup_{i=n+1}^{\infty} A_{k_i}$ and $h_n = \mathbb{1}_{B_n} g_{k_n}$ for $n \in \mathbb{N}$. Then the sets $B_1, B_2, \ldots$ are pairwise disjoint and $\widetilde{h}_n \le \widetilde{g}_{k_n} \le \widetilde{f}$ for $n \in \mathbb{N}$. By Proposition 2.2 we get $\|T(h_n)\|_Y \longrightarrow 0$, which contradicts $(*)$. ∎

THEOREM 2.4. *If a linear operator $T : E(X) \longrightarrow Y$ is absolutely continuous, then it is $(\|\cdot\|_{E(X)}, \|\cdot\|_Y)$-continuous.*

Proof. By Proposition 1.1 it is enough to show that $T(I_u)$ is bounded in Y for every $u \in E^+$. If $T(I_u)$ were not bounded, then we would find a sequence (f_n) in I_u such that $\|T(f_n)\|_Y \longrightarrow \infty$. Thus we have $\|T(f_{n_1})\|_Y > 2$ for some $n_1 \in \mathbb{N}$. Using the absolute continuity of T we can find $k \in \mathbb{N}$ such that $\|T(\mathbb{1}_A f_{n_1})\|_Y < 1$ for every $A \in \Sigma$ with $\mu(A) \le \frac{\mu(\Omega)}{k}$. Since the measure μ is atomless, there exist pairwise disjoint sets $A_1, \ldots, A_k$ in Σ such that $\Omega = \bigcup_{i=1}^{k} A_i$ and $\mu(A_i) = \frac{\mu(\Omega)}{k}$ for $i = 1, \ldots, k$. Thus we have $\|T(\mathbb{1}_{A_i} f_{n_1})\|_Y < 1$ for $i = 1, \ldots, k$ and $\sup_n \|T(\mathbb{1}_{A_j} f_n)\|_Y = \infty$ for some $j \in \{1, \ldots, k\}$. Putting $g_1 = \mathbb{1}_{\Omega \setminus A_j} f_{n_1}$, we obtain $\|T(g_1)\|_Y > 1$. Moreover, we can find a natural number $n_2 > n_1$ such that $\|T(\mathbb{1}_{A_j} f_{n_2}\|_Y > 2$. Using the absolute continuity of T again, we can find $m \in \mathbb{N}$ such that $\|T(\mathbb{1}_A f_{n_2})\|_Y < 1$ for every $A \in \Sigma$, $A \subset A_j$ with $\mu(A) \le \frac{\mu(A_j)}{m}$. Obviously, there exist pairwise disjoint sets $B_1, \ldots, B_m$ in Σ such that $A_j = \bigcup_{i=1}^{m} B_i$ and $\mu(B_i) = \frac{\mu(A_j)}{m}$ for $i = 1, \ldots, m$. Then we have $\|T(\mathbb{1}_{B_i} f_{n_2})\|_Y < 1$ for $i = 1, \ldots, m$ and $\sup_n \|T(\mathbb{1}_{B_l} f_n)\|_Y = \infty$ for some $j \in \{1, \ldots, m\}$. Let us put $g_2 = \mathbb{1}_{A_j \setminus B_l} f_{n_2}$. Note that $\|T(g_2)\|_Y > 1$ and $\widetilde{g}_1 \wedge \widetilde{g}_2 = 0$.

By induction we can define a sequence (g_n) of pairwise disjoint functions in I_u with $\|T(g_n)\|_Y > 1$ for $n = 1, 2, \ldots$.

The last inequality contradicts Proposition 2.2 and it proves that T is continuous. ∎

Now we distinguish some classes of linear operators acting from $E(X)$ to Y.

DEFINITION 2.5.

(i) A linear operator $T : E(X) \longrightarrow Y$ is said to be *σ-smooth* if $\widetilde{f}_n \xrightarrow{(o)} 0$ in E implies $\|T(f_n)\|_Y \longrightarrow 0$.

(ii) A linear operator $T : E(X) \longrightarrow Y$ is said to be *smooth* if $\widetilde{f}_\alpha \xrightarrow{(o)} 0$ in E implies $\|T(f_\alpha)\|_Y \longrightarrow Y$.

THEOREM 2.6. *Assume that $L^\infty \subset E$. For a linear operator $T : E(X) \longrightarrow Y$ the following statements are equivalent:*

(i) *T is absolutely continuous.*
(ii) *T is σ-smooth.*

Proof. (i)$\Longrightarrow$(ii) Assume that T is absolutely continuous. Choose a sequence (f_n) in $E(X)$ with $\widetilde{f}_n \xrightarrow{(o)} 0$ in E. Then there exists a decreasing sequence (u_n) in E^+ such that $\widetilde{f}_n \leq u_n \downarrow 0$ in E. Fix $\varepsilon > 0$. By Lemma 2.3 for u_1 there exists $\delta > 0$ such that $\sup_n \|T(\mathbb{1}_A f_n)\|_y \leq \frac{\varepsilon}{2}$ whenever $A \in \Sigma$ with $\mu(A) \leq \delta$. According to the Egoroff theorem there exists a set $A_0 \in \Sigma$ with $\mu(\Omega \setminus A_0) \leq \delta$ such that $\mathbb{1}_{A_0} f_n(\omega) \longrightarrow 0$ uniformly on Ω. It follows that $\mathbb{1}_{A_0} f_n \longrightarrow 0$ in $(E(X), \|\cdot\|)$. Thus $\|T(\mathbb{1}_{A_0} f_n)\|_Y \longrightarrow 0$ because by Theorem 2.4 T is continuous. Hence

$$\|T(f_n)\|_Y \leq \|T(\mathbb{1}_{A_0} f_n)\|_Y + \|T(\mathbb{1}_{\Omega \setminus A_0} f_n)\|_Y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for sufficiently large n's, i.e., $\|T(f_n)\|_Y \longrightarrow 0$.

(ii)$\Longrightarrow$(i) It is sufficient to show that for every $f \in E(X)$ $\|T(\mathbb{1}_{A_n} f)\|_Y \longrightarrow 0$ for every (A_n) in Σ with $\mu(A_n) \longrightarrow 0$. Let us take $f \in E(X)$ and a sequence (A_n) in Σ such that $\mu(A_n) \longrightarrow 0$. Obviously $\mathbb{1}_{A_n} \widetilde{f} \xrightarrow{(o)} 0$ and it follows that $\|T(\mathbb{1}_{A_n} f)\|_Y \longrightarrow 0$ because T is σ-smooth. ∎

Now we briefly recall terminology concerning locally solid topologies and the duality of $E(X)$ (see [FN], [N2]). A subset H of $E(X)$ is said to be *solid* whenever $\widetilde{f}_1 \leq \widetilde{f}_2$ and $f_1 \in E(X)$, $f_2 \in H$ implies $f_1 \in H$. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets. A locally solid topology τ on $E(X)$ is said to be a Lebesgue topology whenever for any net (f_α) in $E(X)$, $\widetilde{f}_\alpha \xrightarrow{(o)} 0$ in E implies $f_\alpha \longrightarrow 0$ for τ.

A linear functional F on $E(X)$ is said to be *order continuous* whenever $\widetilde{f}_\alpha \xrightarrow{(o)} 0$ in E implies $F(f_\alpha) \longrightarrow 0$. The set consisting of all order continuous linear functionals on $E(X)$ will be denoted by $E(X)_n^\sim$ and called the *order* continuous dual of $E(X)$. Then $E(X)^* = E(X)_n^\sim$ if and only if the norm $\|\cdot\|_E$ is order continuous.

The following theorem will be of importance (see [N3, Theorem 4.1]).

THEOREM 2.7. *Assume that X^* has the Radon-Nikodym property. Then the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is a locally convex-solid Lebesgue topology on $E(X)$.*

Now we are ready to state the following corollary.

COROLLARY 2.8. *Assume that $L^\infty \subset E$ and X^* has the Radon-Nikodym property. Then for a linear operator $T : E(X) \longrightarrow Y$ the following statements are equivalent:*

(i) *$y^* \circ T \in E(X)_n^\sim$ for every $y^* \in Y^*$.*
(ii) *T is $(\sigma(E(X), E(X)_n^\sim), \sigma(Y, Y^*))$-continuous.*
(iii) *T is $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$-continuous.*
(iv) *T is smooth.*
(v) *T is σ-smooth.*
(vi) *T is absolutely continuous.*

Proof. (i)$\Longleftrightarrow$(ii) See [AB, Theorem 9.26].

(ii)$\Longleftrightarrow$(iii) See [AB, Example 11, p. 149].

(iii)$\Longrightarrow$(iv) It follows from Theorem 2.7.

(iv)$\Longrightarrow$(v) It is obvious.

(v)$\Longleftrightarrow$(vi) See Theorem 2.6.

(v)$\Longrightarrow$(i) It is obvious. ∎

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