

SINGULAR INTEGRAL OPERATORS ON NAKANO SPACES WITH WEIGHTS HAVING FINITE SETS OF DISCONTINUITIES

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To the memory of Professor Israel Gohberg (23.08.1928–12.10.2009)

Abstract. In 1968, Gohberg and Krupnik found a Fredholm criterion for singular integral operators of the form $aP + bQ$, where a, b are piecewise continuous functions and P, Q are complementary projections associated to the Cauchy singular integral operator, acting on Lebesgue spaces over Lyapunov curves. We extend this result to the case of Nakano spaces (also known as variable Lebesgue spaces) with certain weights having finite sets of discontinuities on arbitrary Carleson curves.

1. Introduction. We say that a rectifiable curve Γ in the complex plane is simple if it is homeomorphic to a segment or to a circle. We equip Γ with Lebesgue length measure $|d\tau|$. The *Cauchy singular integral* of $f \in L^1(\Gamma)$ is defined by

$$(Sf)(t) := \frac{1}{\pi i} \int_{\Gamma} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma).$$

This integral is understood in the principal value sense, that is,

$$\int_{\Gamma} \frac{f(\tau)}{\tau - t} d\tau := \lim_{R \rightarrow 0} \int_{\Gamma \setminus \Gamma(t, R)} \frac{f(\tau)}{\tau - t} d\tau,$$

where $\Gamma(t, R) := \{\tau \in \Gamma : |\tau - t| < R\}$ for $R > 0$. David [3] (see also [1, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator S on the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$, if and only if Γ is a *Carleson (Ahlfors–David)*

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regular) curve, that is,

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{|\Gamma(t, R)|}{R} < \infty,$$

where for any measurable set $\Omega \subset \Gamma$ the symbol $|\Omega|$ denotes its measure.

A measurable function $w : \Gamma \rightarrow [0, \infty]$ is referred to as a *weight function* or simply a *weight* if $0 < w(\tau) < \infty$ for almost all $\tau \in \Gamma$. Suppose $p : \Gamma \rightarrow [1, \infty]$ is a measurable a.e. finite function. Denote by $L^{p(\cdot)}(\Gamma, w)$ the set of all measurable complex-valued functions f on Γ such that

$$\int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| < \infty$$

for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot), w} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| \leq 1 \right\}.$$

If p is constant, then $L^{p(\cdot)}(\Gamma, w)$ is nothing else but the weighted Lebesgue space. Therefore, it is natural to refer to $L^{p(\cdot)}(\Gamma, w)$ as a *weighted generalized Lebesgue space with variable exponent* or simply as a *weighted variable Lebesgue space*. This is a special case of Musielak-Orlicz spaces [33] (see also [28]). Nakano [34] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma, w)$ are referred to as weighted Nakano spaces.

THEOREM 1.1 (Kokilashvili, Paataashvili, S. Samko). *Suppose Γ is a simple rectifiable curve and $p : \Gamma \rightarrow (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition*

$$|p(\tau) - p(t)| \leq -C_{\Gamma} / \log |\tau - t| \quad \text{whenever} \quad |\tau - t| \leq 1/2, \quad (1)$$

where C_{Γ} is a positive constant depending only on Γ . Let $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. The Cauchy singular integral operator S is bounded on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight given by

$$w(\tau) = \prod_{j=1}^n |\tau - t_j|^{\lambda_j} \quad (\tau \in \Gamma) \quad (2)$$

if and only if Γ is a Carleson curve and $0 < 1/p(t_j) + \lambda_j < 1$ for all $j \in \{1, \dots, n\}$.

For the case of constant p and sufficiently smooth curves, the sufficiency portion of the above result was obtained more than fifty years ago by Khvedelidze [17] (see also [8, Chap. 1, Theorem 4.1]). The necessity portion for constant p goes back to Gohberg and Krupnik [7]. For the complete solution of the boundedness problem for the operator S on weighted standard Lebesgue spaces $L^p(\Gamma, w)$ we refer to the survey paper by Dynkin [4], to the monographs by Böttcher and Yu. Karlovich [1], by Khuskivadze, Kokilashvili, and Paataashvili [16], and by Genebashvili, Gogatishvili, Kokilashvili, and Krbec [5].

Theorem 1.1 was proved in [21, Theorem A]. Later on, Kokilashvili, N. Samko, and S. Samko [24, Theorem 4.3] generalized the sufficiency portion of Theorem 1.1 to the case

of radial oscillating weights

$$w(\tau) = \prod_{j=1}^n \omega_j(|\tau - t_j|) \quad (\tau \in \Gamma), \quad (3)$$

where $\omega_j : (0, |\Gamma|] \rightarrow (0, \infty)$ are some continuous functions oscillating at zero. Those sufficient boundedness conditions are expressed in terms of the Matuszewska-Orlicz indices [31, 32] (see also [29, 30]) of the functions ω_j . The author observed that those conditions are also necessary for the boundedness of the operator S on the weighted Nakano space $L^{p(\cdot)}(\Gamma, w)$ in the case of Jordan curves Γ (see [12, Corollary 4.3] and also [13]). Recall that a rectifiable curve in the complex plane is said to be *Jordan* if it is homeomorphic to a circle.

Now fix $t \in \Gamma$ and assume that w is a weight such that the operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$. In the spectral theory of one-dimensional singular integral operators it is important to know whether the operator S is also bounded on the space $L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma} w)$, where

$$\varphi_{t,\gamma}(\tau) := |(\tau - t)^\gamma|$$

and γ is an arbitrary complex number. For standard Lebesgue spaces and arbitrary Muckenhoupt weights such γ are completely characterized by Böttcher and Yu. Karlovich [1, Chap. 3]. Notice that if γ is the imaginary unit, then $\varphi_{t,i}$ coincides with

$$\eta_t(\tau) := e^{-\arg(\tau-t)}$$

(here and in what follows we choose a continuous branch of the argument on $\Gamma \setminus \{t\}$), and this function lies beyond the class of radial oscillating weights considered by Kokilashvili, N. Samko, and S. Samko [23, 24]. The author [15, Theorem 2.1] found necessary and sufficient conditions for the boundedness of the operator S on the space $L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma})$.

Our first aim in this paper is to generalize known boundedness results for the operator S on the space $L^{p(\cdot)}(\Gamma, w)$ to the case of weights of the form $w(\tau) = \prod_{j=1}^n \psi_j(\tau)$ where each ψ_j is a continuous positive function on $\Gamma \setminus \{t_j\}$ and $t_1, \dots, t_n \in \Gamma$ are pairwise distinct points. In particular, we allow functions ψ_j of the form $\psi_j(\tau) = (\eta_{t_j}(\tau))^x \omega_j(|\tau - t_j|)$ where $x \in \mathbb{R}$ and ω_j is an oscillating function as in [12, 13, 23, 24].

To formulate our first main result explicitly, we need some definitions. Following [1, Section 1.4], a function $\varrho : (0, \infty) \rightarrow (0, \infty]$ is said to be *regular* if it is bounded from above in some open neighborhood of the point 1. A function $\varrho : (0, \infty) \rightarrow (0, \infty]$ is said to be *submultiplicative* if $\varrho(xy) \leq \varrho(x)\varrho(y)$ for all $x, y \in (0, \infty)$. Clearly, if $\varrho : (0, \infty) \rightarrow (0, \infty]$ is regular and submultiplicative, then $\varrho(x)$ is finite for all $x \in (0, \infty)$. Given a regular submultiplicative function $\varrho : (0, \infty) \rightarrow (0, \infty)$, one defines

$$\alpha(\varrho) := \sup_{x \in (0,1)} \frac{\log \varrho(x)}{\log x}, \quad \beta(\varrho) := \inf_{x \in (1,\infty)} \frac{\log \varrho(x)}{\log x}. \quad (4)$$

One can show (see Theorem 2.1) that $-\infty < \alpha(\varrho) \leq \beta(\varrho) < +\infty$. Thus it is natural to call $\alpha(\varrho)$ and $\beta(\varrho)$ the lower and upper indices of ϱ , respectively.

Fix $t \in \Gamma$ and $d_t := \max_{\tau \in \Gamma} |\tau - t|$. Following [1, Section 1.5], for every continuous function

$\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$, we define

$$(W_t \psi)(x) := \begin{cases} \sup_{0 < R \leq d_t} \left(\max_{\tau \in \Gamma: |\tau-t|=xR} \psi(\tau) / \min_{\tau \in \Gamma: |\tau-t|=R} \psi(\tau) \right) & \text{for } x \in (0, 1], \\ \sup_{0 < R \leq d_t} \left(\max_{\tau \in \Gamma: |\tau-t|=R} \psi(\tau) / \min_{\tau \in \Gamma: |\tau-t|=x^{-1}R} \psi(\tau) \right) & \text{for } x \in [1, \infty) \end{cases}$$

and

$$\begin{aligned} (W_t^0 \psi)(x) &= \limsup_{R \rightarrow 0} \left(\max_{\tau \in \Gamma: |\tau-t|=xR} \psi(\tau) / \min_{\tau \in \Gamma: |\tau-t|=R} \psi(\tau) \right) \\ &= \limsup_{R \rightarrow 0} \left(\max_{\tau \in \Gamma: |\tau-t|=R} \psi(\tau) / \min_{\tau \in \Gamma: |\tau-t|=x^{-1}R} \psi(\tau) \right) \end{aligned}$$

for $x \in \mathbb{R}$. The function $W_t \psi$ is always submultiplicative. Moreover, if $W_t \psi$ is regular, then $W_t^0 \psi$ is also regular and submultiplicative and

$$\alpha(W_t \psi) = \alpha(W_t^0 \psi), \quad \beta(W_t \psi) = \beta(W_t^0 \psi)$$

(see [1, Lemmas 1.15 and 1.16]). Our first main result is the following.

THEOREM 1.2. *Suppose Γ is a simple rectifiable curve and $p : \Gamma \rightarrow (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition (1). Let $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points and $\psi_j : \Gamma \setminus \{t_j\} \rightarrow (0, \infty)$ be continuous functions such that the functions $W_{t_j} \psi_j$ are regular for all $j \in \{1, \dots, n\}$.*

(a) *If Γ is a simple Carleson curve and*

$$0 < 1/p(t_j) + \alpha(W_{t_j}^0 \psi_j), \quad 1/p(t_j) + \beta(W_{t_j}^0 \psi_j) < 1 \quad \text{for all } j \in \{1, \dots, n\}, \quad (5)$$

then the operator S is bounded on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight w given by

$$w(\tau) := \prod_{j=1}^n \psi_j(\tau) \quad (\tau \in \Gamma). \quad (6)$$

(b) *If the operator S is bounded on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight w given by (6), then Γ is a Carleson curve and*

$$0 \leq 1/p(t_j) + \alpha(W_{t_j}^0 \psi_j), \quad 1/p(t_j) + \beta(W_{t_j}^0 \psi_j) \leq 1 \quad \text{for all } j \in \{1, \dots, n\}.$$

(c) *If Γ is a rectifiable Jordan curve and the operator S is bounded on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight w given by (6), then Γ is a Carleson curve and conditions (5) are fulfilled.*

A bounded linear operator on a Banach space X is said to be Fredholm if its image $\text{Im } A$ is closed in X and the numbers $\dim \text{Ker } A$ and $\dim(X/\text{Im } A)$ are finite.

We equip a rectifiable Jordan curve Γ with the counter-clockwise orientation. Without loss of generality we will assume that the origin lies inside the domain bounded by Γ . By $PC(\Gamma)$ we denote the set of all $a \in L^\infty(\Gamma)$ for which the one-sided limits

$$a(t \pm 0) := \lim_{\tau \rightarrow t \pm 0} a(\tau)$$

exist and are finite at each point $t \in \Gamma$; here $\tau \rightarrow t-0$ means that τ approaches t following the orientation of Γ , while $\tau \rightarrow t+0$ means that τ goes to t in the opposite direction. Functions in $PC(\Gamma)$ are called *piecewise continuous* functions.

In 1968, Gohberg and Krupnik [6, Theorem 4] (see also [8, Chap. 9, Theorem 3.1]) found criteria for one-sided invertibility of one-dimensional singular integral operators of the form

$$A = aP + bQ, \quad \text{where } a, b \in PC(\Gamma), \quad P := (I + S)/2, \quad Q := (I - S)/2$$

acting on standard Lebesgue spaces $L^p(\Gamma)$ over Lyapunov curves. Their Fredholm theory for one-dimensional singular integral operators on standard Lebesgue spaces $L^p(\Gamma, w)$ with Khvedelidze weights (2) over Lyapunov curves is presented in the monograph [8] first published in Russian in 1973. Generalizations of this theory to the case of arbitrary Muckenhoupt weights and arbitrary Carleson curves are contained in the monograph by Böttcher and Yu. Karlovich [1].

Fredholmness of one-dimensional singular integral operators on Nakano spaces (variable Lebesgue spaces) over sufficiently smooth curves was studied for the first time by Kokilashvili and S. Samko [25]. The closely related Riemann–Hilbert boundary value problem in weighted classes of Cauchy type integrals with density in $L^{p(\cdot)}(\Gamma)$ was considered by Kokilashvili, Paataashvili, and S. Samko [18, 19, 20, 22]. The author [12] found a Fredholm criterion for an arbitrary operator in the Banach algebra of one-dimensional singular integral operators with piecewise continuous coefficients acting on Nakano spaces $L^{p(\cdot)}(\Gamma, w)$ with radial oscillating weights (3) over so-called logarithmic Carleson curves. Roughly speaking, logarithmic Carleson curves are Carleson curves Γ for which the weight $\eta_t(\tau)$ is equivalent to a power weight $|\tau - t|^{\lambda_t}$ with some $\lambda_t \in \mathbb{R}$ for each $t \in \Gamma$. Further, this technical assumption on Carleson curves was removed in [15] but only in the non-weighted case.

The aim of this paper is to prove an analogue of the Gohberg-Krupnik Fredholm criterion for the operator $aP + bQ$ acting on $L^{p(\cdot)}(\Gamma, w)$ in the case of arbitrary Carleson curves and a wide class of weights, in particular, including radial oscillating weights (3). Having this result at hands, one can construct a Fredholm theory for the Banach algebra of singular integral operators with piecewise continuous coefficients by using the machinery developed in [1] exactly in the same way as it was done in [12, 15]. We will not present these results in this paper.

Now we prepare the formulation of our main result. Let $L^{p(\cdot)}(\Gamma, w)$ be as in Theorem 1.2. One can show (see Section 6.1) that the functions $\alpha_t^*, \beta_t^* : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\alpha_{t_j}^*(x) := \alpha(W_{t_j}^0(\eta_t^x \psi_j)), \quad \beta_{t_j}^*(x) := \beta(W_{t_j}^0(\eta_t^x \psi_j)) \quad (7)$$

for $j \in \{1, \dots, n\}$ and by

$$\alpha_t^*(x) := \alpha(W_t^0(\eta_t^x)), \quad \beta_t^*(x) := \beta(W_t^0(\eta_t^x)) \quad (8)$$

for $t \notin \Gamma \setminus \{t_1, \dots, t_n\}$ are well-defined and the set

$$Y(p(t), \alpha_t^*, \beta_t^*) := \{\gamma = x + iy \in \mathbb{C} : 1/p(t) + \alpha_t^*(x) \leq y \leq 1/p(t) + \beta_t^*(x)\}$$

is connected and contains points with arbitrary real parts. Given $z_1, z_2 \in \mathbb{C}$, let

$$\mathcal{L}(z_1, z_2; p(t), \alpha_t^*, \beta_t^*) := \{M_{z_1, z_2}(e^{2\pi\gamma}) : \gamma \in Y(p(t), \alpha_t^*, \beta_t^*)\} \cup \{z_1, z_2\},$$

where

$$M_{z_1, z_2}(\zeta) := (z_2\zeta - z_1)/(\zeta - 1) \quad (9)$$

is the Möbius transform. The set $\mathcal{L}(z_1, z_2; p(t), \alpha_t^*, \beta_t^*)$ is referred to as the *leaf* about (or between) z_1 and z_2 determined by $p(t), \alpha_t^*, \beta_t^*$. This is a connected set containing z_1 and z_2 for every $t \in \Gamma$.

For $a \in PC(\Gamma)$, denote by $\mathcal{R}(a)$ the essential range of a , i.e. let $\mathcal{R}(a)$ be the set

$$\bigcup_{t \in \Gamma} \{a(t-0), a(t+0)\}.$$

Let J_a stand for the set of all points at which a has a jump. Clearly, we may write

$$\mathcal{R}(a) = \bigcup_{t \in \Gamma \setminus J_a} \{a(t)\} \cup \bigcup_{t \in J_a} \{a(t-0), a(t+0)\}.$$

We will say that a function $a \in PC(\Gamma)$ is $L^{p(\cdot)}(\Gamma, w)$ -nonsingular if

$$0 \notin \mathcal{R}(a) \cup \bigcup_{t \in J_a} \mathcal{L}(a(t-0), a(t+0); p(t), \alpha_t^*, \beta_t^*).$$

Our second main result reads as follows.

THEOREM 1.3. *Suppose Γ is a Carleson Jordan curve, $a, b \in PC(\Gamma)$, and $p : \Gamma \rightarrow (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition (1). Let $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points and $\psi_j : \Gamma \setminus \{t_j\} \rightarrow (0, \infty)$ be continuous functions such that the functions $W_{t_j} \psi_j$ are regular and conditions (5) are fulfilled for all $j \in \{1, \dots, n\}$. The operator $aP + bQ$ is Fredholm on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight w given by (6) if and only if $\inf_{t \in \Gamma} |b(t)| > 0$ and the function a/b is $L^{p(\cdot)}(\Gamma, w)$ -nonsingular.*

For $b = 1$, the above result generalizes [12, Theorem 4.5], where the weights of the form (3) were considered over so-called logarithmic Carleson curves, and [15, Theorem 2.2], where underlying curves were arbitrary Carleson curves but no weights were involved.

Although the main results of this paper are new, the methods of their proofs are not new and known to experts in the field. We decided to provide selfcontained proofs with complete formulations of auxiliary results taken from other publications. So, this paper can be considered as a short survey on the topic.

The paper is organized as follows. In Section 2, we collect some auxiliary results on indices of submultiplicative functions associated with curves and weights. In Section 3, following the approach of Kokilashvili, N. Samko, and S. Samko [23], we prove that conditions (5) are sufficient for the boundedness of the maximal operator on Nakano spaces $L^{p(\cdot)}(\Gamma, w)$ with weights of the form (6). With the aid of this result, we prove Theorem 1.2 in Section 4. Section 5 contains basic results on singular integral operators with L^∞ coefficients on weighted Nakano spaces. Their proofs are analogous to the case of standard weighted Lebesgue spaces (see e.g. [8, Chap. 7-8]). In Section 6 we prove Theorem 1.3 following the approach of Böttcher and Yu. Karlovich [1, Chap. 7] (see also [12, 15]). Note that Theorem 1.2 plays a crucial role in the proof of Theorem 1.3.

2. Indices of submultiplicative functions

2.1. Indices as limits. The indices of a regular submultiplicative function defined by (4) can be calculated as limits as $x \rightarrow 0$ and $x \rightarrow \infty$, respectively. The proof of the following result can be found e.g. in [1, Theorem 1.13].

THEOREM 2.1 (well-known). *If a function $\varrho : (0, \infty) \rightarrow (0, \infty)$ is regular and submultiplicative, then*

$$\alpha(\varrho) = \lim_{x \rightarrow 0} \frac{\log \varrho(x)}{\log x}, \quad \beta(\varrho) = \lim_{x \rightarrow \infty} \frac{\log \varrho(x)}{\log x}$$

and $-\infty < \alpha(\varrho) \leq \beta(\varrho) < \infty$.

2.2. Spirality indices. The following result was proved in [1, Theorem 1.18] and [1, Proposition 3.1].

THEOREM 2.2 (Böttcher, Yu. Karlovich). *Let Γ be a simple Carleson curve and $t \in \Gamma$. For every $x \in \mathbb{R}$, the functions $W_t(\eta_t^x)$ and $W_t^0(\eta_t^x)$ are regular and submultiplicative and*

$$\begin{aligned} \alpha(W_t(\eta_t^x)) &= \alpha(W_t^0(\eta_t^x)) = \min\{\delta_t^- x, \delta_t^+ x\}, \\ \beta(W_t(\eta_t^x)) &= \beta(W_t^0(\eta_t^x)) = \max\{\delta_t^- x, \delta_t^+ x\}, \end{aligned}$$

where

$$\delta_t^- := \alpha(W_t^0 \eta_t), \quad \delta_t^+ := \beta(W_t^0 \eta_t).$$

The numbers δ_t^- and δ_t^+ are called the lower and upper spirality indices of Γ at t . If Γ is locally smooth at t , then $\delta_t^- = \delta_t^+ = 0$. One says that Γ is a logarithmic Carleson curve if

$$\arg(\tau - t) = -\delta_t \log |\tau - t| + O(1) \quad \text{as } \tau \rightarrow t$$

for every $t \in \Gamma$. It is not difficult to see that for such curves $\delta_t^- = \delta_t^+ = \delta_t$ for every $t \in \Gamma$. However, arbitrary Carleson curves have much more complicated behavior. Indeed, for given numbers $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, one can construct a Carleson curve such that $\alpha = \delta_t^-$ and $\beta = \delta_t^+$ at some point $t \in \Gamma$ (see [1, Proposition 1.21]).

2.3. Indices of powerlikeness. Fix $t \in \Gamma$. Let $w : \Gamma \rightarrow [0, \infty]$ be a weight on Γ such that $\log w \in L^1(\Gamma(t, R))$ for every $R \in (0, d_t]$. Put

$$H_{w,t}(R_1, R_2) := \frac{\exp\left(\frac{1}{|\Gamma(t, R_1)|} \int_{\Gamma(t, R_1)} \log w(\tau) |d\tau|\right)}{\exp\left(\frac{1}{|\Gamma(t, R_2)|} \int_{\Gamma(t, R_2)} \log w(\tau) |d\tau|\right)}, \quad R_1, R_2 \in (0, d_t].$$

Following [1, Section 3.2], we define

$$(V_t w)(x) := \begin{cases} \sup_{0 < R \leq d_t} H_{w,t}(xR, R) & \text{for } x \in (0, 1], \\ \sup_{0 < R \leq d_t} H_{w,t}(R, x^{-1}R) & \text{for } x \in [1, \infty) \end{cases}$$

and

$$(V_t^0 w)(x) := \limsup_{R \rightarrow 0} H_{w,t}(xR, R) = \limsup_{R \rightarrow 0} H_{w,t}(R, x^{-1}R)$$

for $x \in \mathbb{R}$.

A function $f : \Gamma \rightarrow [-\infty, \infty]$ is said to have *bounded mean oscillation* at a point $t \in \Gamma$ if $f \in L^1(\Gamma)$ and

$$\sup_{R>0} \frac{1}{|\Gamma(t, R)|} \int_{\Gamma(t, R)} |f(\tau) - \Delta_t(f, R)| |d\tau| < \infty,$$

where

$$\Delta_t(f, R) := \frac{1}{|\Gamma(t, R)|} \int_{\Gamma(t, R)} f(\tau) |d\tau| \quad (R > 0).$$

The class of all functions of bounded mean oscillation at $t \in \Gamma$ is denoted by $BMO(\Gamma, t)$.

The following result gives sufficient conditions for the regularity of $V_t w$ and $V_t^0 w$. It was proved in [1, Theorem 3.3(a)] and [1, Lemma 3.5(a)].

THEOREM 2.3 (Böttcher, Yu. Karlovich). *Suppose Γ is a simple Carleson curve and $t \in \Gamma$. If $w : \Gamma \rightarrow [0, \infty]$ is a weight such that $\log w \in BMO(\Gamma, t)$, then the functions $V_t w$ and $V_t^0 w$ are regular and submultiplicative and*

$$\alpha(V_t w) = \alpha(V_t^0 w), \quad \beta(V_t w) = \beta(V_t^0 w).$$

The numbers $\alpha(V_t^0 w)$ and $\beta(V_t^0 w)$ are called the *lower and upper indices of powerlikeness* of w at $t \in \Gamma$, respectively. This terminology can be explained by the simple fact that for the power weight $w(\tau) = |\tau - t|^\lambda$ its indices of powerlikeness coincide and are equal to λ .

LEMMA 2.4. *Let Γ be a simple Carleson curve and $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points. Suppose $\psi_j : \Gamma \setminus \{t_j\} \rightarrow (0, \infty)$ are continuous functions for $j \in \{1, \dots, n\}$ and w is the weight given by (6). If $V_{t_j}^0 w$ is regular for some $j \in \{1, \dots, n\}$, then $V_{t_j}^0 \psi_j$ is also regular and*

$$\alpha(V_{t_j}^0 w) = \alpha(V_{t_j}^0 \psi_j), \quad \beta(V_{t_j}^0 w) = \beta(V_{t_j}^0 \psi_j).$$

Proof. Without loss of generality, assume that $V_{t_1}^0 w$ is regular. Suppose $\Gamma_1 \subset \Gamma$ is an arc that contains the point t_1 but does not contain the points t_2, \dots, t_n . Assume that Γ_1 is homeomorphic to a segment. Then the functions ψ_2, \dots, ψ_n are continuous on the compact set Γ_1 . Therefore there exist constants C_1 and C_2 such that

$$0 < C_1 \leq \psi_2(\tau) \dots \psi_n(\tau) \leq C_2 < +\infty \quad \text{for all } \tau \in \Gamma_1.$$

Then $C_1 \psi_1(\tau) \leq w(\tau) \leq C_2 \psi_1(\tau)$ for all $\tau \in \Gamma_1$ and

$$\frac{C_1}{C_2} H_{\psi_1, t_1}(R_1, R_2) \leq H_{w, t_1}(R_1, R_2) \leq \frac{C_2}{C_1} H_{\psi_1, t_1}(R_1, R_2)$$

for all $R_1, R_2 \in (0, \max_{\tau \in \Gamma_1} |\tau - t_1|)$. Thus,

$$\frac{C_1}{C_2} (V_{t_1}^0 \psi_1)(x) \leq (V_{t_1}^0 w)(x) \leq \frac{C_2}{C_1} (V_{t_1}^0 \psi_1)(x) \quad \text{for all } x \in \mathbb{R}.$$

These inequalities imply that if $V_{t_1}^0 w$ is regular, then $V_{t_1}^0 \psi_1$ is also regular and their indices coincide: $\alpha(V_{t_1}^0 \psi_1) = \alpha(V_{t_1}^0 w)$ and $\beta(V_{t_1}^0 \psi_1) = \beta(V_{t_1}^0 w)$ ■

2.4. Relations between indices of submultiplicative functions. The following statement is proved by analogy with [1, Proposition 3.1].

LEMMA 2.5. *Let Γ be a simple rectifiable curve and $t \in \Gamma$. Suppose $\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ is a continuous function and $W_t\psi$ is regular. Then, for every $s \in \mathbb{R}$, the functions $W_t(\psi^s)$ and $W_t^0(\psi^s)$ are regular and*

$$\alpha(W_t(\psi^s)) = \alpha(W_t^0(\psi^s)) = \begin{cases} s\alpha(W_t^0\psi) & \text{if } s \geq 0, \\ s\beta(W_t^0\psi) & \text{if } s < 0, \end{cases}$$

$$\beta(W_t(\psi^s)) = \beta(W_t^0(\psi^s)) = \begin{cases} s\beta(W_t^0\psi) & \text{if } s \geq 0, \\ s\alpha(W_t^0\psi) & \text{if } s < 0. \end{cases}$$

The next statement is certainly known to experts, however we were unable to find the precise reference.

LEMMA 2.6. *Let Γ be a simple rectifiable curve, $t \in \Gamma$, and $\psi_1, \psi_2 : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ be continuous functions such that the functions $W_t\psi_1$ and $W_t\psi_2$ are regular. Then the functions $W_t(\psi_1\psi_2)$ and $W_t^0(\psi_1\psi_2)$ are regular and submultiplicative and*

$$\alpha(W_t\psi_1) + \alpha(W_t\psi_2) \leq \alpha(W_t(\psi_1\psi_2)) \leq \min\{\alpha(W_t\psi_1) + \beta(W_t\psi_2), \beta(W_t\psi_1) + \alpha(W_t\psi_2)\},$$

$$\beta(W_t\psi_1) + \beta(W_t\psi_2) \geq \beta(W_t(\psi_1\psi_2)) \geq \max\{\alpha(W_t\psi_1) + \beta(W_t\psi_2), \beta(W_t\psi_1) + \alpha(W_t\psi_2)\}.$$

The same inequalities hold with W_t replaced by W_t^0 in each occurrence.

Proof. Let $R \in (0, d_t]$ and $x \in (0, 1]$. Then

$$\frac{\max_{\tau \in \Gamma: |\tau-t|=xR} (\psi_1(\tau)\psi_2(\tau))}{\min_{\tau \in \Gamma: |\tau-t|=R} (\psi_1(\tau)\psi_2(\tau))} \leq \frac{\max_{\tau \in \Gamma: |\tau-t|=xR} \psi_1(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=R} \psi_1(\tau)} \cdot \frac{\max_{\tau \in \Gamma: |\tau-t|=xR} \psi_2(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=R} \psi_2(\tau)}.$$

Taking the supremum over all $R \in (0, d_t]$, we obtain

$$(W_t(\psi_1\psi_2))(x) \leq (W_t\psi_1)(x)(W_t\psi_2)(x) \quad (10)$$

for all $x \in (0, 1]$. Analogously it can be shown that this inequality holds for $x \in (1, \infty)$. In particular, this implies that the function $W_t(\psi_1\psi_2)$ is regular. Further, taking the logarithms of both sides of (10), dividing by $\log x$, and then passing to the limits as $x \rightarrow 0$ and $x \rightarrow \infty$, we obtain

$$\alpha(W_t\psi_1) + \alpha(W_t\psi_2) \leq \alpha(W_t(\psi_1\psi_2)), \quad \beta(W_t(\psi_1\psi_2)) \leq \beta(W_t\psi_1) + \beta(W_t\psi_2), \quad (11)$$

respectively. Notice that the passage to the limits is justified by Theorem 2.1.

Similarly,

$$\frac{\max_{\tau \in \Gamma: |\tau-t|=xR} (\psi_1(\tau)\psi_2(\tau))}{\min_{\tau \in \Gamma: |\tau-t|=R} (\psi_1(\tau)\psi_2(\tau))} \geq \frac{\min_{\tau \in \Gamma: |\tau-t|=xR} \psi_1(\tau)}{\max_{\tau \in \Gamma: |\tau-t|=R} \psi_1(\tau)} \cdot \frac{\max_{\tau \in \Gamma: |\tau-t|=xR} \psi_2(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=R} \psi_2(\tau)}$$

$$\geq \left(\inf_{R \in (0, d_t]} \frac{\min_{\tau \in \Gamma: |\tau-t|=xR} \psi_1(\tau)}{\max_{\tau \in \Gamma: |\tau-t|=R} \psi_1(\tau)} \right) \cdot \frac{\max_{\tau \in \Gamma: |\tau-t|=xR} \psi_2(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=R} \psi_2(\tau)}$$

$$\begin{aligned}
 &= \left(\sup_{R \in (0, d_t]} \frac{\max_{\tau \in \Gamma: |\tau-t|=R} \psi_1(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=xR} \psi_1(\tau)} \right)^{-1} \cdot \frac{\max_{\tau \in \Gamma: |\tau-t|=xR} \psi_2(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=R} \psi_2(\tau)} \\
 &= \frac{1}{(W_t \psi_1)(x^{-1})} \cdot \frac{\max_{\tau \in \Gamma: |\tau-t|=xR} \psi_2(\tau)}{\min_{\tau \in \Gamma: |\tau-t|=R} \psi_2(\tau)}.
 \end{aligned}$$

Taking the supremum over all $R \in (0, d_t]$, we obtain

$$(W_t(\psi_1 \psi_2))(x) \geq \frac{(W_t \psi_2)(x)}{(W_t \psi_1)(x^{-1})} \tag{12}$$

for $x \in (0, 1]$. Then

$$\frac{\log(W_t(\psi_1 \psi_2))(x)}{\log x} \leq \frac{\log(W_t \psi_2)(x)}{\log x} + \frac{\log(W_t \psi_1)(x^{-1})}{\log(x^{-1})}.$$

Passing to the limit as $x \rightarrow 0$, we obtain

$$\alpha(W_t(\psi_1 \psi_2)) \leq \alpha(W_t \psi_2) + \beta(W_t \psi_1). \tag{13}$$

In the same way it can be shown that (12) is also fulfilled for $x \in (1, \infty)$. This implies that

$$\beta(W_t(\psi_1 \psi_2)) \geq \beta(W_t \psi_2) + \alpha(W_t \psi_1). \tag{14}$$

Replacing ψ_1 by ψ_2 and vice versa, we also get

$$\alpha(W_t(\psi_1 \psi_2)) \leq \alpha(W_t \psi_1) + \beta(W_t \psi_2), \quad \beta(W_t(\psi_1 \psi_2)) \geq \beta(W_t \psi_1) + \alpha(W_t \psi_2). \tag{15}$$

Combining inequalities (11) and (13)–(15) we arrive at the statement of the lemma for W_t . The statement for W_t^0 follows from the statement for W_t and [1, Lemma 1.16]. ■

From [1, Theorem 3.3(c)] and [1, Lemma 3.16] we get the following.

LEMMA 2.7. *Let Γ be a simple Carleson curve and $t \in \Gamma$. If $\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ is a continuous function such that $W_t \psi$ is regular, then the functions $W_t^0 \psi$ and $V_t^0 \psi$ are regular and submultiplicative and*

$$\alpha(W_t^0 \psi) = \alpha(V_t^0 \psi), \quad \beta(W_t^0 \psi) = \beta(V_t^0 \psi).$$

The next statement is taken from [1, Lemma 3.17].

LEMMA 2.8. *Let Γ be a simple Carleson curve and $t \in \Gamma$. Suppose $\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ is a continuous function such that the function $W_t \psi$ is regular and $w : \Gamma \rightarrow [0, \infty]$ is a weight such that $\log w \in BMO(\Gamma, t)$. Then the function $V_t^0(\psi w)$ is regular and submultiplicative and*

$$\begin{aligned}
 \alpha(V_t^0 w) + \alpha(W_t \psi) &\leq \alpha(V_t^0(\psi w)) \leq \min\{\alpha(V_t^0 w) + \beta(W_t \psi), \beta(V_t^0 w) + \alpha(W_t \psi)\}, \\
 \beta(V_t^0 w) + \beta(W_t \psi) &\geq \beta(V_t^0(\psi w)) \geq \max\{\alpha(V_t^0 w) + \beta(W_t \psi), \beta(V_t^0 w) + \alpha(W_t \psi)\}.
 \end{aligned}$$

2.5. Estimates of weights with one singularity by power weights. Fix $t_0 \in \Gamma$. Let $\omega(t_0, \delta)$ denote the open arc on Γ which contains t_0 and whose endpoints lie on the circle

$$\{\tau \in \mathbb{C} : |\tau - t_0| = \delta\}.$$

It is clear that $\omega(t_0, \delta) \subset \Gamma(t_0, \delta)$, however, it may happen that $\omega(t_0, \delta) \neq \Gamma(t_0, \delta)$. The following lemma was obtained in [14, Lemma 3.2].

LEMMA 2.9. *Let Γ be a simple Carleson curve and $t_0 \in \Gamma$. Suppose $\psi : \Gamma \setminus \{t_0\} \rightarrow (0, \infty)$ is a continuous function and $W_{t_0}\psi$ is regular. Let $\varepsilon > 0$ and δ be such that $0 < \delta < d_{t_0}$. Then there exist positive constants $C_j = C_j(\varepsilon, \delta, \psi)$, where $j = 1, 2$, such that*

$$\frac{\psi(t)}{\psi(\tau)} \leq C_1 \left| \frac{t - t_0}{\tau - t_0} \right|^{\beta(W_{t_0}\psi) + \varepsilon} \quad (16)$$

for all $t \in \Gamma \setminus \omega(t_0, \delta)$ and all $\tau \in \omega(t_0, \delta)$; and

$$\frac{\psi(t)}{\psi(\tau)} \leq C_2 \left| \frac{t - t_0}{\tau - t_0} \right|^{\alpha(W_{t_0}\psi) - \varepsilon} \quad (17)$$

for all $t \in \omega(t_0, \delta)$ and all $\tau \in \Gamma \setminus \omega(t_0, \delta)$.

3. The boundedness of the maximal operator on weighted Nakano spaces

3.1. Muckenhoupt weights on Carleson curves. For a function $f \in L^1(\Gamma)$ the maximal function Mf of f on Γ is defined by

$$(Mf)(t) := \sup_{R>0} \frac{1}{|\Gamma(t, R)|} \int_{\Gamma(t, R)} |f(\tau)| |d\tau| \quad (t \in \Gamma).$$

The map $M : f \mapsto Mf$ is referred to as the *maximal operator*.

The boundedness of the operators M and S on standard weighted Lebesgue spaces is well understood (see e.g. [1, 4, 5, 16, 39]).

THEOREM 3.1 (well-known). *Suppose Γ is a simple Carleson curve. If T is one of the operators M or S and $1 < p < \infty$, then T is bounded on $L^p(\Gamma, w)$ if and only if w is a Muckenhoupt weight, $w \in A_p(\Gamma)$, that is,*

$$\sup_{t \in \Gamma} \sup_{R>0} \left(\frac{1}{R} \int_{\Gamma(t, R)} w^p(\tau) |d\tau| \right)^{1/p} \left(\frac{1}{R} \int_{\Gamma(t, R)} w^{-q}(\tau) |d\tau| \right)^{1/q} < \infty$$

where $1/p + 1/q = 1$.

We now consider weights ψ which are continuous and nonzero on Γ minus a point t . If the function $W_t\psi$ is regular, then its indices are well defined. The following theorem is due to Böttcher and Yu. Karlovich [1, Theorem 2.33]. It provides us with a very useful tool for checking the Muckenhoupt condition once the indices of $W_t\psi$ are available.

THEOREM 3.2 (Böttcher, Yu. Karlovich). *Let $1 < p < \infty$ and Γ be a simple Carleson curve and $t \in \Gamma$. Suppose $\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ is a continuous function and $W_t\psi$ is regular. Then $\psi \in A_p(\Gamma)$ if and only if*

$$0 < 1/p + \alpha(W_t^0\psi), \quad 1/p + \beta(W_t^0\psi) < 1.$$

3.2. The boundedness of M on Nakano spaces with Khvedelidze weights. The proof of Theorem 1.1 in [21] is based on the following result proved in [26, Theorem A]. It will also play an essential role in our proof of Theorem 1.2.

THEOREM 3.3 (Kokilashvili, S. Samko). *Suppose Γ is a simple Carleson curve and $p : \Gamma \rightarrow (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition (1). Let $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. The maximal operator M is bounded on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with the Khvedelidze weight w given by (2) if and only if $0 < 1/p(t_j) + \lambda_j < 1$ for all $j \in \{1, \dots, n\}$.*

3.3. Sufficient condition for the boundedness of M involving Muckenhoupt weights. Although a complete characterization of weights for which M is bounded on weighted variable Lebesgue spaces is still unknown in the setting of arbitrary Carleson curves (see [9] for the setting of \mathbb{R}^n), one of the most significant recent results to achieve this aim is the following sufficient condition (see [23, Theorem A’]).

THEOREM 3.4 (Kokilashvili, N. Samko, S. Samko). *Let Γ be a simple Carleson curve, $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1), and $w : \Gamma \rightarrow [0, \infty]$ be a weight such that w^{p/p_*} belongs to the Muckenhoupt class $A_{p_*}(\Gamma)$, where*

$$p_* := p_*(\Gamma) := \min_{\tau \in \Gamma} p(\tau). \tag{18}$$

Then M is bounded on $L^{p(\cdot)}(\Gamma, w)$.

This theorem does not contain the sufficiency portion of Theorem 3.3 whenever p is variable because for the weight $\varrho(\tau) = |\tau - t|^\lambda$ the condition $\varrho^{p/p_*} \in A_{p_*}(\Gamma)$ is equivalent to $-1/p(t) < \lambda < (p_* - 1)/p(t)$, while the “correct” interval for λ is wider:

$$-1/p(t) < \lambda < (p(t) - 1)/p(t).$$

This means that the conditions of Theorem 3.4 cannot be necessary unless p is constant.

3.4. Sufficient conditions for the boundedness of M on weighted Nakano spaces. Recall that two weights w_1 and w_2 on Γ are said to be equivalent if there is a bounded and bounded away from zero function f on Γ such that $w_1 = fw_2$.

Now we will apply Theorem 3.4 to the weight w given by (6).

LEMMA 3.5. *Let Γ be a simple Carleson curve, $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1), and $t \in \Gamma$. Suppose $\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ is a continuous functions such that the function $W_t\psi$ is regular. If*

$$0 < 1/p(t) + \alpha(W_t^0\psi), \quad 1/p(t) + \beta(W_t^0\psi) < p_*/p(t), \tag{19}$$

where p_ is defined by (18), then the operator M is bounded on $L^{p(\cdot)}(\Gamma, \psi)$.*

Proof. The proof is analogous to the proof of [14, Lemma 2.2]. Taking into account Lemma 2.5, we see that the function $W_t(\psi^{p(t)/p_*})$ is regular and inequalities (19) are equivalent to

$$\begin{aligned} 0 &< \frac{1}{p_*} + \frac{p(t)}{p_*} \alpha(W_t^0\psi) = \frac{1}{p_*} + \alpha(W_t^0(\psi^{p(t)/p_*})), \\ 1 &> \frac{1}{p_*} + \frac{p(t)}{p_*} \beta(W_t^0\psi) = \frac{1}{p_*} + \beta(W_t^0(\psi^{p(t)/p_*})). \end{aligned}$$

By Theorem 3.2, the latter inequalities are equivalent to $\psi^{p(t)/p_*} \in A_{p_*}(\Gamma)$.

Let us show that the weights ψ^{p/p^*} and $\psi^{p(t)/p^*}$ are equivalent. Fix $\varepsilon > 0$. Since $\psi : \Gamma \setminus \{t\} \rightarrow (0, \infty)$ is continuous, from Lemma 2.9 it follows that there exist $C_1, C_2 > 0$ such that

$$C_1 |\tau - t|^{\beta(W_t^0 \psi) + \varepsilon} \leq \psi(\tau) \leq C_2 |\tau - t|^{\alpha(W_t^0 \psi) - \varepsilon}$$

for all $\tau \in \Gamma \setminus \{t\}$. Then

$$\log C_1 + (\beta(W_t^0 \psi) + \varepsilon) \log |\tau - t| \leq \log \psi(\tau), \quad (20)$$

$$\log C_2 + (\alpha(W_t^0 \psi) - \varepsilon) \log |\tau - t| \geq \log \psi(\tau) \quad (21)$$

for all $\tau \in \Gamma \setminus \{t\}$. By the Dini-Lipschitz condition (1),

$$-\frac{C_\Gamma}{-\log |\tau - t|} \leq p(\tau) - p(t) \leq \frac{C_\Gamma}{-\log |\tau - t|} \quad (22)$$

for all $\tau \in \Gamma \setminus \{t\}$ such that $|\tau - t| \leq 1/2$. Multiplying inequalities (20)–(22), we see that the function

$$F_t(\tau) := \frac{p(\tau) - p(t)}{p^*} \log \psi(\tau)$$

is bounded on $\Gamma(t, 1/2) \setminus \{t\}$. Obviously, it is also bounded on $\Gamma \setminus \Gamma(t, 1/2)$. Therefore

$$\frac{\psi(\tau)^{p(\tau)/p^*}}{\psi(\tau)^{p(t)/p^*}} = \exp(F_t(\tau))$$

is bounded and bounded away from zero on $\Gamma \setminus \{t\}$. Thus the weights ψ^{p/p^*} and $\psi^{p(t)/p^*}$ are equivalent. In particular, this implies that $\psi^{p/p^*} \in A_{p^*}(\Gamma)$ if and only if $\psi^{p(t)/p^*} \in A_{p^*}(\Gamma)$. Thus, inequalities (19) imply that $\psi^{p/p^*} \in A_{p^*}(\Gamma)$. Applying Theorem 3.4, we finally conclude that the maximal operator M is bounded on $L^{p(\cdot)}(\Gamma, \psi)$. ■

THEOREM 3.6. *Suppose Γ is a simple Carleson curve and $p : \Gamma \rightarrow (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition (1). Let $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points and $\psi_j : \Gamma \setminus \{t_j\} \rightarrow (0, \infty)$ be continuous functions such that the functions $W_{t_j} \psi_j$ are regular for all $j \in \{1, \dots, n\}$. If for all $j \in \{1, \dots, n\}$,*

$$0 < 1/p(t_j) + \alpha(W_{t_j}^0 \psi_j), \quad 1/p(t_j) + \beta(W_{t_j}^0 \psi_j) < 1, \quad (23)$$

then the maximal operator M is bounded on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight w given by (6).

Proof. The idea of the proof is borrowed from [23, Theorem B] (see also [14, Theorem 1.4]).

We start the proof with a kind of separation of singularities of the weight. Let arcs $\Gamma_1, \dots, \Gamma_n \subset \Gamma$ form a partition of Γ , that is, each two arcs Γ_i and Γ_k may have only endpoints in common and $\Gamma_1 \cup \dots \cup \Gamma_n = \Gamma$. We will assume that each arc Γ_j is homeomorphic to a segment. Suppose that this partition has the following property: each point t_j belongs to Γ_j and all other points in $\{t_1, \dots, t_n\} \setminus \{t_j\}$ do not belong to Γ_j .

Obviously, the function

$$w/\psi_j := \psi_1 \dots \psi_{j-1} \widetilde{\psi_j} \psi_{j+1} \dots \psi_n,$$

where \tilde{a} denotes that the term a is absent, is continuous on the closed set Γ_j . Therefore,

$$\inf_{\tau \in \Gamma_j} \left(\frac{w}{\psi_j} \right) (\tau) =: c_j > 0, \quad \sup_{\tau \in \Gamma_j} \left(\frac{w}{\psi_j} \right) (\tau) =: C_j < +\infty.$$

Hence, for every $f \in L^{p(\cdot)}(\Gamma, w)$, we have

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(\Gamma, w)} &\leq \sum_{j=1}^n \|f\chi_{\Gamma_j}\|_{L^{p(\cdot)}(\Gamma, w)} = \sum_{j=1}^n \left\| f \left(\frac{w}{\psi_j} \right) \psi_j \chi_{\Gamma_j} \right\|_{L^{p(\cdot)}(\Gamma)} \\ &\leq \sum_{j=1}^n C_j \|f\|_{\Gamma_j} \| \chi_{\Gamma_j} \|_{L^{p(\cdot)}(\Gamma_j, \psi_j|_{\Gamma_j})} \end{aligned}$$

and

$$\begin{aligned} \|f\|_{\Gamma_j} \| \chi_{\Gamma_j} \|_{L^{p(\cdot)}(\Gamma_j, \psi_j|_{\Gamma_j})} &= \|(f\psi_j)|_{\Gamma_j}\|_{L^{p(\cdot)}(\Gamma_j)} = \left\| (fw)|_{\Gamma_j} \left(\frac{w}{\psi_j} \right)^{-1} \right\|_{\Gamma_j} \| \chi_{\Gamma_j} \|_{L^{p(\cdot)}(\Gamma_j)} \\ &\leq \frac{1}{c_j} \|(fw)|_{\Gamma_j}\|_{L^{p(\cdot)}(\Gamma_j)} = \frac{1}{c_j} \|fw\chi_{\Gamma_j}\|_{L^{p(\cdot)}(\Gamma)} \\ &\leq \frac{1}{c_j} \|fw\|_{L^{p(\cdot)}(\Gamma)} = \frac{1}{c_j} \|f\|_{L^{p(\cdot)}(\Gamma, w)} \end{aligned}$$

for every $j \in \{1, \dots, n\}$. From these estimates it follows that it is sufficient to prove that M is bounded on $L^{p(\cdot)}(\Gamma_j, \psi_j|_{\Gamma_j})$ for each $j \in \{1, \dots, n\}$.

Fix $j \in \{1, \dots, n\}$. For simplicity of notation, assume that $\Gamma_j = \Gamma$. This does not cause any problem because

$$(W_{t_j}(\psi_j|_{\Gamma_j}))(x) \leq (W_{t_j}\psi_j)(x), \quad (W_{t_j}^0(\psi_j|_{\Gamma_j}))(x) = (W_{t_j}^0\psi_j)(x)$$

for all $x \in \mathbb{R}$. Therefore, $W_{t_j}(\psi_j|_{\Gamma_j})$ and $W_{t_j}^0(\psi_j|_{\Gamma_j})$ are regular and

$$\alpha := \alpha(W_{t_j}^0(\psi_j|_{\Gamma_j})) = \alpha(W_{t_j}^0\psi_j), \quad \beta := \beta(W_{t_j}^0(\psi_j|_{\Gamma_j})) = \beta(W_{t_j}^0\psi_j).$$

It is easily seen that M is bounded on $L^{p(\cdot)}(\Gamma, \psi_j)$ if and only if the operator

$$(M_j f)(t) := \sup_{R>0} \frac{\psi_j(t)}{|\Gamma(t, R)|} \int_{\Gamma(t, R)} \frac{|f(\tau)|}{\psi_j(\tau)} |d\tau| \quad (t \in \Gamma)$$

is bounded on $L^{p(\cdot)}(\Gamma)$. From (23) it follows that there is a small $\varepsilon > 0$ such that

$$0 < 1/p(t_j) + \alpha - \varepsilon \leq 1/p(t_j) + \beta + \varepsilon < 1. \quad (24)$$

Since $p : \Gamma \rightarrow (1, \infty)$ is continuous and $1/p(t_j) + \beta < 1$, we can choose a number $\delta \in (0, d_{t_j})$ such that the arc $\omega(t_j, \delta)$, which contains t_j and has the endpoints on the circle $\{\tau \in \mathbb{C} : |\tau - t_j| = \delta\}$, is so small that $1 + \beta p(t_j) < p_*$, where

$$p_* := p_*(\omega(t_j, \delta)) = \min_{\tau \in \omega(t_j, \delta)} p(\tau).$$

Hence

$$0 < 1/p(t_j) + \alpha \leq 1/p(t_j) + \beta < p_*/p(t_j). \quad (25)$$

For $f \in L^{p(\cdot)}(\Gamma)$, we have

$$\begin{aligned} M_j f &\leq \chi_{\omega(t_j, \delta)} M_j \chi_{\omega(t_j, \delta)} f + \chi_{\Gamma \setminus \omega(t_j, \delta)} M_j \chi_{\omega(t_j, \delta)} f \\ &\quad + \chi_{\omega(t_j, \delta)} M_j \chi_{\Gamma \setminus \omega(t_j, \delta)} f + \chi_{\Gamma \setminus \omega(t_j, \delta)} M_j \chi_{\Gamma \setminus \omega(t_j, \delta)} f. \end{aligned} \quad (26)$$

From (25) and Lemma 3.5 it follows that M_j is bounded on $L^{p(\cdot)}(\omega(t_j, \delta))$. Consequently, the operator $\chi_{\omega(t_j, \delta)} M_j \chi_{\omega(t_j, \delta)} I$ is bounded on $L^{p(\cdot)}(\Gamma)$.

For $\lambda \in \mathbb{R}$, by M_j^λ denote the weighted maximal operator defined by

$$(M_j^\lambda f)(t) := \sup_{R>0} \frac{|t - t_j|^\lambda}{|\Gamma(t, R)|} \int_{\Gamma(t, R)} \frac{|f(\tau)|}{|\tau - t_j|^\lambda} |d\tau|.$$

From Lemma 2.9 it follows that

$$\chi_{\Gamma \setminus \omega(t_j, \delta)} M_j \chi_{\omega(t_j, \delta)} f \leq C_1 \chi_{\Gamma \setminus \omega(t_j, \delta)} M_j^{\beta+\varepsilon} \chi_{\omega(t_j, \delta)} f \leq C_1 M_j^{\beta+\varepsilon} f \quad (27)$$

and

$$\chi_{\omega(t_j, \delta)} M_j \chi_{\Gamma \setminus \omega(t_j, \delta)} f \leq C_2 \chi_{\omega(t_j, \delta)} M_j^{\alpha-\varepsilon} \chi_{\Gamma \setminus \omega(t_j, \delta)} f \leq C_2 M_j^{\alpha-\varepsilon} f, \quad (28)$$

where C_1 and C_2 are positive constants depending only on ε, δ , and ψ_j . From (24) and Theorem 3.3 it follows that the operators $M_j^{\alpha-\varepsilon}$ and $M_j^{\beta+\varepsilon}$ are bounded on $L^{p(\cdot)}(\Gamma)$. From here and (27)–(28) we conclude that $\chi_{\Gamma \setminus \omega(t_j, \delta)} M_j \chi_{\omega(t_j, \delta)} I$ and $\chi_{\omega(t_j, \delta)} M_j \chi_{\Gamma \setminus \omega(t_j, \delta)} I$ are bounded on $L^{p(\cdot)}(\Gamma)$.

Finally, since $\Gamma \setminus \omega(t_j, \delta)$ does not contain the singularity of the weight ψ_j (which is continuous on $\Gamma \setminus \{t_j\}$), there exists a constant $C_3 > 0$ such that

$$\chi_{\Gamma \setminus \omega(t_j, \delta)} M_j \chi_{\Gamma \setminus \omega(t_j, \delta)} f \leq C_3 M f.$$

Theorem 3.3 and the above estimate yield the boundedness of $\chi_{\Gamma \setminus \omega(t_j, \delta)} M_j \chi_{\Gamma \setminus \omega(t_j, \delta)} I$ on $L^{p(\cdot)}(\Gamma)$. Thus, all operators on the right-hand side of (26) are bounded on $L^{p(\cdot)}(\Gamma)$. Therefore, the operator on the left-hand side of (26) is bounded, too. This completes the proof of the boundedness of M on $L^{p(\cdot)}(\Gamma_j, \psi_j|_{\Gamma_j})$. ■

4. The Cauchy singular integral operator on weighted Nakano spaces

4.1. Necessary conditions for the boundedness of the operator S . We will need the following necessary condition for the boundedness of S on weighted Nakano spaces.

THEOREM 4.1. *Let Γ be a simple rectifiable curve and let $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1). If $w : \Gamma \rightarrow [0, \infty]$ is an arbitrary weight such that the operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$, then Γ is a Carleson curve, $\log w \in BMO(\Gamma, t)$, the functions $V_t w$ and $V_t^0 w$ are regular and submultiplicative, and*

$$0 \leq 1/p(t) + \alpha(V_t^0 w), \quad 1/p(t) + \beta(V_t^0 w) \leq 1 \quad (29)$$

for every $t \in \Gamma$. If, in addition, Γ is a rectifiable Jordan curve, then

$$0 < 1/p(t) + \alpha(V_t^0 w), \quad 1/p(t) + \beta(V_t^0 w) < 1 \quad (30)$$

for every $t \in \Gamma$.

Proof. For simple curves, the statement follows from [10, Lemma 4.9] and [10, Theorems 5.9 and 6.1]. For Jordan curves, inequality (30) was proved in [12, Corollary 4.2]. ■

4.2. The boundedness of M implies the boundedness of S . One of the main ingredients of the proof of Theorem 1.2 is the following recent result by Kokilashvili and S. Samko [27, Theorem 4.21].

THEOREM 4.2 (Kokilashvili, S. Samko). *Let Γ be a simple Carleson curve. Suppose that $p : \Gamma \rightarrow (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition (1) and $w : \Gamma \rightarrow [1, \infty]$ is a weight. If there exists a number p_0 such that*

$$1 < p_0 < \min_{\tau \in \Gamma} p(\tau)$$

and M is bounded on $L^{p(\cdot)/(p(\cdot)-p_0)}(\Gamma, w^{-p_0})$, then S is bounded on $L^{p(\cdot)}(\Gamma, w)$.

4.3. Proof of Theorem 1.2. (a) This part is proved by analogy with [15, Theorem 2.1]. Since the function $p : \Gamma \rightarrow (1, \infty)$ is continuous and Γ is compact, we deduce that $\min_{\tau \in \Gamma} p(\tau) > 1$. If the inequalities

$$1/p(t_j) + \beta(W_{t_j}^0 \psi_j) < 1, \quad j \in \{1, \dots, n\},$$

are fulfilled, then there exists a number p_0 such that

$$1 < p_0 < \min_{\tau \in \Gamma} p(\tau)$$

and

$$1/p(t_j) + \beta(W_{t_j}^0 \psi_j) < 1/p_0, \quad j \in \{1, \dots, n\}.$$

Taking into account Lemma 2.5, we see that the functions $W_{t_j}(\psi_j^{-p_0})$ are regular and the latter inequalities are equivalent to

$$0 < 1 - \frac{p_0}{p(t_j)} - p_0 \beta(W_{t_j}^0 \psi_j) = \frac{p(t_j) - p_0}{p(t_j)} + \alpha(W_{t_j}^0(\psi_j^{-p_0})), \quad j \in \{1, \dots, n\}. \quad (31)$$

Analogously, the inequalities

$$0 < 1/p(t_j) + \alpha(W_{t_j}^0 \psi_j), \quad j \in \{1, \dots, n\},$$

are equivalent to

$$1 > 1 - \frac{p_0}{p(t_j)} - p_0 \alpha(W_{t_j}^0 \psi_j) = \frac{p(t_j) - p_0}{p(t_j)} + \beta(W_{t_j}^0(\psi_j^{-p_0})), \quad j \in \{1, \dots, n\}. \quad (32)$$

From inequalities (31)–(32) and Theorem 3.6 it follows that the maximal operator M is bounded on $L^{p(\cdot)/(p(\cdot)-p_0)}(\Gamma, w^{-p_0})$. To finish the proof of part (a), it remains to apply Theorem 4.2.

(b) If the operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$, then from (29) it follows that

$$0 \leq 1/p(t_j) + \alpha(V_{t_j}^0 w), \quad 1/p(t_j) + \beta(V_{t_j}^0 w) \leq 1 \quad \text{for all } j \in \{1, \dots, n\}.$$

Then, by Lemma 2.4,

$$0 \leq 1/p(t_j) + \alpha(V_{t_j}^0 \psi_j), \quad 1/p(t_j) + \beta(V_{t_j}^0 \psi_j) \leq 1 \quad \text{for all } j \in \{1, \dots, n\}.$$

Applying Lemma 2.7 to the above inequalities, we see that

$$0 \leq 1/p(t_j) + \alpha(W_{t_j}^0 \psi_j), \quad 1/p(t_j) + \beta(W_{t_j}^0 \psi_j) \leq 1 \quad \text{for all } j \in \{1, \dots, n\}.$$

Part (b) is proved. The proof of part (c) follows the same lines with inequalities (30) in place of (29). ■

5. Singular integral operators with L^∞ coefficients

5.1. Necessary conditions for Fredholmness. In this section we will suppose that Γ is a Carleson Jordan curve, $p : \Gamma \rightarrow (1, \infty)$ is a continuous function, and $w : \Gamma \rightarrow [0, \infty]$ is an arbitrary weight (not necessarily of the form (6)) such that S is bounded on $L^{p(\cdot)}(\Gamma, w)$. Under these assumptions,

$$P := (I + S)/2, \quad Q := (I - S)/2$$

are bounded projections on $L^{p(\cdot)}(\Gamma, w)$ (see [10, Lemma 6.4]). The operators of the form $aP + bQ$, where $a, b \in L^\infty(\Gamma)$, are called *singular integral operators* (SIOs).

THEOREM 5.1. *Suppose $a, b \in L^\infty(\Gamma)$. If $aP + bQ$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$, then $a^{-1}, b^{-1} \in L^\infty(\Gamma)$.*

This result can be proved in the same way as [11, Theorem 5.4] where the case of Khvedelidze weights (2) was considered.

5.2. The local principle. Two functions $a, b \in L^\infty(\Gamma)$ are said to be locally equivalent at a point $t \in \Gamma$ if

$$\inf\{\|(a - b)c\|_\infty : c \in C(\Gamma), c(t) = 1\} = 0.$$

THEOREM 5.2. *Suppose $a \in L^\infty(\Gamma)$ and for each $t \in \Gamma$ there exists a function $a_t \in L^\infty(\Gamma)$ which is locally equivalent to a at t . If the operators $a_t P + Q$ are Fredholm on $L^{p(\cdot)}(\Gamma, w)$ for all $t \in \Gamma$, then $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$.*

For weighted Lebesgue spaces this theorem is known as Simonenko's local principle [37]. It follows from [10, Theorem 6.13].

5.3. Wiener-Hopf factorization. The curve Γ divides the complex plane \mathbb{C} into the bounded simply connected domain D^+ and the unbounded domain D^- . Recall that without loss of generality we assumed that $0 \in D^+$. We say that a function $a \in L^\infty(\Gamma)$ admits a *Wiener-Hopf factorization on $L^{p(\cdot)}(\Gamma, w)$* if $a^{-1} \in L^\infty(\Gamma)$ and a can be written in the form

$$a(t) = a_-(t)t^\kappa a_+(t) \quad \text{a.e. on } \Gamma, \tag{33}$$

where $\kappa \in \mathbb{Z}$, and the factors a_\pm enjoy the following properties:

- (i) $a_- \in QL^{p(\cdot)}(\Gamma, w) \dot{+} \mathbb{C}$, $a_-^{-1} \in QL^{q(\cdot)}(\Gamma, 1/w) \dot{+} \mathbb{C}$,
 $a_+ \in PL^{q(\cdot)}(\Gamma, 1/w)$, $a_+^{-1} \in PL^{p(\cdot)}(\Gamma, w)$,
- (ii) the operator $a_+^{-1} S a_+ I$ is bounded on $L^{p(\cdot)}(\Gamma, w)$,

where $1/p(t) + 1/q(t) = 1$ for all $t \in \Gamma$. One can prove that the number κ is uniquely determined.

THEOREM 5.3. *A function $a \in L^\infty(\Gamma)$ admits a Wiener-Hopf factorization (33) on $L^{p(\cdot)}(\Gamma, w)$ if and only if the operator $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$.*

This theorem goes back to Simonenko [36, 38]. For more about this topic we refer to [1, Section 6.12], [2, Section 5.5], [8, Section 8.3] in the case of weighted Lebesgue spaces. Theorem 5.3 follows from [10, Theorem 6.14].

6. Singular integral operators with PC coefficients

6.1. Indicator functions. Combining Theorems 2.2 and 4.1 with Lemma 2.7, we arrive at the following.

LEMMA 6.1. *Let Γ be a Carleson Jordan curve, $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1), and $w : \Gamma \rightarrow [0, \infty]$ be a weight such that the operator S is bounded on the weighted Nakano space $L^{p(\cdot)}(\Gamma, w)$. Then, for every $x \in \mathbb{R}$ and every $t \in \Gamma$, the function $V_t^0(\eta_t^x w)$ is regular and submultiplicative.*

The above lemma says that the functions

$$\alpha_t(x) := \alpha(V_t^0(\eta_t^x w)), \quad \beta_t(x) := \beta(V_t^0(\eta_t^x w)) \quad (x \in \mathbb{R})$$

are well-defined for every $t \in \Gamma$. The shape of these functions can be described with the aid of the following theorem.

THEOREM 6.2. *Let Γ be a Carleson Jordan curve, $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1), $w : \Gamma \rightarrow [0, \infty]$ be a weight such that the operator S is bounded on the weighted Nakano space $L^{p(\cdot)}(\Gamma, w)$, and $t \in \Gamma$. Then the functions α_t and β_t enjoy the following properties:*

- (a) $-\infty < \alpha_t(x) \leq \beta_t(x) < +\infty$ for all $x \in \mathbb{R}$;
- (b) $0 < 1/p(t) + \alpha_t(0) \leq 1/p(t) + \beta_t(0) < 1$;
- (c) α_t is concave and β_t is convex;
- (d) $\alpha_t(x)$ and $\beta_t(x)$ have asymptotes as $x \rightarrow \pm\infty$ and the convex regions

$$\{x + iy \in \mathbb{C} : y < \alpha_t(x)\} \quad \text{and} \quad \{x + iy \in \mathbb{C} : y > \beta_t(x)\}$$

may be separated by parallels to each of these asymptotes; to be more precise, there exist real numbers $\mu_t^-, \mu_t^+, \nu_t^-, \nu_t^+$ such that

$$\begin{aligned} 0 < 1/p(t) + \mu_t^- \leq 1/p(t) + \nu_t^- < 1, \quad 0 < 1/p(t) + \mu_t^+ \leq 1/p(t) + \nu_t^+ < 1, \\ \beta_t(x) = \nu_t^+ + \delta_t^+ x + o(1) \quad \text{as } x \rightarrow +\infty, \\ \beta_t(x) = \nu_t^- + \delta_t^- x + o(1) \quad \text{as } x \rightarrow -\infty, \\ \alpha_t(x) = \mu_t^+ + \delta_t^+ x + o(1) \quad \text{as } x \rightarrow +\infty, \\ \alpha_t(x) = \mu_t^- + \delta_t^- x + o(1) \quad \text{as } x \rightarrow -\infty. \end{aligned}$$

Proof. Part (a) follows from Lemma 6.1. Theorem 4.1 yields part (b). Part (c) is proved in [1, Proposition 3.20] under the assumption that p is constant and $w \in A_p(\Gamma)$. In our case the proof is literally the same. Again, part (d) is proved in [1, Theorem 3.31] for $w \in A_p(\Gamma)$ and constant p . This proof works equally in our case because in view of Theorem 4.1 we can apply Lemma 2.8 under the assumption that the operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$. ■

From Theorem 6.2(a),(c) we immediately deduce that the set

$$Y(p(t), \alpha_t, \beta_t) := \{\gamma = x + iy \in \mathbb{C} : 1/p(t) + \alpha_t(x) \leq y \leq 1/p(t) + \beta_t(x)\}$$

is a connected set containing points with arbitrary real parts. Hence the set

$$\{e^{2\pi\gamma} : \gamma \in Y(p(t), \alpha_t, \beta_t)\}$$

is connected and contains points arbitrarily close to the origin and to the infinity. Let $z_1, z_2 \in \mathbb{C}$. The Möbius transform M_{z_1, z_2} given by (9) maps 0 and ∞ to z_1 and z_2 , respectively. Thus, the leaf $\mathcal{L}(z_1, z_2; p(t), \alpha_t, \beta_t)$ is a connected set containing z_1 and z_2 . More information about leaves (with many examples and computer plots) can be found in [1, Chap. 7].

LEMMA 6.3. *Let Γ be a Carleson Jordan curve, $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1), and $t_1, \dots, t_n \in \Gamma$ be pairwise distinct points. Suppose $\psi_j : \Gamma \setminus \{t_j\} \rightarrow (0, \infty)$ are continuous functions such that the functions $W_{t_j} \psi_j$ are regular, conditions (5) are fulfilled, and the weight w is given by (6).*

- (a) *If $t \in \Gamma \setminus \{t_1, \dots, t_n\}$, then for every $x \in \mathbb{R}$, the functions $W_t^0(\eta_t^x)$ and $V_t^0(\eta_t^x w)$ are regular and submultiplicative and*

$$\begin{aligned} \alpha(W_t^0(\eta_t^x)) &= \alpha(V_t^0(\eta_t^x w)) = \min\{\delta_t^- x, \delta_t^+ x\}, \\ \beta(W_t^0(\eta_t^x)) &= \beta(V_t^0(\eta_t^x w)) = \max\{\delta_t^- x, \delta_t^+ x\}. \end{aligned}$$

- (b) *If $j \in \{1, \dots, n\}$, then for every $x \in \mathbb{R}$, the functions $W_{t_j}^0(\eta_{t_j}^x \psi_j)$ and $V_{t_j}^0(\eta_{t_j}^x w)$ are regular and submultiplicative and*

$$\alpha(W_{t_j}^0(\eta_{t_j}^x \psi_j)) = \alpha(V_{t_j}^0(\eta_{t_j}^x w)), \quad \beta(W_{t_j}^0(\eta_{t_j}^x \psi_j)) = \beta(V_{t_j}^0(\eta_{t_j}^x w)). \quad (34)$$

Proof. Let us prove a slightly more difficult part (b). Fix $t_j \in \{t_1, \dots, t_n\}$ and $x \in \mathbb{R}$. By Theorem 2.2, the functions $W_{t_j}(\eta_{t_j}^x)$ and $W_{t_j}^0(\eta_{t_j}^x)$ are regular and submultiplicative. Then, in view of Lemma 2.6, the functions $W_{t_j}(\eta_{t_j}^x \psi_j)$ and $W_{t_j}^0(\eta_{t_j}^x \psi_j)$ are regular and submultiplicative. By Lemma 2.7, the function $V_{t_j}^0(\eta_{t_j}^x \psi_j)$ is regular and submultiplicative and

$$\alpha(W_{t_j}^0(\eta_{t_j}^x \psi_j)) = \alpha(V_{t_j}^0(\eta_{t_j}^x \psi_j)), \quad \beta(W_{t_j}^0(\eta_{t_j}^x \psi_j)) = \beta(V_{t_j}^0(\eta_{t_j}^x \psi_j)). \quad (35)$$

From Theorem 4.1 we know that $\log w \in BMO(\Gamma, t)$. Therefore, in view of Lemma 2.8, the function $V_{t_j}^0(\eta_{t_j}^x w)$ is regular and submultiplicative. By Lemma 2.4,

$$\alpha(V_{t_j}^0(\eta_{t_j}^x w)) = \alpha(V_{t_j}^0(\eta_{t_j}^x \psi_j)), \quad \beta(V_{t_j}^0(\eta_{t_j}^x w)) = \beta(V_{t_j}^0(\eta_{t_j}^x \psi_j)). \quad (36)$$

Combining (35)–(36), we arrive at (34). Part (b) is proved. The proof of part (a) is analogous. ■

This lemma says that, under the assumptions of Theorem 1.2, the functions α_t^* and β_t^* are well-defined by (7)–(8) and

$$\alpha_t^*(x) = \alpha_t(x), \quad \beta_t^*(x) = \beta_t(x) \quad (x \in \mathbb{R}) \quad (37)$$

for all $t \in \Gamma$. We say that the functions α_t^* and β_t^* are the indicator functions of the triple (Γ, p, w) at the point $t \in \Gamma$.

6.2. Necessary conditions for Fredholmness. The following necessary conditions for Fredholmness were obtained by the author [10, Theorem 8.1].

THEOREM 6.4. *Let Γ be a Carleson Jordan curve and let $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1). Suppose $w : \Gamma \rightarrow [0, \infty]$ is an arbitrary weight such that the operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$. If the operator $aP + Q$,*

where $a \in PC(\Gamma)$, is Fredholm on the weighted Nakano space $L^{p(\cdot)}(\Gamma, w)$, then $a(t \pm 0) \neq 0$ and

$$-\frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} + \frac{1}{p(t)} + \theta \alpha_t \left(\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| \right) + (1-\theta) \beta_t \left(\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| \right) \notin \mathbb{Z}$$

for all $\theta \in [0, 1]$ and all $t \in \Gamma$.

6.3. Wiener-Hopf factorization of local representatives. Fix $t \in \Gamma$. For a function $a \in PC(\Gamma)$ such that $a^{-1} \in L^\infty(\Gamma)$, we construct a “canonical” function $g_{t,\gamma}$ which is locally equivalent to a at the point $t \in \Gamma$. The interior and the exterior of the unit circle can be conformally mapped onto D^+ and D^- of Γ , respectively, so that the point 1 is mapped to t , and the points $0 \in D^+$ and $\infty \in D^-$ remain fixed. Let Λ_0 and Λ_∞ denote the images of $[0, 1]$ and $[1, \infty) \cup \{\infty\}$ under this map. The curve $\Lambda_0 \cup \Lambda_\infty$ joins 0 to ∞ and meets Γ at exactly one point, namely t . Let $\arg z$ be a continuous branch of argument in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. For $\gamma \in \mathbb{C}$, define the function $z^\gamma := |z|^\gamma e^{i\gamma \arg z}$, where $z \in \mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. Clearly, z^γ is an analytic function in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. The restriction of z^γ to $\Gamma \setminus \{t\}$ will be denoted by $g_{t,\gamma}$. Obviously, $g_{t,\gamma}$ is continuous and nonzero on $\Gamma \setminus \{t\}$. Since $a(t \pm 0) \neq 0$, we can define $\gamma_t = \gamma \in \mathbb{C}$ by the formulas

$$\operatorname{Re} \gamma_t := \frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)}, \quad \operatorname{Im} \gamma_t := -\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right|, \tag{38}$$

where we can take any value of $\arg(a(t-0)/a(t+0))$, which implies that any two choices of $\operatorname{Re} \gamma_t$ differ by an integer only. Clearly, there is a constant $c_t \in \mathbb{C} \setminus \{0\}$ such that $a(t \pm 0) = c_t g_{t,\gamma_t}(t \pm 0)$, which means that a is locally equivalent to $c_t g_{t,\gamma_t}$ at the point $t \in \Gamma$.

For $t \in \Gamma$ and $\gamma \in \mathbb{C}$, consider the weight

$$\varphi_{t,\gamma}(\tau) := |(\tau - t)^\gamma|, \quad \tau \in \Gamma \setminus \{t\}.$$

From [10, Lemma 7.1] we get the following.

LEMMA 6.5. *Let Γ be a Carleson Jordan curve and let $p : \Gamma \rightarrow (1, \infty)$ be a continuous function. Suppose $w : \Gamma \rightarrow [0, \infty]$ is an arbitrary weight such that the operator S is bounded on $L^{p(\cdot)}(\Gamma, w)$. If, for some $k \in \mathbb{Z}$ and $\gamma \in \mathbb{C}$, the operator $\varphi_{t,k-\gamma} S \varphi_{t,\gamma-k} I$ is bounded on $L^{p(\cdot)}(\Gamma, w)$, then the function $g_{t,\gamma}$ admits a Wiener-Hopf factorization on $L^{p(\cdot)}(\Gamma, w)$.*

6.4. Sufficient conditions for Fredholmness. The following result is one of the main ingredients of the proof of Theorem 1.3. The idea of its proof is borrowed from the proof of [1, Proposition 7.3].

THEOREM 6.6. *Let Γ be a Carleson Jordan curve and $p : \Gamma \rightarrow (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition (1). Suppose $t_1, \dots, t_n \in \Gamma$ are pairwise distinct points and $\psi_j : \Gamma \setminus \{t_j\} \rightarrow (0, \infty)$ are continuous functions such that the functions $W_{t_j} \psi_j$ are regular and conditions (5) are fulfilled for all $j \in \{1, \dots, n\}$. If $a \in PC(\Gamma)$ is such that $a(t \pm 0) \neq 0$ and*

$$-\frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} + \frac{1}{p(t)} + \theta \alpha_t^* \left(\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| \right) + (1-\theta) \beta_t^* \left(\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| \right) \notin \mathbb{Z}$$

for all $\theta \in [0, 1]$ and all $t \in \Gamma$, then the operator $aP + Q$ is Fredholm on the Nakano space $L^{p(\cdot)}(\Gamma, w)$ with weight w given by (6).

Proof. We will follow the proof of [12, Theorem 4.5] and [15, Theorem 2.2]. If $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$, then, by Theorem 5.1, $a(t \pm 0)$ for all $t \in \Gamma$. Fix an arbitrary $t \in \Gamma$ and choose $\gamma = \gamma_t$ as in (38). Then the function a is locally equivalent to $c_t g_{t, \gamma_t}$ at the point $t \in \Gamma$, where $c_t \in \mathbb{C} \setminus \{0\}$ is some constant. In this case the main condition of the theorem has the form

$$1/p(t) - \operatorname{Re} \gamma_t + \theta \alpha_t^*(-\operatorname{Im} \gamma_t) + (1 - \theta) \beta_t^*(-\operatorname{Im} \gamma_t) \notin \mathbb{Z} \quad \text{for all } \theta \in [0, 1].$$

Therefore, there exists a number $k_t \in \mathbb{Z}$ such that

$$0 < 1/p(t) + k_t - \operatorname{Re} \gamma_t + \theta \alpha_t^*(-\operatorname{Im} \gamma_t) + (1 - \theta) \beta_t^*(-\operatorname{Im} \gamma_t) < 1 \quad \text{for all } \theta \in [0, 1].$$

In particular, if $\theta = 1$, then

$$0 < 1/p(t) + \operatorname{Re}(k_t - \gamma_t) + \alpha_t^*(\operatorname{Im}(k_t - \gamma_t)); \quad (39)$$

if $\theta = 0$, then

$$1/p(t) + \operatorname{Re}(k_t - \gamma_t) + \beta_t^*(\operatorname{Im}(k_t - \gamma_t)) < 1. \quad (40)$$

Consider the weights $\omega_t(\tau) := |\tau - t|$ and

$$w_t := \varphi_{t, k_t - \gamma_t} w = \omega_t^{\operatorname{Re}(k_t - \gamma_t)} \eta_t^{\operatorname{Im}(k_t - \gamma_t)} \psi_1 \dots \psi_n.$$

If $t \in \Gamma \setminus \{t_1, \dots, t_n\}$, then the weight $w = \psi_1 \dots \psi_n$ has no singularity at t and

$$\varphi_{t, k_t - \gamma_t} = \omega_t^{\operatorname{Re}(k_t - \gamma_t)} \eta_t^{\operatorname{Im}(k_t - \gamma_t)}$$

is a continuous function on $\Gamma \setminus \{t\}$. If $t = t_j \in \{t_1, \dots, t_n\}$, then the weight w/ψ_j has no singularity at t_j and

$$\varphi_{t_j, k_{t_j} - \gamma_{t_j}} \psi_j = \omega_{t_j}^{\operatorname{Re}(k_{t_j} - \gamma_{t_j})} \eta_{t_j}^{\operatorname{Im}(k_{t_j} - \gamma_{t_j})} \psi_j$$

is a continuous function on $\Gamma \setminus \{t_j\}$. Thus, in both cases, the weight w_t is of the same form as the weight w .

It is easy to see that the function $W_t(\omega_t^{\operatorname{Re}(k_t - \gamma_t)})$ is regular and submultiplicative and

$$\alpha(W_t^0(\omega_t^{\operatorname{Re}(k_t - \gamma_t)})) = \beta(W_t^0(\omega_t^{\operatorname{Re}(k_t - \gamma_t)})) = \operatorname{Re}(k_t - \gamma_t) \quad \text{for every } t \in \Gamma.$$

Then, by Lemma 2.6, the functions $W_t(\varphi_{t, k_t - \gamma_t})$ and $W_t^0(\varphi_{t, k_t - \gamma_t})$ are regular and submultiplicative and

$$\alpha(W_t^0(\varphi_{t, k_t - \gamma_t})) = \operatorname{Re}(k_t - \gamma_t) + \alpha(W_t(\eta_t^{\operatorname{Im}(k_t - \gamma_t)})) = \operatorname{Re}(k_t - \gamma_t) + \alpha_t^*(\operatorname{Im}(k_t - \gamma_t)), \quad (41)$$

$$\beta(W_t^0(\varphi_{t, k_t - \gamma_t})) = \operatorname{Re}(k_t - \gamma_t) + \beta(W_t(\eta_t^{\operatorname{Im}(k_t - \gamma_t)})) = \operatorname{Re}(k_t - \gamma_t) + \beta_t^*(\operatorname{Im}(k_t - \gamma_t)) \quad (42)$$

for all $t \in \Gamma \setminus \{t_1, \dots, t_n\}$. Analogously, if $t = t_j \in \{t_1, \dots, t_n\}$, then the function $W_{t_j}(\varphi_{t_j, k_{t_j} - \gamma_{t_j}} \psi_j)$ is regular and submultiplicative and

$$\alpha(W_{t_j}^0(\varphi_{t_j, k_{t_j} - \gamma_{t_j}} \psi_j)) = \operatorname{Re}(k_{t_j} - \gamma_{t_j}) + \alpha_{t_j}^*(\operatorname{Im}(k_{t_j} - \gamma_{t_j})), \quad (43)$$

$$\beta(W_{t_j}^0(\varphi_{t_j, k_{t_j} - \gamma_{t_j}} \psi_j)) = \operatorname{Re}(k_{t_j} - \gamma_{t_j}) + \beta_{t_j}^*(\operatorname{Im}(k_{t_j} - \gamma_{t_j})). \quad (44)$$

Combining relations (39)–(44) with conditions (5), we see that, by Theorem 1.2, the operator S is bounded on $L^{p(\cdot)}(\Gamma, w_t) = L^{p(\cdot)}(\varphi_{t, k_t - \gamma_t} w)$, where $t \in \Gamma$. Therefore the operator $\varphi_{t, k_t - \gamma_t} S \varphi_{t, \gamma_t - k_t} I$ is bounded on $L^{p(\cdot)}(\Gamma, w)$.

Then, in view of Lemma 6.5, the function g_{t,γ_t} admits a Wiener-Hopf factorization on $L^{p(\cdot)}(\Gamma, w)$. From Theorem 5.3 we deduce that the operator $g_{t,\gamma_t}P + Q$ is Fredholm. It is not difficult to see that in this case the operator $c_t g_{t,\gamma_t}P + Q$ is also Fredholm. Thus, for all local representatives $c_t g_{t,\gamma_t}$ of the coefficient a , the operators $c_t g_{t,\gamma_t}P + Q$ are Fredholm. To finish the proof, it remains to apply the local principle (Theorem 5.2), which says that the operator $aP + Q$ is Fredholm. ■

6.5. Proof of Theorem 1.3. Necessity. If $aP + bQ$ is Fredholm, then $a^{-1}, b^{-1} \in L^\infty(\Gamma)$ by Theorem 5.1. Put $c := a/b$. Then $c(t \pm 0) \neq 0$ for all $t \in \Gamma$. Further, the operator bI is invertible on $L^{p(\cdot)}(\Gamma, w)$. Therefore, the operator $cP + Q = (bI)^{-1}(aP + bQ)$ is Fredholm. From Theorem 6.4 and equalities (37) it follows that

$$-\frac{1}{2\pi} \arg \frac{c(t-0)}{c(t+0)} + \frac{1}{p(t)} + \theta \alpha_t^* \left(\frac{1}{2\pi} \log \left| \frac{c(t-0)}{c(t+0)} \right| \right) + (1-\theta) \beta_t^* \left(\frac{1}{2\pi} \log \left| \frac{c(t-0)}{c(t+0)} \right| \right) \notin \mathbb{Z} \quad (45)$$

for all $\theta \in [0, 1]$ and all $t \in \Gamma$. The latter condition in conjunction with $c(t \pm 0) \neq 0$ for all $t \in \Gamma$ is equivalent to $0 \notin \bigcup_{t \in \Gamma} \mathcal{L}(c(t-0), c(t+0); p(t), \alpha_t^*, \beta_t^*)$. Thus, the function $c = a/b$ is $L^{p(\cdot)}(\Gamma, w)$ -nonsingular. Necessity is proved.

Sufficiency. The $L^{p(\cdot)}(\Gamma, w)$ -nonsingularity of $c = a/b$ implies that $c(t \pm 0) \neq 0$ and (45) holds for all $\theta \in [0, 1]$ and all $t \in \Gamma$. Then the operator $cP + Q$ is Fredholm by Theorem 6.6. Since $\inf_{t \in \Gamma} |b(t)| > 0$, we see that the operator bI is invertible. Thus, the operator $aP + bQ = (bI)(cP + Q)$ is Fredholm. ■

After the initial submission of this paper in March of 2010, the work [35] has appeared, where an alternative approach to the Fredholm theory of singular integral operators on Nakano spaces is presented. It allows one to consider a large (but proper) subclass of composed Carleson curves. On the other hand, our approach is powerful enough to treat arbitrary Carleson curves within the class of non-composed (Jordan) curves.

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