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# AN ITERATIVE ALGORITHM BY VISCOSITY APPROXIMATION METHOD FOR MIXED EQUILIBRIUM PROBLEMS, VARIATIONAL INCLUSION AND FIXED POINT OF AN INFINITE FAMILY OF PSEUDO-CONTRACTIVE MAPPINGS

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**Abstract.** The purpose of this paper is to investigate the problem of finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of the variational inclusion problems for inverse strongly monotone mappings and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings in the setting of Hilbert spaces. We prove the strong convergence theorem by using the viscosity approximation method for finding the common element of the above four sets. Our results improve and extend the corresponding results of Peng and Yao [Math. Comput. Modelling 49 (2009), 1816–1828], Plubtieng and Sriprad [Fixed Point Theory Appl. 2009, Article ID 567147] and some well-known results in the literature.

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**1. Introduction.** Throughout this paper, we assume that $H$ is a real Hilbert space with inner product and norm which are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively, $C$ is a closed convex subset of $H$, $\mathbf{R}$ is the set of real numbers and $\mathbf{N}$ is the set of natural numbers. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of $T$, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a self-mapping $f : C \to C$ is a contraction on $C$ if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$. Let $B$ be a strongly positive bounded linear operator on $H$, that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Bx, x \rangle \geq \overline{\gamma}\|x\|^2 \text{ for all } x \in H. \tag{1}$$

Let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper extended real-valued function and $F$ be a bifunction of $C \times C$ into $\mathbf{R}$. Ceng and Yao [CY] considered the mixed equilibrium problem for finding $x \in C$ such that

$$F(x, y) + \varphi(y) \geq \varphi(x) \text{ for all } y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $MEP(F, \varphi)$. We see that $x$ is a solution of problem (2) which implies that $x \in \operatorname{dom} \varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi = 0$, then the mixed equilibrium problem (2) becomes the following equilibrium problem: find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{3}$$

The set of solutions of (3) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., $z$ is a solution of the variational inequality. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems and the equilibrium problem as special cases. Numerous problems in physics, optimization and economics are reduced to find a solution of (3). Some methods have been proposed to solve the equilibrium problem (see [BO, FA, K1, K2, K3, KK, MT]).

Let $A : H \to H$ be a mapping. Then $A$ is called:

(1) *monotone* if
$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in H;$$

(2) *$\sigma$-strongly monotone* if there exists a positive real number $\sigma$ such that
$$\langle Ax - Ay, x - y \rangle \geq \sigma\|x - y\|^2 \quad \forall x, y \in H.$$
For constant $\sigma > 0$, this implies that
$$\|Ax - Ay\| \geq \sigma\|x - y\|,$$
that is, $A$ is $\sigma$-expansive and when $\sigma = 1$, it is expansive;

(3) *$\sigma$-inverse-strongly monotone* if there exists a positive real number $\sigma$ such that
$$\langle Ax - Ay, x - y \rangle \geq \sigma\|Ax - Ay\|^2 \quad \forall x, y \in H;$$

(4) *$k$-strictly pseudo-contractive*, if there exists a constant $k \in [0, 1)$ such that
$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + k\|(I - A)x - (I - A)y\|^2 \quad \forall x, y \in H.$$

Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. We consider the following *variational inclusion problem*, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{4}$$

where $\theta$ is the zero vector in $H$. The set of solutions of problem (4) is denoted by $I(A, M)$.

If $M = \partial\delta_C$, where $C$ is a nonempty closed convex subset of $H$ and $\delta_C : H \to [0, \infty]$ is the indicator function of $C$, i.e., $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = +\infty$ for $x \notin C$, then the variational inclusion problem (4) is equivalent to finding $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \quad \forall v \in H. \tag{5}$$

This problem is called the *Hartman-Stampacchia variational problem* ([Bro, HS, LS]). The set of solutions of problem (5) is denoted by $VI(C, A)$.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph of $G(M)$ of $M$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $M$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Let the set-valued mapping $M : H \to 2^H$ be maximal monotone. We define the *resolvent operator* $J_{M,\lambda}$ associated with $M$ and $\lambda$ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u) \quad u \in H, \tag{6}$$

where $\lambda$ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive and 1-inverse strongly monotone ([Bré, PWSY, PS, ZLC]).

In this paper, we introduce a new iterative process (16) below for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of the variational inclusion problems for inverse strongly monotone mappings and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings in a Hilbert space. Then, we prove strong convergence theorems which extend and improve the corresponding results of Peng and Yao [PY] and Plubtieng and Sriprad [PS].

**2. Preliminaries.** Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let $C$ be a closed convex subset of $H$. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{7}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{8}$$

for all $x, y \in H$ and $\lambda \in \mathbf{R}$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \ \text{ for all } y \in C.$$

$P_C$ is called the *metric projection* of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{9}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{10}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{11}$$

for all $x \in H$, $y \in C$.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction $F$, $\varphi$ and the set $C$:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous;
(A5) for each $y \in C, x \mapsto F(x, y)$ is weakly upper semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x \rangle < \varphi(z);$$

(B2) $C$ is a bounded set.

Lemma 2.1 (Peng and Yao [PY]). *Let $C$ be a nonempty closed convex subset of $H$. Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying* (A1)–(A5) *and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \Big\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x \rangle \geq \varphi(z) \quad \forall y \in C \Big\}$$

*for all $z \in H$. Then*

1. *For each $x \in H$, $T_r(x) \neq \emptyset$*;
2. *$T_r$ is single-valued*;
3. *$T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$*;
4. *$F(T_r) = MEP(F, \varphi)$*;
5. *$MEP(F, \varphi)$ is closed and convex.*

In order to prove our main results, we need the following lemmas.

Lemma 2.2 (Zhou [Z]). *If $V : C \to H$ is a $k$-strict pseudo-contraction, then*

(1) *the fixed point set $F(V)$ of $V$ is closed convex, so that the projection $P_{F(V)}$ is well defined*;
(2) *define a mapping $T : C \to H$ by*

$$Tx = tx + (1-t)Vx \quad \forall x \in C. \tag{12}$$

*If $t \in [k, 1)$, then $T$ is a nonexpansive mapping such that $F(V) = F(T)$.*

A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly $k$-strict pseudo-contractions*, if there exists a constant $k \in [0, 1)$ such that

$$\|V_i x - V_i y\|^2 \leq \|x - y\|^2 + k\|(I - V_i)x - (I - V_i)y\|^2 \quad \forall x, y \in C \ \forall i \geq 1.$$

Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly $k$-strict pseudo-contractions. Let $\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (12), i.e.,

$$T_i x = tx + (1-t)V_i x \quad \forall x \in C\ \forall i \geq 1,\ t \in [k,1). \tag{13}$$

Let $\{T_i\}$ be a sequence of nonexpansive mappings of $C$ into itself defined by (13) and let $\{\mu_i\}$ be a sequence of nonnegative numbers in $[0,1]$. For each $n \geq 1$, define a mapping $W_n$ of $C$ into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \quad k = n, n-1, \ldots, 2, \\ W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I. \end{aligned} \tag{14}$$

Such a mapping $W_n$ is nonexpansive from $C$ to $C$ and it is called the *$W$-mapping generated by $T_1, T_2, \ldots, T_n$* and *$\mu_1, \mu_2, \ldots, \mu_n$*.

Let for each $n, k \in \mathbf{N}$ the mapping $U_{n,k}$ be defined by (14). Then we can have the following crucial conclusions concerning $W_n$ which can be found in [ST]. Now we only need the following similar version in Hilbert spaces.

LEMMA 2.3 (Shimoji and Takahashi [ST]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $C$ into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then*

(1) *$W_n$ is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for every $n \geq 1$;*
(2) *for every $x \in C$ and $k \in \mathbf{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *a mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \quad \forall x \in C \tag{15}$$

*is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and it is called the $W$-mapping generated by $T_1, T_2, \ldots$ and $\mu_1, \mu_2, \ldots$.*

LEMMA 2.4 (Chang [C]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1$ for $i \geq 1$. If $D$ is any bounded subset of $C$, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

LEMMA 2.5 (Xu [X]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0,$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.6 (Osilike and Igbokwe [OI]). *Let $(E, \langle ., . \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

LEMMA 2.7 (Suzuki [S]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

LEMMA 2.8 (Marino and Xu [MX]). *Assume $B$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I-\rho B\| \leq 1-\rho\bar{\gamma}$.*

LEMMA 2.9 (Opial [O]). *Each Hilbert space $H$ satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

*holds for each $y \in H$ with $y \neq x$.*

LEMMA 2.10 (Brézis [Bré]). *Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

REMARK 2.11. Lemma 2.10 implies that $I(A, M)$ is closed and convex if $M : H \to 2^H$ is a maximal monotone mapping and $A : H \to H$ is a Lipschitz continuous mapping.

LEMMA 2.12 (Zhang et al. [ZLC]). *$u \in H$ is a solution of variational inclusion* (4) *if and only if $u = J_{M,\lambda}(u - \lambda Au)$ for each $\lambda > 0$, i.e.,*

$$I(A, M) = F(J_{M,\lambda}(I - \lambda A)) \quad \forall\lambda > 0.$$

**3. Main results.** In this section, we show a strong convergence theorem for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of the variational inclusion problems for inverse strongly monotone mappings and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings in a Hilbert space by using the viscosity approximation method.

THEOREM 3.1. *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction of $C \times C$ into real numbers $\mathbf{R}$ satisfying* (A1)–(A5) *and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $f$ be a contraction of $H$ into itself with coefficient $\alpha \in (0,1)$ and $B$ be a strongly bounded linear operator on $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $M_1, M_2 : H \to 2^H$ be maximal monotone mappings and $A_1, A_2 : H \to H$ be $\sigma_1, \sigma_2$-inverse-strongly monotone mappings, respectively. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly $k$-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x$, for all $x \in C$, $i \geq 1$, $t \in [k, 1)$. Let $W_n$ be the $W$-mapping defined by* (14) *and $W$ be a mapping defined by* (15) *with $F(W) \neq \emptyset$. Assume that either* (B1) *or* (B2) *holds and $\Omega := \bigcap_{n=1}^{N} F(T_i) \cap I(A_1, M_1) \cap I(A_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let*

*$\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \qquad \forall y \in C, \\ y_n = J_{M_2,\delta}(u_n - \delta A_2 u_n), \\ v_n = J_{M_1,\tau}(y_n - \tau A_1 y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n \end{cases} \tag{16}$$

*for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, $\tau \in (0, 2\sigma_1)$ and $\delta \in (0, 2\sigma_2)$ satisfy:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

*Then $\{x_n\}$ converges strongly to $z \in \Omega$ which is the unique solution of the variational inequality*

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \quad x \in \Omega.$$

*Equivalently, we have $z = P_\Omega(I - B + \gamma f)(z)$.*

*Proof.* First, we show that $I - \tau A_1$ and $I - \delta A_2$ are nonexpansive. Indeed, for all $x, y \in H$ and $\tau \in (0, 2\sigma_1)$, we note that

$$\begin{aligned} \|(I - \tau A_1)u - (I - \tau A_1)v\|^2 &= \|(u - v) - \tau(A_1 u - A_1 v)\|^2 \\ &= \|u - v\|^2 - 2\tau\langle u - v, A_1 u - A_1 v\rangle + \tau^2\|A_1 u - A_1 v\|^2 \\ &\leq \|u - v\|^2 + \tau(\tau - 2\sigma_1)\|A_1 u - A_1 v\|^2 \leq \|u - v\|^2, \end{aligned} \tag{17}$$

which implies that the mapping $I - \tau A_1$ is nonexpansive. So is $I - \delta A_2$.

By condition (i), we may assume, without loss of generality, that $\alpha_n < \|B\|^{-1}$ for all $n$. We assume that $\|I - B\| \leq 1 - \bar{\gamma}$. Since $B$ is a strongly positive bounded linear operator on $H$, we have

$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \ \|x\| = 1\}.$$

Observe that

$$\langle ((1-\beta_n)I - \alpha_n B)x, x\rangle = 1 - \beta_n - \alpha_n\langle Bx, x\rangle \geq 1 - \beta_n - \alpha_n\|B\| \geq 0,$$

this shows that $(1-\beta_n)I - \alpha_n B$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n B\| &= \sup\big\{\big|\langle((1-\beta_n)I - \alpha_n B)x, x\rangle\big| : x \in H, \|x\| = 1\big\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Bx, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\bar{\gamma}. \end{aligned}$$

Let $p \in \Omega$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1 and $u_n = T_{r_n}x_n$. For any $n \in \mathbf{N}$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|.$$

Since $p \in I(A_1, M_1)$ and $p \in I(A_2, M_2)$, we have $p = J_{M_1,\tau}(p - \tau A_1 p) = J_{M_2,\delta}(p - \delta A_2 p)$.

Because $I - \tau A_1$, $I - \delta A_2$, $J_{M_1,\tau}$ and $J_{M_2,\delta}$ are nonexpansive mappings, we obtain

$$\begin{aligned}\|v_n - p\| &= \|J_{M_1,\tau}(y_n - \tau A_1 z_n) - J_{M_1,\tau}(p - \tau A_1 p)\| \\ &\leq \|(I - \tau A_1)y_n - (I - \tau A_1)p\| \leq \|y_n - p\| \\ &= \|J_{M_2,\delta}(u_n - \delta A_2 u_n) - J_{M_2,\delta}(p - \delta A_2 p)\| \\ &\leq \|(I - \delta A_2)u_n - (I - \delta A_2)p\| \\ &\leq \|u_n - p\| \leq \|x_n - p\|. \end{aligned} \tag{18}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Bp) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n B)(W_n v_n - p)\| \\ &\leq \alpha_n\|\gamma f(x_n) - Bp\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|v_n - p\| \\ &\leq \alpha_n\|\gamma f(x_n) - Bp\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\leq \alpha_n\|\gamma f(x_n) - \gamma f(p)\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &= (1 - (\bar{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\| + (\bar{\gamma} - \gamma\alpha)\alpha_n\frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma\alpha} \end{aligned} \tag{19}$$

for every $n \in \mathbf{N}$. It follows by mathematical induction that

$$\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Bp\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad n \geq 1.$$

Therefore $\{x_n\}$ is bounded, so $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{W_n v_n\}$ are all bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - W_n v_n\| \to 0$ as $n \to 0$. Observing that $u_n = T_{r_n}x_n \in \operatorname{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \operatorname{dom}\varphi$ we get

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C \tag{20}$$

and

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \text{ for all } y \in C. \tag{21}$$

Take $y = u_{n+1}$ in (20) and $y = u_n$ in (21), by using condition (A2), we obtain

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

Thus $\langle u_{n+1} - u_n, u_n - u_{n+1} + x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \geq 0$. Without loss of generality, let us assume that there exists a real number $c$ such that $r_n > c$, for $n \geq 1$. Then we have

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1, \end{aligned} \tag{22}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbf{N}\}$. On the other hand, again since $I - \tau A_1$, $I - \delta A_2$, $J_{M_1,\tau}$ and $J_{M_2,\delta}$ are nonexpansive, we obtain

$$\begin{aligned}\|v_{n+1} - v_n\| &= \|J_{M_1,\tau}(y_{n+1} - \tau A_1 y_{n+1}) - J_{M_1,\tau}(y_n - \tau A_1 y_n)\| \\ &\leq \|(y_{n+1} - \tau A_1 y_{n+1}) - (y_n - \tau A_1 y_n)\| \leq \|y_{n+1} - y_n\| \\ &= \|J_{M_2,\delta}(u_{n+1} - \delta A_2 u_{n+1}) - J_{M_2,\delta}(u_n - \delta A_2 u_n)\| \\ &\leq \|(u_{n+1} - \delta A_2 u_{n+1}) - (u_n - \delta B_2 u_n)\| \leq \|u_{n+1} - u_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}\,|r_{n+1} - r_n| M_1. \end{aligned} \tag{23}$$

Since $T_i$ and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}\|W_{n+1}v_n - W_n v_n\| &= \|\mu_1 T_1 U_{n+1,2} v_n - \mu_1 T_1 U_{n,2} v_n\| \\ &\leq \mu_1 \|U_{n+1,2} v_n - U_{n,2} v_n\| \\ &= \mu_1 \|\mu_2 T_2 U_{n+1,3} v_n - \mu_2 T_2 U_{n,3} v_n\| \\ &\leq \mu_1 \mu_2 \|U_{n+1,3} v_n - U_{n,3} v_n\| \\ &\ \ \vdots \\ &\leq \mu_1 \mu_2 \cdots \mu_n \|U_{n+1,n+1} v_n - U_{n,n+1} v_n\| \\ &\leq M_2 \prod_{i=1}^{n} \mu_i, \end{aligned} \tag{24}$$

where $M_2 \geq 0$ is a constant such that $\|U_{n+1,n+1} v_n - U_{n,n+1} v_n\| \leq M_2$ for all $n \geq 0$. It follows from (23) and (24) that

$$\begin{aligned}\|W_{n+1}v_{n+1} - W_n v_n\| &\leq \|W_{n+1}v_{n+1} - W_{n+1}v_n\| + \|W_{n+1}v_n - W_n v_n\| \\ &\leq \|v_{n+1} - v_n\| + M_2 \prod_{i=1}^{n} \mu_i \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{c}\,|r_{n+1} - r_n| M_1 + M_2 \prod_{i=1}^{n} \mu_i. \end{aligned} \tag{25}$$

Define the sequence $\{z_n\}$ by $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$, for each $n \geq 1$. Then, observe that

$$\begin{aligned} z_n &= \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n B) W_n v_n - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n B) W_n v_n}{1 - \beta_n} \end{aligned}$$

and hence

$$\begin{aligned} z_{n+1} - z_n = {}& \frac{\alpha_{n+1} \gamma f(x_{n+1}) + ((1 - \beta_{n+1}) I - \alpha_{n+1} B) W_{n+1} v_{n+1}}{1 - \beta_{n+1}} \\ & - \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n B) W_n v_n}{1 - \beta_n} \end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})W_{n+1}v_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}BW_{n+1}v_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)W_nv_n}{1-\beta_n} + \frac{\alpha_nBW_nv_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - BW_{n+1}v_{n+1}) + \frac{\alpha_n}{1-\beta_n}(BW_nv_n - \gamma f(x_n)) \\
&\quad + W_{n+1}v_{n+1} - W_nv_n. \qquad (26)
\end{aligned}$$

Combining this with (25), we obtain

$$\begin{aligned}
\|z_{n+1}-z_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|BW_{n+1}v_{n+1}\|\big) \\
&\quad + \frac{\alpha_n}{1-\beta_n}\big(\|BW_nv_n\| + \|\gamma f(x_n)\|\big) + \|W_{n+1}v_{n+1} - W_nv_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|BW_{n+1}v_{n+1}\|\big) \\
&\quad + \frac{\alpha_n}{1-\beta_n}\big(\|BW_nv_n\| + \|\gamma f(x_n)\|\big) \\
&\quad + \|x_{n+1}-x_n\| + \frac{1}{c}|r_{n+1}-r_n|M_1 + M_2\prod_{i=1}^{n}\mu_i.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|BW_{n+1}v_{n+1}\|\big) \\
&\quad + \frac{\alpha_n}{1-\beta_n}\big(\|BW_nv_n\| + \|\gamma f(x_n)\|\big) + \frac{1}{c}|r_{n+1}-r_n|M_1 + M_2\prod_{i=1}^{n}\mu_i.
\end{aligned}$$

Assumptions (i)–(iii) imply that

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|\big) \le 0.$$

Hence, by Lemma 2.7, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \qquad (27)$$

From (ii), (23) and (27), we have $\|u_{n+1}-u_n\| \to 0$, $\|y_{n+1}-y_n\| \to 0$ and $\|v_{n+1}-v_n\| \to 0$ as $n \to \infty$. We note that

$$\begin{aligned}
x_{n+1}-x_n &= \alpha_n\gamma f(x_n) + \beta_nx_n + \big((1-\beta_n)I - \alpha_nB\big)W_nv_n - x_n \\
&= \alpha_n\gamma f(x_n) - \alpha_nBx_n + \alpha_nBx_n + \beta_nx_n + \big((1-\beta_n)I - \alpha_nB\big)W_nv_n \\
&\quad - \big((1-\beta_n)I - \alpha_nB\big)x_n + \big((1-\beta_n)I - \alpha_nB\big)x_n - x_n \\
&= \alpha_n(\gamma f(x_n) - Bx_n) + \big((1-\beta_n)I - \alpha_nB\big)(W_nv_n - x_n),
\end{aligned}$$

hence

$$(1-\beta_n-\alpha_n\bar{\gamma})\|x_n - W_nv_n\| \le \alpha_n\|\gamma f(x_n) - Bx_n\| + \|x_n - x_{n+1}\|.$$

From (i)–(iii) and (27) we obtain

$$\lim_{n\to\infty} \|W_n v_n - x_n\| = 0. \tag{28}$$

Next, we shall show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. For any $p \in \Omega$ and $T_{r_n}$ is firmly nonexpansive, and we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle = \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2).\end{aligned}$$

It follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|v_n - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|u_n - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\ &\qquad + (1-\beta_n-\alpha_n\bar{\gamma})(\|x_n - p\|^2 - \|u_n - x_n\|^2) \\ &= \alpha_n\|\gamma f(x_n) - Bp\|^2 + (1-\alpha_n\bar{\gamma})\|x_n - p\|^2 \\ &\qquad - (1-\beta_n-\alpha_n\bar{\gamma})\|u_n - x_n\|^2.\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}(1-\beta_n-\alpha_n\bar{\gamma})\|u_n - x_n\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + (1-\alpha_n\bar{\gamma})\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).\end{aligned}$$

Assumptions (i)–(iii) and formula (27) imply that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{30}$$

and by (ii) we have

$$\lim_{n\to\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0.$$

We note that, by (29), nonexpansiveness of $J_{M_1,\tau}$, $J_{M_2,\delta}$ and the inverse-strong monotonicity of $A_1$, $A_2$ imply that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|v_n - p\|^2 \\ &= \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\ &\qquad + (1-\beta_n-\alpha_n\bar{\gamma})\big\|J_{M_1,\tau}(y_n - \tau A_1 y_n) - J_{M_1,\tau}(p - \tau A_1 p)\big\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\ &\qquad + (1-\beta_n-\alpha_n\bar{\gamma})\big\|(I - \tau A_1)y_n - (I - \tau A_1)p\big\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\ &\qquad + (1-\beta_n-\alpha_n\bar{\gamma})\big\{\|y_n - p\|^2 + \tau(\tau - 2\sigma_1)\|A_1 y_n - A_1 p\|^2\big\} \\ &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|x_n - p\|^2 \\ &\qquad + (1-\beta_n-\alpha_n\bar{\gamma})\tau(\tau - 2\sigma_1)\|A_1 y_n - A_1 p\|^2\end{aligned}$$

$$\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2$$
$$+ (1-\beta_n-\alpha_n\bar{\gamma})\tau(\tau - 2\sigma_1)\|A_1 y_n - A_1 p\|^2$$

and

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|y_n - p\|^2 \\
&= \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\big\|J_{M_2,\delta}(u_n - \delta A_2 u_n) - J_{M_2,\delta}(p - \delta A_2 p)\big\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\big\|(I - \delta A_2)u_n - (I - \delta A_2)p\big\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\big\{\|u_n - p\|^2 + \delta(\delta - 2\sigma_2)\|A_2 u_n - A_2 p\|^2\big\} \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\delta(\delta - 2\sigma_2)\|A_2 u_n - A_2 p\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\delta(\delta - 2\sigma_2)\|A_2 u_n - A_2 p\|^2
\end{aligned}$$

which imply that

$$\begin{aligned}
0 &\leq (1-\beta_n-\alpha_n\bar{\gamma})\tau(2\sigma_1 - \tau)\|A_1 y_n - A_1 p\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq (1-\beta_n-\alpha_n\bar{\gamma})\delta(2\sigma_2 - \delta)\|A_2 u_n - A_2 p\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

It follows from (i), (iii) and (27) that

$$\lim_{n\to\infty}\|A_1 y_n - A_1 p\| = 0 \tag{31}$$

and

$$\lim_{n\to\infty}\|A_2 u_n - A_2 p\| = 0. \tag{32}$$

On the other hand, since $J_{M_1,\tau}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|v_n - p\|^2 &= \big\|J_{M_1,\tau}(y_n - \tau A_1 y_n) - J_{M_1,\tau}(p - \tau A_1 p)\big\|^2 \\
&\leq \langle (y_n - \tau A_1 y_n) - (p - \tau A_1 p), v_n - p\rangle \\
&= \frac{1}{2}\big\{\|(y_n - \tau A_1 y_n) - (p - \tau A_1 p)\|^2 + \|v_n - p\|^2 \\
&\qquad - \|(y_n - \tau A_1 y_n) - (p - \tau A_1 p) - (v_n - p)\|^2\big\} \\
&\leq \frac{1}{2}\big\{\|y_n - p\|^2 + \|v_n - p\|^2 - \|(y_n - v_n) - \tau(A_1 y_n - A_1 p)\|^2\big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big\{\|y_n - p\|^2 + \|v_n - p\|^2 \\
&\qquad - \|y_n - v_n\|^2 + 2\tau\langle y_n - v_n, A_1 y_n - A_1 p\rangle - \tau^2\|A_1 y_n - A_1 p\|^2\big\} \\
&\le \frac{1}{2}\big\{\|y_n - p\|^2 + \|v_n - p\|^2 \\
&\qquad - \|y_n - v_n\|^2 + 2\tau\|y_n - v_n\|\|A_1 y_n - A_1 p\| - \tau^2\|A_1 y_n - A_1 p\|^2\big\} \\
&\le \frac{1}{2}\big\{\|x_n - p\|^2 + \|v_n - p\|^2 - \|y_n - v_n\|^2 + 2\tau\|y_n - v_n\|\|A_1 y_n - A_1 p\|\big\},
\end{aligned}
$$

which yields that

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\tau\|y_n - v_n\|\|A_1 y_n - A_1 p\|. \tag{33}$$

Similarly, since $J_{M_2,\delta}$ is firmly nonexpansive, we also have

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta\|u_n - y_n\|\|A_2 u_n - A_2 p\|. \tag{34}$$

Substituting (33) into (29), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|W_n v_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|v_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1 - \beta_n - \alpha_n\bar{\gamma})\big(\|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\tau\|y_n - v_n\|\|A_1 y_n - A_1 p\|\big) \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - v_n\|^2 \\
&\quad + 2\tau(1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - v_n\|\|A_1 y_n - A_1 p\|
\end{aligned} \tag{35}
$$

and substituting (34) into (29), we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|W_n v_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|v_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - p\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + (1 - \beta_n - \alpha_n\bar{\gamma})\big(\|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta\|u_n - y_n\|\|A_2 u_n - A_2 p\|\big) \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|^2 \\
&\quad + 2\delta(1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|\|A_2 u_n - A_2 p\|.
\end{aligned} \tag{36}
$$

Therefore, by (35) and (36), we have

$$
\begin{aligned}
(1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - v_n\|^2 &\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\tau(1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - v_n\|\|A_1 y_n - A_1 p\| \\
&\le \alpha_n\|\gamma f(x_n) - Bp\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad + 2\tau(1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - v_n\|\|A_1 y_n - A_1 p\|
\end{aligned} \tag{37}
$$

and

$$\begin{aligned}(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-y_n\|^2 &\leq \alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2\delta(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-y_n\|\|A_2u_n-A_2p\|\\&\leq \alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\&\quad+2\delta(1-\beta_n-\alpha_n\bar{\gamma})\|u_n-y_n\|\|A_2u_n-A_2p\|. \end{aligned}\tag{38}$$

It follows from (i), (iii) and (27) that

$$\lim_{n\to\infty}\|y_n-v_n\|=0 \tag{39}$$

and

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{40}$$

From (28), (30), (39) and (40), we have

$$\|W_nv_n-v_n\|\leq\|W_nv_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|+\|y_n-v_n\|\to 0 \quad \text{as} \quad n\to\infty \tag{41}$$

and also

$$\|v_n-x_n\|\leq\|v_n-y_n\|+\|y_n-u_n\|+\|u_n-x_n\|\to 0 \quad \text{as} \quad n\to\infty. \tag{42}$$

Observe that $P_\Omega(I-B+\gamma f)$ is a contraction of $H$ into itself. Indeed, for all $x,y\in H$, we have

$$\begin{aligned}\big\|P_\Omega(I-B+\gamma f)(x)-P_\Omega(I-B+\gamma f)(y)\big\| &\leq \|(I-B+\gamma f)(x)-(I-B+\gamma f)(y)\|\\&\leq\|I-B\|\|x-y\|+\gamma\|f(x)-f(y)\|\\&\leq(1-\bar{\gamma})\|x-y\|+\gamma\alpha\|x-y\|\\&=(1-(\bar{\gamma}-\gamma\alpha))\|x-y\|.\end{aligned}$$

Since $H$ is complete, there exists a unique fixed point $z\in H$ such that

$$z=P_\Omega(I-B+\gamma f)(z).$$

Next, we show that

$$\limsup_{n\to\infty}\langle(B-\gamma f)z,z-x_n\rangle\leq 0.$$

Indeed, we can choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\lim_{i\to\infty}\langle(B-\gamma f)z,z-v_{n_i}\rangle=\limsup_{n\to\infty}\langle(B-\gamma f)z,z-v_n\rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to $v\in C$. Without loss of generality, we can assume that $v_{n_i}\rightharpoonup v$. From $\|W_nv_n-v_n\|\to 0$, we obtain $W_nv_{n_i}\rightharpoonup v$. Let us show that $v\in MEP(F,\varphi)$. Since $u_n=T_{r_n}x_n\in\operatorname{dom}\varphi$, we have

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0 \quad \forall y\in C.$$

From (A2) we also have

$$\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq F(y,u_n) \quad \forall y\in C,$$

and hence

$$\varphi(y)-\varphi(u_n)+\Big\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\Big\rangle\geq F(y,u_{n_i}) \quad \forall y\in C.$$

From $\|u_n - x_n\| \to 0$, $\|x_n - W_n v_n\| \to 0$, and $\|W_n v_n - v_n\| \to 0$, we get $u_{n_i} \rightharpoonup v$. Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$, it follows by (A4) and the weak lower semicontinuity of $\varphi$ that

$$F(y, v) + \varphi(v) - \varphi(y) \le 0 \quad \forall y \in C.$$

For $t$ with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)v$. Since $y \in C$ and $v \in C$, we have $y_t \in C$ and hence $F(y_t, v) + \varphi(v) - \varphi(y_t) \le 0$. So, from (A1), (A4) and the convexity of $\varphi$, we have

$$\begin{aligned}
0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\le tF(y_t, y) + (1-t)F(y_t, v) + t\varphi(y) + (1-t)\varphi(v) - \varphi(y_t) \\
&\le t(F(y_t, y) + \varphi(y) - \varphi(y_t)).
\end{aligned}$$

Dividing by $t$, we get $F(y_t, y) + \varphi(y) - \varphi(y_t) \ge 0$. From (A3) and the weak lower semicontinuity of $\varphi$, we have $F(v, y) + \varphi(y) - \varphi(v) \ge 0$ for all $y \in C$ and hence $v \in MEP(F, \varphi)$.

Next, we show that $v \in F(W) = \bigcap_{n=1}^{\infty} F(W_n)$, where $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for every $n \ge 1$, and $F(W_{n+1}) \subset F(W_n)$. Assume that $v \notin F(W)$, then there exists a positive integer $m$ such that $v \notin F(T_m)$ and so $v \notin \bigcap_{i=1}^{m} F(T_i)$. Hence for any $n \ge m$, $v \notin \bigcap_{i=1}^{n} F(T_i) = F(W_n)$, i.e., $v \ne W_n v$. It follows from Opial's condition that

$$\begin{aligned}
\liminf_{i\to\infty} \|v_{n_i} - v\| &< \liminf_{i\to\infty} \|v_{n_i} - W_n v\| \\
&\le \liminf_{i\to\infty} (\|v_{n_i} - W_n v_{n_i}\| + \|W_n v_{n_i} - W_n v\|) \\
&\le \liminf_{i\to\infty} \|v_{n_i} - v\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $v \in F(W)$.

Next, we show that $v \in I(A_1, M_1)$ and $v \in I(A_2, M_2)$. The fact that $A_1$ is a $\sigma_1$-inverse-strongly monotone mapping implies that $A_1$ is a $\frac{1}{\sigma_1}$-Lipschitz continuous monotone mapping and the domain of $A_1$ equals $H$. It follows from Lemma 2.10 that $M_1 + A_1$ is maximal monotone. Let $(y, g) \in G(M_1 + A_1)$, that is, $g - A_1 y \in M_1(y)$. Since $v_{n_i} = J_{M_1,\tau}(y_{n_i} - \tau A_1 y_{n_i})$, we have $y_{n_i} - \tau A_1 y_{n_i} \in (I + \tau M_1)(v_{n_i})$, that is,

$$\frac{1}{\tau}(y_{n_i} - v_{n_i} - \tau A_1 y_{n_i}) \in M_1(v_{n_i}). \tag{43}$$

Since $M_1 + A_1$ is maximal monotone, we have

$$\left\langle y - v_{n_i}, g - A_1 y - \frac{1}{\lambda}(y_{n_i} - v_{n_i} - \tau A_1 y_{n_i}) \right\rangle \ge 0, \tag{44}$$

and so

$$\begin{aligned}
\langle y - v_{n_i}, g \rangle &\ge \left\langle y - v_{n_i}, A_1 y + \frac{1}{\tau}(y_{n_i} - v_{n_i} - \tau A_1 y_{n_i}) \right\rangle \\
&= \left\langle y - v_{n_i}, A_1 y - A_1 v_{n_i} + A_1 v_{n_i} - A_1 y_{n_i} + \frac{1}{\tau}(y_{n_i} - v_{n_i}) \right\rangle \\
&\ge 0 + \langle y - v_{n_i}, A_1 v_{n_i} - A_1 y_{n_i} \rangle + \left\langle y - v_{n_i}, \frac{1}{\tau}(y_{n_i} - v_{n_i}) \right\rangle
\end{aligned} \tag{45}$$

It follows from (39) and $v_{n_i} \rightharpoonup v$ that

$$\lim_{i\to\infty} \langle y - v_{n_i}, g \rangle = \langle y - v, g \rangle \ge 0. \tag{46}$$

It follows from the maximal monotonicity of $M_1 + A_1$ that $\theta \in (M_1 + A_1)(v)$, that is, $v \in I(A_1, M_1)$. By the same way, from (40) and $y_{n_i} \rightharpoonup v$, we obtain $v \in I(A_2, M_2)$. Hence

$v \in \Omega$ is proved.

Since $z = P_\Omega(I - B + \gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle (B-\gamma f)z, z - x_n\rangle &= \limsup_{n\to\infty}\langle (B-\gamma f)z, z - v_n\rangle \\ &= \lim_{i\to\infty}\langle (B-\gamma f)z, z - v_{n_i}\rangle = \langle (B-\gamma f)z, z - v\rangle \le 0. \quad (47)\end{aligned}$$

By (41), (42) and the last inequality, we conclude that

$$\limsup_{n\to\infty}\langle \gamma f z - Bz, W_n v_n - z\rangle \le 0. \quad (48)$$

Finally, we show that $\{x_n\}$ converges strongly to $z$. Indeed, from (16) we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \big\|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n - z\big\|^2 \\
&= \big\|\alpha_n(\gamma f(x_n) - Bz) + \beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n B)(W_n v_n - z)\big\|^2 \\
&= \alpha_n^2\|\gamma f(x_n) - Bz\|^2 + \big\|\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n B)(W_n v_n - z)\big\|^2 \\
&\quad + 2\big\langle \beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n B)(W_n v_n - z), \alpha_n(\gamma f(x_n) - Bz)\big\rangle \\
&\le \alpha_n^2\|\gamma f(x_n) - Bz\|^2 + \big(\beta_n\|x_n - z\| + (1-\beta_n-\alpha_n\bar\gamma)\|v_n - z\|\big)^2 \\
&\quad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - Bz\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle W_n v_n - z, \gamma f(x_n) - Bz\rangle \\
&\le \alpha_n^2\|\gamma f(x_n) - Bz\|^2 + \big(\beta_n\|x_n - z\| + (1-\beta_n-\alpha_n\bar\gamma)\|x_n - z\|\big)^2 \\
&\quad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - \gamma f(z)\rangle + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Bz\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle W_n v_n - z, \gamma f(x_n) - \gamma f(z)\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle W_n v_n - z, \gamma f(z) - Bz\rangle \\
&\le \alpha_n^2\|\gamma f(x_n) - Bz\|^2 + (1-\alpha_n\bar\gamma)^2\|x_n - z\|^2 \\
&\quad + 2\alpha_n\beta_n\gamma\|x_n - z\|\|f(x_n) - f(z)\| + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Bz\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\gamma\|W_n v_n - z\|\|f(x_n) - f(z)\| \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle W_n v_n - z, \gamma f(z) - Bz\rangle \\
&\le \alpha_n^2\|\gamma f(x_n) - Bz\|^2 + (1-\alpha_n\bar\gamma)^2\|x_n - z\|^2 \\
&\quad + 2\alpha_n\beta_n\gamma\alpha\|x_n - z\|^2 + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Bz\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\gamma\alpha\|x_n - z\|^2 \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle W_n v_n - z, \gamma f(z) - Bz\rangle \\
&= \alpha_n^2\|\gamma f(x_n) - Bz\|^2 + (1 - 2\alpha_n\bar\gamma + \alpha_n^2\bar\gamma^2 - 2\alpha_n^2\bar\gamma\gamma\alpha)\|x_n - z\|^2 \\
&\quad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Bz\rangle + 2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle W_n v_n - z, \gamma f(z) - Bz\rangle \\
&\le (1 - \alpha_n(2\bar\gamma + \alpha_n\bar\gamma^2))\|x_n - z\|^2 + 2\alpha_n\sigma_n \quad (49)
\end{aligned}$$

where $\sigma_n = \alpha_n\|\gamma f(x_n) - Bz\|^2 + 2\beta_n\langle x_n - z, \gamma f(z) - Bz\rangle + 2(1-\beta_n-\alpha_n\bar\gamma) \times \langle W_n v_n - z, \gamma f(z) - Bz\rangle$. By (48), we get $\limsup_{n\to\infty}\sigma_n \le 0$. Hence by Lemma 2.5 applied to (49), we conclude that $x_n \to z$. This completes the proof. ■

Using Theorem 3.1, we obtain the following corollaries.

COROLLARY 3.2. *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction of $C \times C$ into real numbers* $\mathbf{R}$ *satisfying* (A1)–(A5)*, $f$ be a contraction of $H$ into itself with coefficient $\alpha \in (0,1)$ and $B$ be a strongly bounded linear operator on $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ a $\sigma$-inverse-strongly monotone mapping. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly $k$-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x$, for $x \in C$, $i \geq 1$, $t \in [k,1)$. Let $W_n$ be the $W$-mapping defined by* (14) *and $W$ be a mapping defined by* (15) *with $F(W) \neq \emptyset$. Assume that $\Omega := \bigcap_{n=1}^{N} F(T_i) \cap I(A,M) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ y_n = J_{M,\tau}(u_n - \tau A u_n), \\ v_n = J_{M,\tau}(y_n - \tau A y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n, \end{cases} \tag{50}$$

*for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$ and $\tau \in (0, 2\sigma)$ satisfy conditions* (i)–(iii) *in Theorem* 3.1*. Then $\{x_n\}$ converges strongly to $z \in \Omega$ which is the unique solution of the variational inequality*

$$\langle (B - \gamma f)z, z - x\rangle \leq 0, \quad x \in \Omega.$$

*Equivalently, we have $z = P_\Omega(I - B + \gamma f)(z)$.*

*Proof.* Taking $\varphi \equiv 0$, $M_1 = M_2 = M$, $A_1 = A_2 = A$ and $\tau = \delta$ in Theorem 3.1, we obtain the desired conclusion easily. This completes the proof. ■

COROLLARY 3.3. *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction of $C \times C$ into real numbers* $\mathbf{R}$ *satisfying* (A1)–(A5) *and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $f$ be a contraction of $H$ into itself with coefficient $\alpha \in (0,1)$ and let $B$ be a strongly bounded linear operator on $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $A_1, A_2 : H \to H$ be $\sigma_1, \sigma_2$-inverse-strongly monotone mappings, respectively. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly $k$-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x$, for $x \in C$, $i \geq 1$, $t \in [k,1)$. Let $W_n$ be the $W$-mapping defined by* (14) *and $W$ be a mapping defined by* (15) *with $F(W) \neq \emptyset$. Assume that either* (B1) *or* (B2) *holds and $\Omega := \bigcap_{n=1}^{N} F(T_i) \cap VI(C, A_1) \cap VI(C, A_2) \cap MEP(F, \varphi) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \qquad \forall y \in C, \\ y_n = P_C(u_n - \delta A_2 u_n), \\ v_n = P_C(y_n - \tau A_1 y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n, \end{cases} \tag{51}$$

*for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, $\tau \in (0, 2\sigma_1)$ and $\delta \in (0, 2\sigma_2)$ satisfy conditions* (i)–(iii) *in Theorem* 3.1*. Then $\{x_n\}$ converges strongly to $z \in \Omega$ which* *is the unique solution of the variational inequality*

$$\langle (B-\gamma f)z, z-x\rangle \leq 0, \quad x \in \Omega.$$

*Equivalently, we have* $z = P_\Omega(I - B + \gamma f)(z)$.

*Proof.* In Theorem 3.1 put $M_1 = M_2 = \partial\delta_C$, then $J_{M_1,\tau} = J_{M_2,\delta} = P_C$. The conclusion can be obtained immediately. ∎

Corollary 3.4. *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $f$ be a contraction of $H$ into itself with coefficient $\alpha \in (0,1)$ and let $B$ be a strongly bounded linear operator on $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $M_1, M_2 : H \to 2^H$ be maximal monotone mappings and $A_1, A_2 : H \to H$ be $\sigma_1, \sigma_2$-inverse-strongly monotone mappings, respectively. Let $\{V_i : C \to C\}_{i=1}^\infty$ be a countable family of uniformly k-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^\infty$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x$ for $x \in C$, $i \geq 1$, $t \in [k,1)$. Let $W_n$ be the $W$-mapping defined by* (14) *and $W$ be a mapping defined by* (15) *with $F(W) \neq \emptyset$. Assume that $\Omega := \bigcap_{n=1}^N F(T_i) \cap I(A_1, M_1) \cap I(A_2, M_2) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} y_n = J_{M_2,\delta}(x_n - \delta A_2 x_n), \\ v_n = J_{M_1,\tau}(y_n - \tau A_1 y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n, \end{cases} \tag{52}$$

*for every $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\tau \in (0, 2\sigma_1)$ and $\delta \in (0, 2\sigma_2)$ satisfy conditions* (i),(iii) *in Theorem* 3.1*. Then $\{x_n\}$ converges strongly to $z \in \Omega$ which is the unique solution of the variational inequality*

$$\langle (B-\gamma f)z, z-x\rangle \leq 0, \quad x \in \Omega.$$

*Equivalently, we have* $z = P_\Omega(I - B + \gamma f)(z)$.

*Proof.* Putting in Theorem 3.1 $\varphi \equiv 0$ and $F(x,y) = 0$ for all $x, y \in C$, we deduce that $u_n = P_C x_n = x_n$. Thus the desired conclusion follows easily. This completes the proof. ∎

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