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APPROXIMATION OF FUNCTIONS FROM $L^p(\widetilde{\omega})_\beta$ BY LINEAR OPERATORS OF CONJUGATE FOURIER SERIES

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Abstract. We show the results corresponding to some theorems of S. Lal and H. K. Nigam [Int. J. Math. Math. Sci. 27 (2001), 555–563] on the norm and pointwise approximation of conjugate functions and to the results of the authors [Acta Comment. Univ. Tartu. Math. 13 (2009), 11–24] also on such approximations.

1. Introduction. Let L^p $(1 \le p < \infty)$ be the class of all 2π-periodic real-valued functions integrable in the Lebesgue sense with p-th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| = \|f(\cdot)\|_{L^p} = \Big(\int_Q |f(t)|^p \, dt\Big)^{1/p} \quad \text{when} \quad 1 \le p < \infty. \tag{1.1}$$

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_o(f)}{2} + \sum_{\nu=1}^{\infty} \big(a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x\big)$$

and the conjugate one

$$\widetilde{S} f(x) := \sum_{\nu=1}^{\infty} \big(a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x\big)$$

with the partial sums $\widetilde{S}_k f$. We know that if $f \in L$ then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{1}{2} \cot \frac{t}{2} \, dt = \lim_{\epsilon \to 0} \widetilde{f}(x, \epsilon),$$

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where

$$\widetilde{f}(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt, \quad \text{with } \psi_x(t) := f(x+t) - f(x-t),$$

exists for almost all x [10, Th. (3.1)IV].

Let $A := (a_{n,k})$ be an infinite lower triangular matrix of real numbers such that

$$a_{n,k} \geq 0 \text{ when } \ k = 0,1,2,\ldots,n, \ a_{n,k} = 0 \text{ when } k > n,$$

$$\sum_{k=0}^{n} a_{n,k} = 1, \text{ where } n = 0,1,2,\ldots,$$

and let, for $m = 0,1,2,\ldots,n$,

$$A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad \text{and} \quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}.$$

Let the A-transformation of $(\widetilde{S}_k f)$ be given by

$$\widetilde{T}_{n,A} f(x) := \sum_{k=0}^{n} a_{n,k}\widetilde{S}_k f(x), \quad n = 0,1,2,\ldots.$$

We define two classes of sequences (see [3]).

A sequence $c := (c_n)$ of nonnegative numbers tending to zero will be called a *Rest Bounded Variation Sequence*, or briefly $c \in RBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c)c_m,$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_n)$ of nonnegative numbers will be called a *Head Bounded Variation Sequence*, or briefly $c \in HBVS$, if it has the property

$$\sum_{k=0}^{m-1} |c_k - c_{k+1}| \leq K(c)c_m,$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only a finite number of nonzero terms and the last nonzero term is c_N.

Now, we define another class of sequences.

Following L. Leindler ([4]), a sequence $c := (c_n)$ of nonnegative numbers tending to zero will be called a *Mean Rest Bounded Variation Sequence*, or briefly $c \in MRBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c)\frac{1}{m+1}\sum_{k=\lceil m/2\rceil}^{m} c_k \tag{1.2}$$

for all natural numbers m (where $\lceil x\rceil$ is the smallest integer greater than or equal to x).

Analogously, a sequence $c := (c_k)$ of nonnegative numbers will be called a *Mean Head Bounded Variation Sequence*, or briefly $c \in MHBVS$, if it has the property

$$\sum_{k=0}^{n-m-1} |c_k - c_{k+1}| \leq K(c)\frac{1}{m+1}\sum_{k=n-m}^{n} c_k, \tag{1.3}$$

for all positive integers $m < n$, where the sequence c has only a finite number of nonzero terms and the last nonzero term is c_n.

It is clear that (see [9])

$$RBVS \subsetneq MRBVS \quad \text{and} \quad HBVS \subsetneq MHBVS.$$

Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, i.e. that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K$$

for all n, where $K(\alpha_n)$ denote the constants appearing in the inequalities (1.2) or (1.3) for the sequences α_n.

Now we can give the conditions to be used later on. We assume that for all n and $0 \leq m < n$

$$\sum_{k=m}^{n-1} |a_{n,k} - a_{n,k+1}| \leq K \frac{1}{m+1} \sum_{k=\lceil m/2 \rceil}^{m} a_{n,k}, \tag{1.4}$$

$$\text{and} \quad \sum_{k=0}^{n-m-1} |a_{n,k} - a_{n,k+1}| \leq K \frac{1}{m+1} \sum_{k=n-m}^{n} a_{n,k} \tag{1.5}$$

if $(a_{n,k})_{k=0}^{n}$ belong to $MRBVS$ and $MHBVS$, respectively.

As a measure of approximation by the above quantities we use the generalized modulus of continuity of f in the space L^p defined for $\beta \geq 0$ by the formula

$$\widetilde{\omega}_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ |\sin t/2|^{\beta p} \int_0^{\pi} |\psi_x(t)|^p \, dx \right\}^{1/p}.$$

It is clear that for $\beta > \alpha \geq 0$

$$\widetilde{\omega}_\beta f(\delta)_{L^p} \leq \widetilde{\omega}_\alpha f(\delta)_{L^p},$$

and it is easily seen that $\widetilde{\omega}_0 f(\cdot)_{L^p} = \widetilde{\omega} f(\cdot)_{L^p}$ is the classical modulus of continuity.

The deviation $\widetilde{T}_{n,A} f - \widetilde{f}$ was pointwise estimated by K. Qureshi [6] (with special matrix A) and in the norm of L^p by S. Lal and H. Nigam [2]. These results were generalized by K. Qureshi [7]. The next generalization was obtained by S. Lal [1] in the following form:

Let $A = (a_{n,k})$ be an infinite regular triangular matrix with nonnegative entries such that $(a_{n,k})_{k=0}^{n}$ are nondecreasing sequences, then the degree of approximation of the function $\widetilde{f}$, conjugate to a periodic function f belonging to the class

$$W(L^p, \omega_0) = \left\{ f \in L^p : \left(\int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{\beta p} x \, dx \right)^{1/p} = O(\omega_0(t)) \right\},$$

is given by

$$\|\widetilde{T}_{n,A} f - \widetilde{f}\,\| = O\Big((n+1)^{\beta + 1/p} \omega_0\Big(\frac{\pi}{n+1}\Big)\Big),$$

provided that ω_0 increases and satisfies

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{t|\psi_x(t)|}{\omega_0(t)} \right)^p \sin^{\beta p} t \, dt \right\}^{1/p} = O((n+1)^{-1})$$

and

$$\left\{\int_{\pi/(n+1)}^{\pi}\Big(\frac{t^{-\gamma}|\psi_x(t)|}{\omega_0(t)}\Big)^p\,dt\right\}^{1/p}=O((n+1)^{\gamma})$$

uniformly in x, where γ is an arbitrary positive number with $q(1-\gamma)-1>0$, and $p^{-1}+q^{-1}=1$, $1\le p\le\infty$.

In the sequel we shall present some comments on the result formulated in [1] and its corrected form. In this note we shall consider the same deviation and additionally the deviation $\widetilde{T}_{n,A}f(\cdot)-\widetilde{f}(\cdot,\frac{2\pi}{n+2})$. In the theorems we formulate the general and precise conditions for the functions and the modulus of continuity. Finally, we also give some results on norm approximation. The results obtained here generalize the results of the authors [5] with the sequences $(a_{n,k})_{k=0}^{n}$ from the classes $RBVS$ and $HBVS$.

We shall write $I_1\ll I_2$ if there exists a positive constant K, sometimes depending on some parameters, such that $I_1\le KI_2$.

2. Statement of the results. Let us consider a function ω of the type of the modulus of continuity on the interval $[0,2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2)$ for any $0\le\delta_1\le\delta_2\le\delta_1+\delta_2\le 2\pi$. It is easy to conclude that the function $\delta^{-1}\omega(\delta)$ is a nonincreasing function of δ. Let

$$L^p(\widetilde{\omega})_\beta=\left\{f\in L^p:\widetilde{\omega}_\beta f(\delta)_{L^p}\ll\widetilde{\omega}(\delta)\right\},$$

where $\widetilde{\omega}$ is a function of modulus of the continuity type. It is clear that for $\beta>\alpha\ge 0$

$$L^p(\widetilde{\omega})_\alpha\subset L^p(\widetilde{\omega})_\beta.$$

We can now formulate our main results.

At the beginning, we formulate the results on the degrees of pointwise summability of conjugate series.

THEOREM 1. *Let $f\in L^p(\widetilde{\omega})_\beta$ with $\beta<1-\frac{1}{p}$, $(a_{n,k})_{k=0}^{n}\in MHBVS$ with the condition $\overline{A}_{n,n-2\tau}=O(\frac{\tau}{n+1})$, where $\tau=[\pi/t]$, $\frac{2\pi}{n+2}\le t\le\pi$, and let $\widetilde{\omega}$ be such that*

$$\left\{\int_0^{2\pi/(n+2)}\Big(\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)}\Big)^p\sin^{\beta p}t/2\,dt\right\}^{1/p}=O_x((n+1)^{-1})\qquad(2.1)$$

$$\textit{and}\ \left\{\int_{\pi/(n+1)}^{\pi}\Big(\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\Big)^p\sin^{\beta p}t/2\,dt\right\}^{1/p}=O_x((n+1)^{\gamma})\qquad(2.2)$$

with $0<\gamma<\beta+\frac{1}{p}$. Then

$$\Big|\widetilde{T}_{n,A}f(x)-\widetilde{f}\Big(x,\frac{2\pi}{n+2}\Big)\Big|=O_x\Big((n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big)\Big),\qquad(2.3)$$

for x under consideration.

THEOREM 2. *Let $f\in L^p(\widetilde{\omega})_\beta$, where*

$$\left\{\int_0^{2\pi/(n+2)}\Big(\frac{\widetilde{\omega}(t)}{t\sin^{\beta}t/2}\Big)^q\,dt\right\}^{1/q}=O\Big((n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big)\Big)\qquad(2.4)$$

with $q = p(p-1)^{-1}$. *If* $(a_{n,k})_{k=0}^{n} \in MHBVS$ *with the condition* $\overline{A}_{n,n-2\tau} = O(\frac{\tau}{n+1})$, *where* $\tau = [\pi/t]$, $\frac{2\pi}{n+2} \le t \le \pi$, *and* $\widetilde{\omega}$ *satisfies* (2.2) *with* $0 < \gamma < \beta + \frac{1}{p}$ *and*

$$\Big\{\int_0^{2\pi/(n+2)} \Big(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\Big)^p \sin^{\beta p} t/2\, dt\Big\}^{1/p} = O_x((n+1)^{-1/p}), \tag{2.5}$$

then

$$\big|\widetilde{T}_{n,A} f(x) - \widetilde{f}(x)\big| = O_x\Big((n+1)^{\beta+1/p}\, \widetilde{\omega}\Big(\frac{\pi}{n+1}\Big)\Big), \tag{2.6}$$

for x *such that* $\widetilde{f}(x)$ *exists.*

THEOREM 3. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $\beta < 1 - \frac{1}{p}$, $(a_{n,k})_{k=0}^{n} \in MRBVS$ *with the condition* $A_{n,\tau} = O(\frac{\tau}{n+1})$, *where* $\tau = [\pi/t]$, $\frac{2\pi}{n+2} \le t \le \pi$, *and let* $\widetilde{\omega}$ *satisfy* (2.1) *and* (2.2) *with* $0 < \gamma < \beta + \frac{1}{p}$. *Then the relation* (2.3) *holds, for* x *under consideration.*

THEOREM 4. *Let* $f \in L^p(\widetilde{\omega})_\beta$, *where* (2.4) *holds with* $q = p(p-1)^{-1}$. *If* $(a_{n,k})_{k=0}^{n} \in MRBVS$ *with the condition* $A_{n,\tau} = O(\frac{\tau}{n+1})$, *where* $\tau = [\pi/t]$, $\frac{2\pi}{n+2} \le t \le \pi$, *and* $\widetilde{\omega}$ *satisfies* (2.2) *with* $0 < \gamma < \beta + \frac{1}{p}$ *and* (2.5) *then the relation* (2.6) *holds for* x *such that* $\widetilde{f}(x)$ *exists.*

Finally, we formulate some remarks.

REMARK 1. Considering the L^p-norms of the deviations from our theorems instead of pointwise ones we can obtain the same estimates without any additional assumptions like (2.2), (2.1) or (2.5).

REMARK 2. If we consider the modulus of continuity in the form

$$\widetilde{\omega} f(\delta)_{L^p_\beta} := \sup_{0\le|t|\le\delta} \Big\{\int_0^{\pi} |\psi_x(t)|^p\ |\sin x/2|^{\beta p}\ dx\Big\}^{1/p}$$

then our theorems will be true under the assumption

$$f \in L^p_\beta(\widetilde{\omega}) = \big\{f \in L^p : \widetilde{\omega} f(\delta)_{L^p_\beta} \ll \widetilde{\omega}(\delta)\big\}$$

and with the following norm

$$\|f\|_{L^p_\beta} := \|f(\cdot)\|_{L^p_\beta} = \Big(\int_Q |f(t)|^p\ |\sin t/2|^{\beta p}\ dt\Big)^{1/p} \quad \text{when} \quad 1 \le p < \infty.$$

REMARK 3. In the paper [1] $\sin t$ is considered instead of $\sin t/2$. This generates some inconvenience since the inequality $\sin t \ge \frac{2}{\pi} t$ does not hold for all $t \in (0, \pi]$ and therefore the proofs in [1] are incorrect. We also note that assumption (2.1) instead of (2.5) without the condition (2.4) leads us to the divergent integral of the form $\int_0^{\pi/(n+1)} t^{-(1+\beta)/(1-1/p)}\, dt$ $(\beta \ge 0)$.

REMARK 4. Under the additional assumption $\widetilde{\omega}(t) = O(t^\alpha)$ $(0 < \alpha \le 1)$, the degrees of approximation in the theorems are equal to $O(n^{1/p-\alpha})$.

REMARK 5. Under the above remarks we can observe that in the special case $\beta = 0$, when our sequences (a_{nk}) are monotonic with respect to k, we also have the corrected form of the result of S. Lal and H. K. Nigam [2].

3. Auxiliary results. We begin this section by some notation following A. Zygmund [10, Section 5 of Chapter II].

It is clear that

$$\widetilde{S}_k f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\widetilde{D}_k(t)\,dt$$

and

$$\widetilde{T}_{n,A} f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k}(t)\,dt,$$

where

$$\widetilde{D_k}(t) = \sum_{\nu=0}^{k}\sin\nu t = \frac{\cos t/2 - \cos(2k+1)t/2}{2\sin t/2}.$$

Hence

$$\widetilde{T}_{n,A} f(x) - \widetilde{f}\Big(x, \frac{2\pi}{n+2}\Big) = -\frac{1}{\pi}\int_{0}^{2\pi/(n+2)} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k}(t)\,dt$$
$$+ \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k^\circ}(t)\,dt$$

and

$$\widetilde{T}_{n,A} f(x) - \widetilde{f}(x) = \frac{1}{\pi}\int_{0}^{\pi} \psi_x(t)\sum_{k=0}^{n} a_{n,k}\widetilde{D_k^\circ}(t)\,dt,$$

where

$$\widetilde{D_k^\circ}(t) = \frac{\cos(2k+1)t/2}{2\sin t/2}.$$

Now, we formulate some estimates for the conjugate Dirichlet kernels.

LEMMA 1 [10]. *If* $0 < |t| \le \pi/2$ *then*

$$|\widetilde{D_k^\circ}(t)| \le \frac{\pi}{2|t|} \quad \textit{and} \quad |\widetilde{D_k}(t)| \le \frac{\pi}{|t|}$$

and for any real t *we have*

$$|\widetilde{D_k}(t)| \le \frac{1}{2}k(k+1)|t| \quad \textit{and} \quad |\widetilde{D_k}(t)| \le k+1.$$

More complicated estimates are given with proofs.

LEMMA 2. *If* $(a_{n,k})_{k=0}^{n} \in MHBVS$, *then*

$$\Big|\sum_{k=0}^{n} a_{n,k}\widetilde{D_k^\circ}(t)\Big| = O(t^{-1}\overline{A}_{n,n-2\tau})$$

and if $(a_{n,k})_{k=0}^{n} \in MRBVS$, *then*

$$\Big|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D_k^\circ}(t)\Big| = O(t^{-1}A_{n,\tau}),$$

for $\frac{2\pi}{n} \le t \le \pi$ $(n = 2, 3, \dots)$, *where* $\tau = [\pi/t]$.

Proof. Using partial summation we obtain

$$\begin{aligned}
&\Big|\sum_{k=0}^{n} a_{n,k}\cos\frac{(2k+1)t}{2}\sin t/2\Big| \\
&\le tA_{n,\tau}+\frac{1}{2}\sum_{k=\tau-1}^{n-1}|a_{n,k}-a_{n,k+1}|\Big|\sum_{l=\tau-1}^{k}2\cos\frac{(2l+1)t}{2}\sin t/2\Big| \\
&\qquad+\frac{1}{2}a_{n,n}\Big|\sum_{l=\tau-1}^{n}2\cos\frac{(2l+1)t}{2}\sin t/2\Big| \\
&\le tA_{n,\tau}+\frac{1}{2}\sum_{k=\tau-1}^{n-1}|a_{n,k}-a_{n,k+1}|\Big|\sum_{l=\tau-1}^{k}(\sin(l+1)t-\sin lt)\Big| \\
&\qquad+\frac{1}{2}a_{n,n}\Big|\sum_{l=\tau-1}^{n}(\sin(l+1)t-\sin lt)\Big| \\
&\le tA_{n,\tau}+\frac{1}{2}\sum_{k=\tau-1}^{n-1}|a_{n,k}-a_{n,k+1}|\big|\sin(k+1)t-\sin(\tau-1)t\big| \\
&\qquad+\frac{1}{2}a_{n,n}\big|\sin(n+1)t-\sin(\tau-1)t\big| \\
&\le tA_{n,\tau}+\sum_{k=\tau}^{n-1}|a_{n,k}-a_{n,k+1}|+a_{n,n}
\end{aligned}$$

or

$$\begin{aligned}
&\Big|\sum_{k=0}^{n} a_{n,k}\cos((2k+1)t/2)\sin t/2\Big| \\
&\le t\overline{A}_{n,n-\tau}+\frac{1}{2}\sum_{k=0}^{n-\tau-1}|a_{n,k}-a_{n,k+1}|\Big|\sum_{l=0}^{k}2\cos((2l+1)t/2)\sin t/2\Big| \\
&\qquad+\frac{1}{2}a_{n,n-\tau}\Big|\sum_{l=0}^{n-\tau}2\cos\frac{(2l+1)t}{2}\sin t/2\Big|+a_{n,n-\tau} \\
&\le t\overline{A}_{n,n-\tau}+\sum_{k=0}^{n-\tau-1}|a_{n,k}-a_{n,k+1}|+2a_{n,n-\tau}.
\end{aligned}$$

Because $(a_{n,k})_{k=0}^{n}\in MRBVS$ we have

$$\begin{aligned}
a_{n,s}-a_{n,m}&\le|a_{n,m}-a_{n,s}|\le\sum_{k=m}^{s-1}|a_{n,k}-a_{n,k+1}| \\
&\le\sum_{k=m}^{n-1}|a_{n,k}-a_{n,k+1}|\ll\frac{1}{m+1}\sum_{k=\lceil m/2\rceil}^{m}a_{n,k}\quad(0\le m<s\le n),
\end{aligned}$$

whence

$$a_{n,s}\ll a_{n,m}+\frac{1}{m+1}\sum_{k=\lceil m/2\rceil}^{m}a_{n,k}\quad(0\le m<s\le n)$$

and therefore

$$\begin{aligned}
&\Big|\sum_{k=0}^{n} a_{n,k}\cos((2k+1)t/2)\sin t/2\Big| \\
&\ll tA_{n,\tau}+\frac{1}{\tau+1}\sum_{k=\lceil\tau/2\rceil}^{\tau} a_{n,k}+\frac{1}{\tau+1}\sum_{m=\lceil\tau/2\rceil}^{\tau} a_{n,n} \\
&\ll tA_{n,\tau}+\frac{1}{\tau+1}\sum_{k=\lceil\tau/2\rceil}^{\tau} a_{n,k}+\frac{1}{\tau+1}\sum_{m=\lceil\tau/2\rceil}^{\tau}\Big(a_{n,m}+\frac{1}{m+1}\sum_{k=\lceil m/2\rceil}^{m} a_{n,k}\Big) \\
&\ll tA_{n,\tau}+\frac{1}{\tau+1}\sum_{k=\lceil\tau/2\rceil}^{\tau} a_{n,k}+\frac{1}{\tau+1}\sum_{k=\lceil\tau/4\rceil}^{\tau} a_{n,k}\ll tA_{n,\tau}.
\end{aligned}$$

Analogously, the relation $(a_{n,k})_{k=0}^{n}\in MHBVS$ implies

$$a_{n,s}-a_{n,m}\le |a_{n,m}-a_{n,s}|\le\sum_{k=m}^{s-1}|a_{n,k}-a_{n,k+1}|$$

$$\sum_{k=0}^{s-1}|a_{n,k}-a_{n,k+1}|\ll\frac{1}{n-s+1}\sum_{k=s}^{n}a_{n,k}\quad(0\le m<s\le n)$$

and

$$a_{n,s}\ll a_{n,m}+\frac{1}{n-s+1}\sum_{k=s}^{n}a_{n,k}\quad(0\le m<s\le n),$$

whence

$$\begin{aligned}
&\Big|\sum_{k=0}^{n} a_{n,k}\cos((2k+1)t/2)\sin t/2\Big| \\
&\ll t\overline{A}_{n,n-\tau}+\frac{1}{\tau+1}\sum_{k=n-\tau}^{n} a_{n,k}+\frac{1}{\tau}\sum_{m=n-2\tau}^{n-\tau-1} a_{n,n-\tau} \\
&\ll t\overline{A}_{n,n-\tau}+\frac{1}{\tau}\sum_{m=n-2\tau}^{n-\tau-1}\Big(a_{n,m}+\frac{1}{\tau+1}\sum_{k=n-\tau}^{n} a_{n,k}\Big) \\
&\ll t\overline{A}_{n,n-\tau}+\frac{1}{\tau}\sum_{m=n-2\tau}^{n-\tau-1} a_{n,m}\ll t\overline{A}_{n,n-2\tau}\quad(\tau\le n/2).
\end{aligned}$$

Thus our proof is complete. ∎

4. Proofs of the results

4.1. Proof of Theorem 1. We start with the obvious relations

$$\begin{aligned}
\widetilde{T}_{n,A}f(x)-\widetilde{f}\Big(x,\frac{2\pi}{n+2}\Big)&=-\frac{1}{\pi}\int_0^{2\pi/(n+2)}\psi_x(t)\sum_{k=0}^{n}a_{nk}\widetilde{D_k}(t)\,dt \\
&\quad+\frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{nk}\widetilde{D_k^{\circ}}(t)=\widetilde{I_1}+\widetilde{I_2^{\circ}}
\end{aligned}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}\Big(x,\frac{2\pi}{n+2}\Big)\right|\le|\widetilde{I}_1|+|\widetilde{I}_2^{\circ}|.$$

By the Hölder inequality $(\frac{1}{p}+\frac{1}{q}=1)$, Lemma 1 and (2.1)

$$\begin{aligned}|\widetilde{I}_1| &\le (n+1)^2\int_0^{2\pi/(n+2)} t|\psi_x(t)|\,dt\\ &\le (n+1)^2\Big\{\int_0^{2\pi/(n+2)}\Big[\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta t/2\Big]^p dt\Big\}^{1/p}\Big\{\int_0^{2\pi/(n+2)}\Big[\frac{\widetilde{\omega}(t)}{\sin^\beta t/2}\Big]^q dt\Big\}^{1/q}\\ &\ll (n+1)\Big\{\int_0^{2\pi/(n+2)}\Big[\frac{\widetilde{\omega}(t)}{t^\beta}\Big]^q dt\Big\}^{1/q}\ll (n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{2\pi}{n+2}\Big)\\ &\ll (n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big),\end{aligned}$$

for $\beta<1-\frac{1}{p}$.

By the Hölder inequality $(\frac{1}{p}+\frac{1}{q}=1)$, Lemma 2 and (2.2)

$$\begin{aligned}|\widetilde{I}_2^{\circ}| &\le \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}|\psi_x(t)|\Big|\sum_{k=0}^n a_{nk}\widetilde{D}_k^{\circ}(t)\Big|\,dt\ll\int_{2\pi/(n+2)}^{\pi}\frac{|\psi_x(t)|}{t}\overline{A}_{n,n-2\tau}\,dt\\ &\ll\frac{1}{n+1}\int_{2\pi/(n+2)}^{\pi}\frac{|\psi_x(t)|}{t^2}\,dt\le\frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\frac{|\psi_x(t)|}{t^2}\,dt\\ &\le\frac{1}{n+1}\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta t/2\Big]^p dt\Big\}^{1/p}\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^\beta t/2}\Big]^q dt\Big\}^{1/q}\\ &\ll\frac{1}{n+1}(n+1)^\gamma\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^\beta t/2}\Big]^q dt\Big\}^{1/q}\\ &\ll\frac{1}{n+1}(n+1)^{\gamma+1}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big)\Big\{\int_{\pi/(n+1)}^{\pi}\big[t^{(\gamma-\beta-1)}\big]^q dt\Big\}^{1/q}\\ &\ll(n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big),\end{aligned}$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates we obtain the desired result. ■

4.2. Proof of Theorem 2. We start with the obvious relations

$$\begin{aligned}\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)&=\frac{1}{\pi}\int_0^{2\pi/(n+2)}\psi_x(t)\sum_{k=0}^n a_{n,k}\widetilde{D}_k^{\circ}(t)\,dt\\ &\quad+\frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\psi_x(t)\sum_{k=0}^n a_{n,k}\widetilde{D}_k^{\circ}(t)\,dt=\widetilde{I}_1^{\circ}+\widetilde{I}_2^{\circ}\end{aligned}$$

and

$$|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)|\le|\widetilde{I}_1^{\circ}|+|\widetilde{I}_2^{\circ}|.$$

By the Hölder inequality $(\frac{1}{p}+\frac{1}{q}=1)$, Lemma 1, (2.5) and (2.4),

$$
\begin{aligned}
|\widetilde{I}_1^{\circ}| &\le \frac{1}{\pi}\int_0^{2\pi/(n+2)}\frac{|\psi_x(t)|}{t}\,dt\\
&\le \frac{1}{\pi}\Big\{\int_0^{2\pi/(n+2)}\Big[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta t/2\Big]^p\,dt\Big\}^{1/p}\Big\{\int_0^{2\pi/(n+2)}\Big[\frac{\widetilde{\omega}(t)}{t^1\sin^\beta t/2}\Big]^q\,dt\Big\}^{1/q}\\
&\ll (n+1)^{-1/p}\Big\{\int_0^{2\pi/(n+2)}\Big[\frac{\omega(t)}{t^{1+\beta}}\Big]^q\,dt\Big\}^{1/q}\ll (n+1)^\beta\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big).
\end{aligned}
$$

By the Hölder inequality $(\frac{1}{p}+\frac{1}{q}=1)$, Lemma 2 and (2.2)

$$
\begin{aligned}
|\widetilde{I}_2^{\circ}| &\le \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\frac{|\psi_x(t)|}{t}\overline{A}_{n,n-2\tau}\,dt\\
&\le \frac{1}{\pi(n+1)}\int_{2\pi/(n+2)}^{\pi}\frac{|\psi_x(t)|}{t^2}\,dt\le\frac{1}{\pi(n+1)}\int_{\pi/(n+1)}^{\pi}\frac{|\psi_x(t)|}{t^2}\,dt\\
&\le \frac{1}{\pi(n+1)}\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta t/2\Big]^p\,dt\Big\}^{1/p}\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^\beta t/2}\Big]^q\,dt\Big\}^{1/q}\\
&\ll \frac{1}{\pi(n+1)}(n+1)^{\gamma+1}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big)\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{1}{t^{1-\gamma+\beta}}\Big]^q\,dt\Big\}^{1/q}\\
&\ll (n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big),
\end{aligned}
$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates we obtain the desired result. ∎

4.3. Proof of Theorem 3. Let as usual

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}\Big(x,\frac{2\pi}{n+2}\Big)=\widetilde{I}_1+\widetilde{I}_2^{\circ}$$

and

$$\Big|\widetilde{T}_{n,A}f(x)-\widetilde{f}\Big(x,\frac{2\pi}{n+2}\Big)\Big|\le|\widetilde{I}_1|+|\widetilde{I}_2^{\circ}|.$$

Analogously to the above, by the Hölder inequality $(\frac{1}{p}+\frac{1}{q}=1)$, Lemma 1 and (2.1),

$$|\widetilde{I}_1|\le(n+1)^2\int_0^{2\pi/(n+2)}t|\psi_x(t)|\,dt\ll(n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big).$$

By the Hölder inequality $(\frac{1}{p}+\frac{1}{q}=1)$, Lemma 2 and (2.2),

$$
\begin{aligned}
|\widetilde{I}_2^{\circ}| &\le \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\frac{|\psi_x(t)|}{t}A_{n,\tau}\,dt\\
&\le \frac{1}{\pi(n+1)}\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta t/2\Big]^p\,dt\Big\}^{1/p}\Big\{\int_{\pi/(n+1)}^{\pi}\Big[\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^\beta t/2}\Big]^q\,dt\Big\}^{1/q}\\
&\ll (n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big),
\end{aligned}
$$

for $0<\gamma<\beta+\frac{1}{p}$.

Collecting these estimates we obtain the desired result. ∎

4.4. Proof of Theorem 4. Let as usual

$$\widetilde{T}_{n,A}f(x) - \widetilde{f}(x) = \widetilde{I}_1^\circ + \widetilde{I}_2^\circ$$

and

$$|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)| \le |\widetilde{I}_1^\circ| + |\widetilde{I}_2^\circ|.$$

Analogously to the previous proofs, for $\beta < 1 - \frac{1}{p}$,

$$|\widetilde{I}_1^\circ| \le \frac{1}{\pi}\int_0^{2\pi/(n+2)} \frac{|\psi_x(t)|}{t}\,dt \ll (n+1)^\beta\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big),$$

and for $0 < \gamma < \beta + \frac{1}{p}$

$$|\widetilde{I}_2^\circ| \le \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi} \frac{|\psi_x(t)|}{t}A_{n,\tau}\,dt \ll (n+1)^{\beta+1/p}\,\widetilde{\omega}\Big(\frac{\pi}{n+1}\Big).$$

Collecting these estimates we obtain the desired result. ■

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