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FRACTIONAL INTEGRAL OPERATORS ON $B^{p,\lambda}$ WITH MORREY-CAMPANATO NORMS

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Dedicated to Professor Takahiro Mizuhara in celebration of his 65th birthday

Abstract. We introduce function spaces $B^{p,\lambda}$ with Morrey-Campanato norms, which unify $B^{p,\lambda}$, $\mathrm{CMO}^{p,\lambda}$ and Morrey-Campanato spaces, and prove the boundedness of the fractional integral operator I_α on these spaces.

1. Introduction. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. It is known that the fractional integral operator I_α $(0 < \alpha < n)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $-n/p+\alpha = -n/q$, from $L^{n/\alpha}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$, from $L^p(\mathbb{R}^n)$ to Lipschitz space $\mathrm{Lip}_{\alpha-n/p}(\mathbb{R}^n)$ for $0 < \alpha - n/p < 1$, from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$, and from $\mathrm{Lip}_\beta(\mathbb{R}^n)$ to $\mathrm{Lip}_{\alpha+\beta}(\mathbb{R}^n)$ for $0 < \beta < \alpha + \beta < 1$. In this paper we introduce $B^{p,\lambda}(\mathbb{R}^n)$ with Morrey-Campanato norms and extend the boundedness of I_α to these function spaces.

The space $B^p(\mathbb{R}^n)$ is introduced by Beurling [3] together with its predual $A^p(\mathbb{R}^n)$. Feichtinger [5] gave an equivalent norm on $B^p(\mathbb{R}^n)$. The space $B^p(\mathbb{R}^n)$ is a special case of Herz spaces $K_p^{\alpha,r}(\mathbb{R}^n)$ introduced by Herz [8]. Lu and Yang [12] proved the boundedness of I_α on Herz spaces. However, it does not cover the space $B^p(\mathbb{R}^n)$ which is an end point

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case. Chen and Lau [4] and García-Cuerva [7] introduced the central mean oscillation space $\mathrm{CMO}^p(\mathbb{R}^n)$ with its predual. Lu and Yang [10, 11] also introduced the central bounded mean oscillation space $\mathrm{CBMO}^p(\mathbb{R}^n)$ with its predual. As an extension of these spaces, Alvarez, Guzmán-Partida and Lakey [2] introduced the spaces $B^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$. Our spaces in this paper unify $B^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ and Campanato spaces $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. Note that $B^{p,\lambda}$ with Lip_α-norms is considered in Komori-Furuya and Matsuoka [9]. For Morrey-Campanato spaces and the boundedness of I_α on them, see Peetre [22]. For Herz spaces, see for example Lu, Yang and Hu [13].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. We denote $B(0,r)$ by B_r and the characteristic function of B_r by χ_r. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy,$$

where $|B|$ is the Lebesgue measure of B.

For $1 \le p < \infty$, $-\infty < \lambda < \infty$ and $0 < \alpha \le 1$, let $B^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$, i.e. the homogeneous $B^{p,\lambda}$ space, $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$, $L_{p,\lambda}(\mathbb{R}^n)$, $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\begin{aligned}
\|f\|_{B^{p,\lambda}} &= \sup_{r\ge 1} \frac{1}{r^\lambda}\Big(\fint_{B_r} |f(y)|^p\,dy\Big)^{1/p},\\
\|f\|_{\mathrm{CMO}^{p,\lambda}} &= \sup_{r\ge 1} \frac{1}{r^\lambda}\Big(\fint_{B_r} |f(y)-f_{B_r}|^p\,dy\Big)^{1/p},\\
\|f\|_{\dot{B}^{p,\lambda}} &= \sup_{r>0} \frac{1}{r^\lambda}\Big(\fint_{B_r} |f(y)|^p\,dy\Big)^{1/p},\\
\|f\|_{\mathrm{CBMO}^{p,\lambda}} &= \sup_{r>0} \frac{1}{r^\lambda}\Big(\fint_{B_r} |f(y)-f_{B_r}|^p\,dy\Big)^{1/p},\\
\|f\|_{L_{p,\lambda}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{r^\lambda}\Big(\fint_{B(x,r)} |f(y)|^p\,dy\Big)^{1/p},\\
\|f\|_{\mathcal{L}_{p,\lambda}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{r^\lambda}\Big(\fint_{B(x,r)} |f(y)-f_{B(x,r)}|^p\,dy\Big)^{1/p}
\end{aligned}$$

and

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup_{x,y\in\mathbb{R}^n,\, x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We regard $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$, and the others as spaces of functions modulo null-functions. Then $B^{p,\lambda}(\mathbb{R}^n)$, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\lambda}(\mathbb{R}^n)$ are Banach spaces equipped with the norm $\|f\|_{B^{p,\lambda}}$, $\|f\|_{\dot{B}^{p,\lambda}}$ and $\|f\|_{L_{p,\lambda}}$, respectively. Let $\mathcal{C}$ be the space of all constant functions. Then $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$, $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$, $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ and $\mathrm{Lip}_\alpha(\mathbb{R}^n)/\mathcal{C}$ are Banach spaces equipped with the norm $\|f\|_{\mathrm{CMO}^{p,\lambda}}$, $\|f\|_{\mathrm{CBMO}^{p,\lambda}}$, $\|f\|_{\mathcal{L}_{p,\lambda}}$ and $\|f\|_{\mathrm{Lip}_\alpha}$, respectively. For the unit ball B_1, $\|f\|_{\mathrm{CMO}^{p,\lambda}} + |f_{B_1}|$, $\|f\|_{\mathrm{CBMO}^{p,\lambda}} + |f_{B_1}|$, $\|f\|_{\mathcal{L}_{p,\lambda}} + |f_{B_1}|$ and for the origin 0, $\|f\|_{\mathrm{Lip}_\alpha} + |f(0)|$ are norms and thereby $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$, $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ are Banach spaces, respectively. Note that, if $p = 1$ and $\lambda = 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ is the usual $\mathrm{BMO}(\mathbb{R}^n)$.

By the definition we have

$$L_{p,\lambda}(\mathbb{R}^n) \subset \dot{B}^{p,\lambda}(\mathbb{R}^n) \subset B^{p,\lambda}(\mathbb{R}^n), \quad \mathcal{L}_{p,\lambda}(\mathbb{R}^n) \subset \mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n) \subset \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n),$$

and, for $p < q$,

$$L_{q,\lambda}(\mathbb{R}^n) \subsetneqq L_{p,\lambda}(\mathbb{R}^n), \quad \dot{B}^{q,\lambda}(\mathbb{R}^n) \subsetneqq \dot{B}^{p,\lambda}(\mathbb{R}^n), \quad B^{q,\lambda}(\mathbb{R}^n) \subsetneqq B^{p,\lambda}(\mathbb{R}^n),$$
$$\mathcal{L}_{q,\lambda}(\mathbb{R}^n) \subsetneqq \mathcal{L}_{p,\lambda}(\mathbb{R}^n), \quad \mathrm{CBMO}^{q,\lambda}(\mathbb{R}^n) \subsetneqq \mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n), \quad \mathrm{CMO}^{q,\lambda}(\mathbb{R}^n) \subsetneqq \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n).$$

Moreover,

$$\begin{aligned} L_{p,\lambda}(\mathbb{R}^n) &\subset \mathcal{L}_{p,\lambda}(\mathbb{R}^n) &&\text{with} \quad \|f\|_{\mathcal{L}_{p,\lambda}} + |f_{B_1}| \le 3\|f\|_{L_{p,\lambda}}, \\ B^{p,\lambda}(\mathbb{R}^n) &\subset \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n) &&\text{with} \quad \|f\|_{\mathrm{CMO}^{p,\lambda}} + |f_{B_1}| \le 3\|f\|_{B^{p,\lambda}}, \\ \dot{B}^{p,\lambda}(\mathbb{R}^n) &\subset \mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n) &&\text{with} \quad \|f\|_{\mathrm{CBMO}^{p,\lambda}} + |f_{B_1}| \le 3\|f\|_{\dot{B}^{p,\lambda}}. \end{aligned}$$

If $\lambda < 0$, then

$$B^{p,\lambda}(\mathbb{R}^n) \cong \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}, \quad \dot{B}^{p,\lambda}(\mathbb{R}^n) \cong \mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}, \quad L_{p,\lambda}(\mathbb{R}^n) \cong \mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C},$$

where $A \cong B$ means that there exists a bijective and bicontinuous map from A to B, and

$$\begin{aligned} \dot{B}^{p,\lambda}(\mathbb{R}^n) &= B^{p,\lambda}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n) &&\text{for} \quad \lambda = -n/p, \\ \dot{B}^{p,\lambda}(\mathbb{R}^n) &= B^{p,\lambda}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) = \{0\} &&\text{for} \quad \lambda < -n/p. \end{aligned}$$

However, for $\lambda \ge 0$, $B^{p,\lambda}(\mathbb{R}^n)$, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$ are quite different from $L_{p,\lambda}(\mathbb{R}^n)$ and $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. For $\lambda = 0$, we denote $B^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$ by $B^p(\mathbb{R}^n)$, $\mathrm{CMO}^p(\mathbb{R}^n)$, $\dot{B}^p(\mathbb{R}^n)$ and $\mathrm{CBMO}^p(\mathbb{R}^n)$, respectively. Then

$$L^\infty(\mathbb{R}^n) \subsetneqq \bigcap_{p\ge1} \dot{B}^p(\mathbb{R}^n) \subsetneqq \bigcap_{p\ge1} B^p(\mathbb{R}^n),$$
$$\mathrm{BMO}(\mathbb{R}^n) \subsetneqq \bigcap_{p\ge1} \mathrm{CBMO}^p(\mathbb{R}^n) \subsetneqq \bigcap_{p\ge1} \mathrm{CMO}^p(\mathbb{R}^n).$$

On the other hand, for every $p \ge 1$,

$$L_{p,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n), \quad \mathcal{L}_{p,0}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n).$$

The first equality follows from the Lebesgue differentiation theorem and the second follows from the John-Nirenberg theorem and Hölder's inequality. Moreover, if $\lambda > 0$, then $L_{p,\lambda}(\mathbb{R}^n) = \{0\}$, while $B^{p,\lambda}(\mathbb{R}^n)$ is a larger space than $B^p(\mathbb{R}^n)$. It is known that $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$ modulo null-functions for $0 < \lambda = \alpha \le 1$, and $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathcal{C}$ for $\lambda > 1$, while $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$ is a larger space than $\mathrm{CMO}^{p,1}(\mathbb{R}^n)$.

In the next section we introduce $B^{p,\lambda}$ with Morrey-Campanato norms which unify $B^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $L_{p,\lambda}(\mathbb{R}^n)$ and $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and state our main results. To prove the results we state several properties of functions in Morrey-Campanato spaces on balls and the whole space $\mathbb{R}^n$ in Section 3. We give proofs of the main results in Section 4.

2. Definitions and main results. First we define Morrey-Campanato spaces on balls. For $1 \le p < \infty$, $-\infty < \lambda < \infty$, $0 < \alpha \le 1$ and the ball B_r, let $L_{p,\lambda}(B_r)$, $\mathcal{L}_{p,\lambda}(B_r)$, $\mathrm{Lip}_\alpha(B_r)$ and $WL^p(B_r)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\|f\|_{L_{p,\lambda}(B_r)} = \sup_{B(x,s)\subset B_r} \frac{1}{s^\lambda}\Big(\fint_{B(x,s)} |f(y)|^p\,dy\Big)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}(B_r)} = \sup_{B(x,s)\subset B_r} \frac{1}{s^\lambda}\Big(\fint_{B(x,s)} |f(y)-f_{B(x,s)}|^p\,dy\Big)^{1/p},$$

$$\|f\|_{\mathrm{Lip}_\alpha(B_r)} = \sup_{x,y\in B_r,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

and

$$\|f\|_{WL^p(B_r)} = \sup_{t>0} t\, m(B_r,f,t)^{1/p} = \sup_{t>0} t\,\big|\{x\in B_r : |f(x)|>t\}\big|^{1/p}.$$

Note that for any ball B,

$$\inf_c \Big(\fint_B |f(y)-c|^p\,dy\Big)^{1/p} \le \Big(\fint_B |f(y)-f_B|^p\,dy\Big)^{1/p} \le 2\inf_c \Big(\fint_B |f(y)-c|^p\,dy\Big)^{1/p}. \quad (2.1)$$

Now we introduce the spaces $B^{p,\lambda}$ with Morrey-Campanato norms.

Definition 2.1. For $1\le p<\infty$, $-\infty<\lambda<\infty$ and $0\le\sigma<\infty$, let $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$, $\dot B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $\dot B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\|f\|_{B^\sigma(L_{p,\lambda})} = \sup_{r\ge1} \frac{1}{r^\sigma}\|f\|_{L_{p,\lambda}(B_r)},$$

$$\|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} = \sup_{r\ge1} \frac{1}{r^\sigma}\|f\|_{\mathcal{L}_{p,\lambda}(B_r)},$$

$$\|f\|_{\dot B^\sigma(L_{p,\lambda})} = \sup_{r>0} \frac{1}{r^\sigma}\|f\|_{L_{p,\lambda}(B_r)}$$

and

$$\|f\|_{\dot B^\sigma(\mathcal{L}_{p,\lambda})} = \sup_{r>0} \frac{1}{r^\sigma}\|f\|_{\mathcal{L}_{p,\lambda}(B_r)}.$$

In the same way we define $B^\sigma(L^p)(\mathbb{R}^n)$, $B^\sigma(WL^p)(\mathbb{R}^n)$, $B^\sigma(\mathrm{BMO})(\mathbb{R}^n)$, $B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$, and $\dot B^\sigma(L^p)(\mathbb{R}^n)$, $\dot B^\sigma(WL^p)(\mathbb{R}^n)$, $\dot B^\sigma(\mathrm{BMO})(\mathbb{R}^n)$, $\dot B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$.

Then $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}$ and $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$, are Banach spaces equipped with the norm $\|f\|_{B^\sigma(L_{p,\lambda})}$, $\|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}$ and $\|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}+|f_{B_1}|$. The same thing can be said about $\dot B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $\dot B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}$ and $\dot B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$.

In the definition, we have equivalent norms if we replace balls B_r by cubes Q_r centered at the origin and of side-length r.

By the John-Nirenberg theorem, for each $1\le p<\infty$,

$$B^\sigma(\mathcal{L}_{p,0})(\mathbb{R}^n) = B^\sigma(\mathrm{BMO})(\mathbb{R}^n), \quad \dot B^\sigma(\mathcal{L}_{p,0})(\mathbb{R}^n) = \dot B^\sigma(\mathrm{BMO})(\mathbb{R}^n)$$

with equivalent norms. By Theorem 3.2 below, if $0<\lambda=\alpha\le1$, then, for each $1\le p<\infty$,

$$B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n), \quad \dot B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \dot B^\sigma(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$$

with equivalent norms. We note that $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ unifies $L_{p,\lambda}(\mathbb{R}^n)$ and $B^{p,\lambda}(\mathbb{R}^n)$ and that $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ unifies $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$. Actually, we have the following relations:

$$B^0(L_{p,\lambda})(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n), \quad B^0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \mathcal{L}_{p,\lambda}(\mathbb{R}^n),$$
$$B^{\lambda+n/p}(L_{p,-n/p})(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n), \quad B^{\lambda+n/p}(\mathcal{L}_{p,-n/p})(\mathbb{R}^n) = \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n). \tag{2.2}$$

In the above, the last equality follows from Theorem 3.3 below. We also have the same properties for the function spaces $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$.

Next we consider the fractional integral operator I_α $(0 < \alpha < n)$ defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

It is known that I_α is bounded from $L_{p,\lambda}(B_r)$ to $L_{q,\mu}(B_r)$ with appropriate indices. However, we cannot use directly this boundedness to prove the boundedness on our function spaces, since $I_\alpha f \neq I_\alpha(f\chi_r)$ on B_r in general.

In general, $I_\alpha f$ is not necessarily well defined for functions f in our spaces. Actually, $I_\alpha|f| \not\equiv \infty$ is equivalent to

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n-\alpha}}\, dy < \infty$$

(see [16]). Therefore we define the modified version of I_α as follows;

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y)\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_1(y)}{|y|^{n-\alpha}}\Big)\, dy.$$

If $I_\alpha f$ is well defined, then $\tilde{I}_\alpha f$ is also well defined and $I_\alpha f - \tilde{I}_\alpha f$ is a constant function.

REMARK 2.1. For the constant function 1, $I_\alpha 1 \equiv \infty$, while $\tilde{I}_\alpha 1$ is well defined and also a constant function. Actually,

$$\begin{aligned}\tilde{I}_\alpha 1(x) &= \int_{\mathbb{R}^n} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_1(y)}{|y|^{n-\alpha}}\Big)\, dy \\ &= \int_{\mathbb{R}^n} \Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}}\Big)\, dy + \int_{B_1} \frac{1}{|y|^{n-\alpha}}\, dy \\ &= \int_{B_1} \frac{1}{|y|^{n-\alpha}}\, dy = C,\end{aligned}$$

since

$$\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \quad \text{with} \quad 0 < \alpha < n$$

is integrable as a function with respect to y and the value of its integral is zero independent of x. This property is important to define operators on function spaces modulo constants.

Our main results are the following.

THEOREM 2.1. *Let $0 < \alpha < n$, $1 < p < \infty$, $-n/p + \alpha \le \lambda + \alpha = \mu < 0$ and $0 \le \sigma < -\lambda - \alpha$. If $1 \le q \le (\lambda/\mu)p$, then I_α is bounded from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $B^\sigma(L_{q,\mu})(\mathbb{R}^n)$, that is, there exists a positive constant C such that*

$$\|I_\alpha f\|_{B^\sigma(L_{q,\mu})} \le C\|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(L_{q,\mu})(\mathbb{R}^n)$.

In the theorem above, if $\sigma = 0$, then I_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$. This is the result of Adams [1] (see Theorem 3.6 in the next section).

If $\lambda = -n/p$, then $\lambda/\mu = n/(n-p\alpha)$ in the theorem above. Hence, by (2.2), we have the following (Fu, Lin and Lu [6, Proposition 1.1]).

COROLLARY 2.2 ([6]). *Let $0 < \alpha < n$, $1 < p < \infty$ and $-n/p \le \lambda < -\alpha$. If $1 \le q \le pn/(n-p\alpha)$ and $\lambda + \alpha = \mu$, then I_α is bounded from $B^{p,\lambda}(\mathbb{R}^n)$ to $B^{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant C such that*

$$\|I_\alpha f\|_{B^{q,\mu}} \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ to $\dot{B}^{q,\mu}(\mathbb{R}^n)$.

THEOREM 2.3. *Let $0 < \alpha < n$, $1 \le p < \infty$, $-n/p + \alpha \le \lambda + \alpha = \mu < 1$ and $0 \le \sigma < -\lambda - \alpha + 1$. Assume that p and q satisfy one of the following conditions:*

(i) *$p = 1$ and $1 \le q < n/(n-\alpha)$;*
(ii) *$1 < p < n/\alpha$ and $1 \le q \le pn/(n-p\alpha)$;*
(iii) *$n/\alpha \le p < \infty$ and $1 \le q < \infty$ (in this case, $0 \le \mu < 1$).*

Then $\tilde{I}_\alpha$ is bounded from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $B^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$, that is, there exists a positive constant C such that

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathcal{L}_{q,\mu})} + |(\tilde{I}_\alpha f)_{B_1}| \le C\|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$.

By (2.2) we have the following (cf. Komori-Furuya and Matsuoka [9]).

COROLLARY 2.4. *Let $0 < \alpha < n$, $1 < p < \infty$ and $-n/p \le \lambda < -\alpha + 1$. If $1 \le q \le pn/(n-p\alpha)$ and $\lambda + \alpha = \mu$, then $\tilde{I}_\alpha$ is bounded from $B^{p,\lambda}(\mathbb{R}^n)$ to $\mathrm{CMO}^{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant C such that*

$$\|\tilde{I}_\alpha f\|_{\mathrm{CMO}^{q,\mu}} + |(\tilde{I}_\alpha f)_{B_1}| \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ to $\mathrm{CBMO}^{q,\mu}(\mathbb{R}^n)$.

By Theorem 3.4 below we have the following.

COROLLARY 2.5. *Let $0 < \alpha < n$, $1 < p < \infty$, $0 \le -n/p + \alpha = \beta < 1$ and $0 \le \sigma < n/p - \alpha + 1$. If $\beta = 0$, then $\tilde{I}_\alpha$ is bounded from $B^\sigma(WL^p)(\mathbb{R}^n)$ to $B^\sigma(\mathrm{BMO})(\mathbb{R}^n)$, and if $\beta > 0$, then $\tilde{I}_\alpha$ is bounded from $B^\sigma(WL^p)(\mathbb{R}^n)$ to $B^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)$, that is, there exists a positive constant C such that*

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathrm{BMO})} + |(\tilde{I}_\alpha f)_{B_1}| \le C\|f\|_{B^\sigma(WL^p)}, \quad f \in B^\sigma(WL^p)(\mathbb{R}^n), \quad \text{if } \beta = 0,$$

and

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathrm{Lip}_\beta)} + |(\tilde{I}_\alpha f)_{B_1}| \le C\|f\|_{B^\sigma(WL^p)}, \quad f \in B^\sigma(WL^p)(\mathbb{R}^n), \quad \text{if } \beta > 0,$$

respectively. The same conclusion holds for the boundedness from $\dot{B}^\sigma(WL^p)(\mathbb{R}^n)$ to $\dot{B}^\sigma(\mathrm{BMO})(\mathbb{R}^n)$ and to $\dot{B}^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)$.

THEOREM 2.6. *Let $0 < \alpha < 1$, $1 \le p < \infty$, $-n/p + \alpha \le \lambda + \alpha = \mu < 1$ and $0 \le \sigma < -\lambda - \alpha + 1$. Assume that p and q satisfy one of the following conditions:*

(i) *$p = 1$ and $1 \le q < n/(n-\alpha)$;*

(ii) $1 < p < n/\alpha$ *and* $1 \le q \le pn/(n-p\alpha)$;
(iii) $n/\alpha \le p < \infty$ *and* $1 \le q < \infty$ (*in this case,* $0 \le \mu < 1$).

Then $\tilde{I}_\alpha$ *is bounded from* $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}$ *to* $B^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)/\mathcal{C}$ *and from* $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ *to* $B^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$, *that is, there exist positive constants* C_1 *and* C_2 *such that*

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathcal{L}_{q,\mu})} \le C_1 \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}, \quad f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathcal{L}_{q,\mu})} + |(\tilde{I}_\alpha f)_{B_1}| \le C_2 \left(\|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |f_{B_1}|\right), \quad f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness from $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}$ *to* $\dot{B}^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)/\mathcal{C}$ *and from* $\dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ *to* $\dot{B}^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$.

By (2.2) we have the following.

Corollary 2.7. *Let* $0 < \alpha < 1$, $1 < p < \infty$ *and* $-n/p \le \lambda < -\alpha + 1$. *If* $1 \le q \le pn/(n-p\alpha)$ *and* $\lambda + \alpha = \mu$, *then* $\tilde{I}_\alpha$ *is bounded from* $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ *to* $\mathrm{CMO}^{q,\mu}(\mathbb{R}^n)/\mathcal{C}$ *and from* $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$ *to* $\mathrm{CMO}^{q,\mu}(\mathbb{R}^n)$, *that is, there exist positive constants* C_1 *and* C_2 *such that*

$$\|\tilde{I}_\alpha f\|_{\mathrm{CMO}^{q,\mu}} \le C_1 \|f\|_{\mathrm{CMO}^{p,\lambda}}, \quad f \in \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde{I}_\alpha f\|_{\mathrm{CMO}^{q,\mu}} + |(\tilde{I}_\alpha f)_{B_1}| \le C_2 \left(\|f\|_{\mathrm{CMO}^{p,\lambda}} + |f_{B_1}|\right), \quad f \in \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness from $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ *to* $\mathrm{CBMO}^{q,\mu}(\mathbb{R}^n)/\mathcal{C}$ *and from* $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$ *to* $\mathrm{CBMO}^{q,\mu}(\mathbb{R}^n)$.

By Theorem 3.2 below we have the following.

Corollary 2.8. *Let* $0 < \beta < \beta + \alpha = \gamma < 1$ *and* $0 \le \sigma < -\beta - \alpha + 1$. *Then* $\tilde{I}_\alpha$ *is bounded from* $B^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)/\mathcal{C}$ *to* $B^\sigma(\mathrm{Lip}_\gamma)(\mathbb{R}^n)/\mathcal{C}$ *and from* $B^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)$ *to* $B^\sigma(\mathrm{Lip}_\gamma)(\mathbb{R}^n)$, *that is, there exist positive constants* C_1 *and* C_2 *such that*

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathrm{Lip}_\gamma)} \le C_1 \|f\|_{B^\sigma(\mathrm{Lip}_\beta)} \quad f \in B^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathrm{Lip}_\gamma)} + |(\tilde{I}_\alpha f)_{B_1}| \le C_2 \Big(\|f\|_{B^\sigma(\mathrm{Lip}_\beta)} + |f_{B_1}|\Big), \quad f \in B^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness from $\dot{B}^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)/\mathcal{C}$ *to* $\dot{B}^\sigma(\mathrm{Lip}_\gamma)(\mathbb{R}^n)/\mathcal{C}$ *and from* $\dot{B}^\sigma(\mathrm{Lip}_\beta)(\mathbb{R}^n)$ *to* $\dot{B}^\sigma(\mathrm{Lip}_\gamma)(\mathbb{R}^n)$.

3. Morrey-Campanato spaces on balls and $\mathbb{R}^n$.

First we state a lemma. See [21, 23] for the proof.

Lemma 3.1. *Let* $1 \le p < \infty$, $-n/p \le \lambda \le 1$ *and* $r > 0$. *Then there exists a positive constant* $C_{n,\lambda}$, *dependent only on* n *and* λ, *such that, for* $B(x,s) \subset B(z,t) \subset B_r$ *and* $f \in \mathcal{L}_{p,\lambda}(B_r)$,

$$|f_{B(x,s)} - f_{B(z,t)}| \le C_{n,\lambda} \int_s^{2t} u^{\lambda-1}\, du \; \|f\|_{\mathcal{L}_{p,\lambda}(B_r)}.$$

The lemma follows from an elementary inequality

$$|f_B - f_{B'}| \le \frac{|B'|}{|B|} \fint_{B'} |f(y) - f_{B'}|\,dy, \quad B \subset B'. \tag{3.1}$$

By the lemma above we can prove the next two theorems. For the proofs, see [14, 23] and [15, 19], respectively.

THEOREM 3.2. *If $1 \le p < \infty$, $0 < \lambda = \alpha \le 1$ and $r > 0$, then $\mathcal{L}_{p,\lambda}(B_r) = \mathrm{Lip}_\alpha(B_r)$ modulo null-functions and there exists a positive constant C, dependent only on n and λ, such that*

$$C^{-1}\|f\|_{\mathcal{L}_{p,\lambda}(B_r)} \le \|f\|_{\mathrm{Lip}_\alpha(B_r)} \le C\|f\|_{\mathcal{L}_{p,\lambda}(B_r)}.$$

THEOREM 3.3. *If $1 \le p < \infty$, $-n/p \le \lambda < 0$ and $r > 0$, then $\mathcal{L}_{p,\lambda}(B_r)/\mathcal{C} \cong L_{p,\lambda}(B_r)$ and there exists a positive constant C, dependent only on n and λ, such that*

$$C^{-1}\|f\|_{\mathcal{L}_{p,\lambda}(B_r)} \le \|f - f_{B_r}\|_{L_{p,\lambda}(B_r)} \le C\|f\|_{\mathcal{L}_{p,\lambda}(B_r)}.$$

REMARK 3.1. Theorems 3.2 and 3.3 are valid for Morrey-Campanato spaces on $\mathbb{R}^n$.

For the following theorem, see also [18, Theorem 3.4] which deals with Orlicz spaces on $\mathbb{R}^n$.

THEOREM 3.4. *If $1 < p < \infty$, $-n/p = \lambda$ and $r > 0$, then $WL^p(B_r) \subset L_{1,\lambda}(B_r)$ and there exists a positive constant C, dependent only on n and p, such that*

$$\|f\|_{L_{1,\lambda}(B_r)} \le C\|f\|_{WL^p(B_r)}, \quad f \in WL^p(B_r).$$

Proof. Let $f \in WL^p(B_r)$. We may assume that $\|f\|_{WL^p(B_r)} = 1$. Then $m(B_r, f, t) \le t^{-p}$. For any ball $B(z,s) \subset B_r$, let $\eta = s^\lambda = s^{-n/p}$ and

$$f = f^\eta + f_\eta, \quad f^\eta(x) = \begin{cases} f(x), & |f(x)| > \eta, \\ 0, & |f(x)| \le \eta. \end{cases}$$

Then

$$\begin{aligned} \frac{1}{s^\lambda} \fint_{B(z,s)} |f^\eta(x)|\,dx &\le \frac{1}{v_n s^{\lambda+n}} \int_0^\infty m(B_r, f^\eta, t)\,dt \\ &\le \frac{1}{v_n s^{\lambda+n}} \Big(\int_0^\eta m(B_r, f, \eta)\,dt + \int_\eta^\infty t^{-p}\,dt \Big) \\ &\le \frac{1}{v_n s^{\lambda+n}} \frac{p}{p-1} \eta^{1-p} = \frac{p}{v_n(p-1)}, \end{aligned}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. By Hölder's inequality we have

$$\begin{aligned} \frac{1}{s^\lambda} \fint_{B(z,s)} |f_\eta(x)|\,dx &\le \frac{1}{s^\lambda} \Big(\fint_{B(z,s)} |f_\eta(x)|^{2p}\,dx \Big)^{1/(2p)} \\ &= \frac{1}{v_n^{1/(2p)} s^{\lambda+n/(2p)}} \Big(\int_0^\eta m(B_r, f, t)(2p) t^{2p-1}\,dt \Big)^{1/(2p)} \\ &\le \frac{1}{v_n^{1/(2p)} s^{\lambda+n/(2p)}} (2\eta^p)^{1/(2p)} = (2/v_n)^{1/(2p)}. \end{aligned}$$

So we get the conclusion. ∎

Next we prove the following lemma (see also [17, Lemma 4.2] for the first part).

LEMMA 3.5. *Let* $1 \le p < \infty$. *For* $r > 0$, *let* χ_r *be the characteristic function of* B_r *and*

$$h_r(x) = h(x/r), \quad h(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases} \quad \|h\|_{\mathrm{Lip}_1(\mathbb{R}^n)} \le 1. \tag{3.2}$$

(i) *If* $\lambda < 0$, *then*

$$\|f\chi_r\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}(B_{3r})}$$

for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $\|f\|_{L_{p,\lambda}(B_{3r})} < \infty$.

(ii) *If* $0 \le \lambda \le 1$, *then there exists a positive constant* C, *dependent only on* n *and* λ, *such that*

$$\|(f - f_{B_{2r}})h_r\|_{\mathcal{L}_{p,\lambda}} \le C\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}$$

for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})} < \infty$.

Proof. (i) We show that, for all balls $B(x,s) \subset \mathbb{R}^n$,

$$\frac{1}{s^{p\lambda}} \fint_{B(x,s)} |f(y)\chi_r(y)|^p \, dy \le \|f\|^p_{L_{p,\lambda}(B_{3r})}. \tag{3.3}$$

We may assume that $B_r \cap B(x,s) \ne \emptyset$. If $s < r$, then $B(x,s) \subset B_{3r}$ and (3.3) holds. If $s \ge r$, then

$$\frac{1}{s^{p\lambda}} \fint_{B(x,s)} |f(y)\chi_r(y)|^p \, dy \le \frac{r^{p\lambda}}{s^{p\lambda}} \frac{|B_r|}{|B(x,s)|} \frac{1}{r^{p\lambda}} \fint_{B_r} |f(y)|^p \, dy \le \|f\|^p_{L_{p,\lambda}(B_{3r})}.$$

(ii) Let $\tilde{f} = f - f_{B_{2r}}$. We may assume that $B_{2r} \cap B(x,s) \ne \emptyset$, since $\operatorname{supp} h_r \subset B_{2r}$. If $s < r/2$, then $B(x,s) \subset B_{3r}$. By (2.1) it is enough to show

$$\begin{aligned}
&\Big(\fint_{B(x,s)} |\tilde{f}(y)h_r(y) - \tilde{f}_{B(x,s)}(h_r)_{B(x,s)}|^p \, dy\Big)^{1/p} \\
&\le \Big(\fint_{B(x,s)} |(\tilde{f}(y) - \tilde{f}_{B(x,s)})h_r(y)|^p \, dy\Big)^{1/p} + \Big(\fint_{B(x,s)} |\tilde{f}_{B(x,s)}(h_r(y) - (h_r)_{B(x,s)})|^p \, dy\Big)^{1/p} \\
&\equiv I_1 + I_2 \le Cs^{\lambda}\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}.
\end{aligned}$$

From $0 \le h \le 1$ it follows that $I_1 \le s^{\lambda}\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}$. By Lemma 3.1 we get

$$\begin{aligned}
|\tilde{f}_{B(x,s)}| = |f_{B(x,s)} - f_{B_{2r}}| &\le |f_{B(x,s)} - f_{B_{3r}}| + |f_{B_{2r}} - f_{B_{3r}}| \\
&\le 2C_{n,\lambda}((6r)^{\lambda}/\lambda)\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}.
\end{aligned}$$

From $\|h_r\|_{\mathrm{Lip}_1(\mathbb{R}^n)} \le 1/r$ it follows that

$$|h_r(y) - (h_r)_{B(x,s)}| \le \fint_{B(x,s)} |h_r(y) - h_r(z)| \, dz \le 2s/r \le 2(s/r)^{\lambda}.$$

Then $I_2 \le 4C_{n,\lambda}(6^{\lambda}/\lambda)s^{\lambda}\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}$. If $s \ge r/2$, then

$$\begin{aligned}
\frac{1}{s^{\lambda}}\Big(\fint_{B(x,s)} |\tilde{f}(y)h_r(y)|^p \, dy\Big)^{1/p} &\le \frac{(2r)^{\lambda}}{s^{\lambda}} \frac{1}{(2r)^{\lambda}} \Big(\frac{|B_{2r}|}{|B(x,s)|} \fint_{B_{2r}} |\tilde{f}(y)|^p \, dy\Big)^{1/p} \\
&\le 4^{n+\lambda}\|f\|_{\mathcal{L}_{p,\lambda}(B_{3r})}.
\end{aligned}$$

By (2.1) we get the conclusion. ∎

At the end of this section we recall the results on boundedness of I_α on Morrey-Campanato spaces.

THEOREM 3.6 ([1]). *Let $0<\alpha<n$, $1<p<\infty$ and $-n/p+\alpha\le\lambda+\alpha=\mu<0$. If $q=(\lambda/\mu)p$, then I_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant C such that*

$$\|I_\alpha f\|_{L_{q,\mu}}\le C\|f\|_{L_{p,\lambda}},\quad f\in L_{p,\lambda}(\mathbb{R}^n).$$

REMARK 3.2. If $q_1<q_2$, then $L_{q_1,\mu}(\mathbb{R}^n)\supset L_{q_2,\mu}(\mathbb{R}^n)$ and $\|f\|_{L_{q_1,\mu}}\le\|f\|_{L_{q_2,\mu}}$. Therefore, if $1\le q\le(\lambda/\mu)p$, then I_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$.

THEOREM 3.7 ([20]). *Let $0<\alpha<n$, $1\le p<\infty$ and $-n/p+\alpha\le\lambda+\alpha=\mu<1$. Assume that p and q satisfy one of the following conditions:*

(i) $p=1$ *and* $1\le q<n/(n-\alpha)$;
(ii) $1<p<n/\alpha$ *and* $1\le q\le pn/(n-p\alpha)$;
(iii) $n/\alpha\le p<\infty$ *and* $1\le q<\infty$ (*in this case*, $0\le\mu<1$).

Then $\tilde I_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant C such that

$$\|\tilde I_\alpha f\|_{\mathcal{L}_{q,\mu}}+|(\tilde I_\alpha f)_{B_1}|\le C\|f\|_{L_{p,\lambda}},\quad f\in L_{p,\lambda}(\mathbb{R}^n).$$

THEOREM 3.8 ([20]). *Let $0<\alpha<1$, $1\le p<\infty$ and $-n/p+\alpha\le\lambda+\alpha=\mu<1$. Assume that p and q satisfy one of the following conditions:*

(i) $p=1$ *and* $1\le q<n/(n-\alpha)$;
(ii) $1<p<n/\alpha$ *and* $1\le q\le pn/(n-p\alpha)$;
(iii) $n/\alpha\le p<\infty$ *and* $1\le q<\infty$ (*in this case*, $0\le\mu<1$).

Then $\tilde I_\alpha$ is bounded from $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)/\mathcal{C}$ and from $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)$, that is, there exist positive constants C_1 and C_2 such that

$$\|\tilde I_\alpha f\|_{\mathcal{L}_{q,\mu}}\le C_1\|f\|_{\mathcal{L}_{p,\lambda}},\quad f\in\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C},$$

and

$$\|\tilde I_\alpha f\|_{\mathcal{L}_{q,\mu}}+|(\tilde I_\alpha f)_{B_1}|\le C_2\left(\|f\|_{\mathcal{L}_{p,\lambda}}+|f_{B_1}|\right),\quad f\in\mathcal{L}_{p,\lambda}(\mathbb{R}^n),$$

respectively.

4. Proofs. In this section, we use the symbol $A\lesssim B$ to denote that there exists a positive constant C such that $A\le CB$. If $A\lesssim B$ and $B\lesssim A$, we then write $A\sim B$. First, we state two lemmas to prove Theorems 2.1, 2.3 and 2.6.

LEMMA 4.1. *Let $1\le p<\infty$ and $\beta,\lambda,\sigma\in\mathbb{R}$. If $\beta+\lambda+\sigma<0$, then there exists a positive constant C such that*

$$\int_{\mathbb{R}^n\setminus B_r}\frac{|f(y)|}{|y|^{n-\beta}}\,dy\le Cr^{\beta+\lambda+\sigma}\|f\|_{B^\sigma(L_{p,\lambda})}\quad\text{for all } f\in B^\sigma(L_{p,\lambda})(\mathbb{R}^n)\text{ and } r\ge1,$$

and

$$\int_{\mathbb{R}^n\setminus B_r}\frac{|f(y)|}{|y|^{n-\beta}}\,dy\le Cr^{\beta+\lambda+\sigma}\|f\|_{\dot B^\sigma(L_{p,\lambda})}\quad\text{for all } f\in\dot B^\sigma(L_{p,\lambda})(\mathbb{R}^n)\text{ and } r>0.$$

Proof. We prove only the case $f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $r \geq 1$.

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}}\,dy &= \sum_{j=0}^\infty \int_{B_{2^{j+1}r}\setminus B_{2^j r}} \frac{|f(y)|}{|y|^{n-\beta}}\,dy \\
&\lesssim \sum_{j=0}^\infty \frac{1}{(2^j r)^{n-\beta}} \int_{B_{2^{j+1}r}\setminus B_{2^j r}} |f(y)|\,dy \\
&\lesssim r^\beta \sum_{j=0}^\infty (2^\beta)^j \fint_{B_{2^{j+1}r}} |f(y)|\,dy \\
&\leq r^\beta \sum_{j=0}^\infty (2^\beta)^j \Big(\fint_{B_{2^{j+1}r}} |f(y)|^p\,dy\Big)^{1/p} \\
&\lesssim r^{\beta+\lambda} \sum_{j=0}^\infty (2^{\beta+\lambda})^j \|f\|_{L_{p,\lambda}(B_{2^{j+1}r})} \\
&\lesssim r^{\beta+\lambda+\sigma} \sum_{j=0}^\infty (2^{\beta+\lambda+\sigma})^j \|f\|_{B^\sigma(L_{p,\lambda})} \\
&\sim r^{\beta+\lambda+\sigma} \|f\|_{B^\sigma(L_{p,\lambda})}.
\end{aligned}$$

The proof for $f \in \dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $r > 0$ is the same as above. ∎

LEMMA 4.2. *Let $1 \leq p < \infty$ and $\lambda, \sigma \in \mathbb{R}$. If $\beta < 0$ and $\beta + \lambda + \sigma < 0$, then there exists a positive constant C such that*

$$\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y) - f_{B_{2r}}|}{|y|^{n-\beta}}\,dy \leq Cr^{\beta+\lambda+\sigma} \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} \quad \textit{for all } f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) \textit{ and } r \geq 1,$$

and

$$\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y) - f_{B_{2r}}|}{|y|^{n-\beta}}\,dy \leq Cr^{\beta+\lambda+\sigma} \|f\|_{\dot{B}^\sigma(\mathcal{L}_{p,\lambda})} \quad \textit{for all } f \in \dot{B}^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) \textit{ and } r > 0.$$

Proof. We prove only the case $f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $r \geq 1$.

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y) - f_{B_{2r}}|}{|y|^{n-\beta}}\,dy = \sum_{j=0}^\infty \int_{B_{2^{j+1}r}\setminus B_{2^j r}} \frac{|f(y) - f_{B_{2r}}|}{|y|^{n-\beta}}\,dy \\
&\lesssim \sum_{j=0}^\infty \frac{1}{(2^j r)^{n-\beta}} \int_{B_{2^{j+1}r}\setminus B_{2^j r}} |f(y) - f_{B_{2r}}|\,dy \\
&\lesssim r^\beta \sum_{j=0}^\infty (2^\beta)^j \fint_{B_{2^{j+1}r}} |f(y) - f_{B_{2r}}|\,dy \\
&\leq r^\beta \sum_{j=0}^\infty (2^\beta)^j \Big(\fint_{B_{2^{j+1}r}} |f(y) - f_{B_{2^{j+1}r}}|\,dy + \sum_{k=1}^j |f_{B_{2^{k+1}r}} - f_{B_{2^k r}}|\Big) \\
&\lesssim r^{\beta+\lambda} \sum_{j=0}^\infty (2^\beta)^j \Big((2^\lambda)^{j+1} \|f\|_{\mathcal{L}_{p,\lambda}(B_{2^{j+1}r})} + \sum_{k=1}^j (2^\lambda)^{k+1} \|f\|_{\mathcal{L}_{p,\lambda}(B_{2^{k+1}r})}\Big)
\end{aligned}$$

$$\begin{aligned}
&\lesssim r^{\beta+\lambda+\sigma}\sum_{j=0}^{\infty}(2^{\beta})^{j}\sum_{k=1}^{j+1}(2^{\lambda+\sigma})^{k}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}\\
&= r^{\beta+\lambda+\sigma}\sum_{k=1}^{\infty}(2^{\lambda+\sigma})^{k}\sum_{j=k-1}^{\infty}(2^{\beta})^{j}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}\\
&\lesssim r^{\beta+\lambda+\sigma}\sum_{k=1}^{\infty}(2^{\beta+\lambda+\sigma})^{k}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}\\
&\sim r^{\beta+\lambda+\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}.
\end{aligned}$$

The proof for $f\in\dot{B}^{\sigma}(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $r>0$ is the same as above. ■

Now we prove the theorems.

Proof of Theorem 2.1. Let $f\in B^{\sigma}(L_{p,\lambda})(\mathbb{R}^n)$. We prove that $I_\alpha f$ is well defined and that

$$\|I_\alpha f\|_{L_{q,\mu}(B_r)}\lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}$$

for any ball B_r with $r\ge 1$.

For $x\in B_r$, let

$$I_\alpha f(x)=I_\alpha(f\chi_{2r})(x)+I_\alpha(f(1-\chi_{2r}))(x).$$

Then $I_\alpha(f\chi_{2r})$ is well defined, since $f\chi_{2r}$ is in $L^p(\mathbb{R}^n)$. We show later that $I_\alpha(f(1-\chi_{2r}))(x)$ is well defined for all $x\in B_r$. Moreover, if $0<s<r$, then, for $x\in B_s$,

$$I_\alpha(f\chi_{2s})(x)+I_\alpha(f(1-\chi_{2s}))(x)=I_\alpha(f\chi_{2r})(x)+I_\alpha(f(1-\chi_{2r}))(x).$$

Therefore, $I_\alpha f$ is well defined on $\mathbb{R}^n$.

Now, by the boundedness of I_α from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ (Theorem 3.6) and (i) of Lemma 3.5, we have

$$\begin{aligned}
\|I_\alpha(f\chi_{2r})\|_{L_{q,\mu}(B_r)}\le\|I_\alpha(f\chi_{2r})\|_{L_{q,\mu}}&\lesssim\|f\chi_{2r}\|_{L_{p,\lambda}}\\
&\le\|f\|_{L_{p,\lambda}(B_{6r})}\lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}. \qquad (4.1)
\end{aligned}$$

Since

$$|I_\alpha(f(1-\chi_{2r}))(x)|\lesssim\int_{\mathbb{R}^n\setminus B_{2r}}\frac{|f(y)|}{|y|^{n-\alpha}}\,dy$$

for $x\in B_r$, using Lemma 4.1, we have

$$|I_\alpha(f(1-\chi_{2r}))(x)|\lesssim r^{\alpha+\lambda+\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}=r^{\mu+\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}.$$

Then $I_\alpha(f(1-\chi_{2r}))(x)$ is well defined for all $x\in B_r$ and

$$\|I_\alpha(f(1-\chi_{2r}))\|_{L_{q,\mu}(B_r)}\le r^{-\mu}\|I_\alpha(f(1-\chi_{2r}))\|_{L^{\infty}(B_r)}\lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})},$$

since $\mu<0$. Therefore, we have

$$\|I_\alpha f\|_{L_{q,\mu}(B_r)}\lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}$$

for any ball B_r. This shows the conclusion.

The proof of the boundedness from $\dot{B}^{\sigma}(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^{\sigma}(L_{q,\mu})(\mathbb{R}^n)$ is the same as above. ■

Proof of Theorem 2.3. Let $f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$. We first prove that $\tilde{I}_\alpha f$ is well defined and that

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}$$

for any ball B_r with $r \geq 1$. Next we prove that $|(\tilde{I}_\alpha f)_{B_1}| \lesssim \|f\|_{B^\sigma(L_{p,\lambda})}$.

For $x \in B_r$, let

$$\tilde{I}_\alpha f(x) = I_\alpha(f\chi_{2r})(x) + J_\alpha(f(1-\chi_{2r}))(x) + C_\alpha(f(\chi_1 - \chi_{2r})),$$

where

$$J_\alpha f(x) = \int_{\mathbb{R}^n} f(y)\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}}\Big)\, dy \tag{4.2}$$

and

$$C_\alpha f = \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{n-\alpha}}\, dy. \tag{4.3}$$

In the above, $I_\alpha(f\chi_{2r})$ is well defined, since $f\chi_{2r}$ is in $L^p(\mathbb{R}^n)$. $C_\alpha(f(\chi_1-\chi_{2r}))$ is also well defined, since $(\chi_1 - \chi_{2r})/|\cdot|^{n-\alpha}$ is in $L^{p'}(\mathbb{R}^n)$. Note that $C_\alpha(f(\chi_1 - \chi_{2r}))$ is a constant. Moreover, if $r = 1$, then

$$|C_\alpha(f(\chi_1 - \chi_2))| \leq \left\|\frac{\chi_1 - \chi_2}{|\cdot|^{n-\alpha}}\right\|_{L^{p'}} \|f\|_{L^p(B_2)} \lesssim \|f\|_{L^p(B_2)} \lesssim \|f\|_{B^\sigma(L_{p,\lambda})}. \tag{4.4}$$

We show later that $J_\alpha(f(1-\chi_{2r}))(x)$ is well defined for all $x \in B_r$. Then $\tilde{I}_\alpha f$ is well defined on $\mathbb{R}^n$ by the same reason as in the proof of Theorem 2.1.

Now, in the same way as (4.1), by the boundedness of $\tilde{I}_\alpha$ from $L_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)$ (Theorem 3.7) and (i) of Lemma 3.5, we have

$$\|I_\alpha(f\chi_{2r})\|_{\mathcal{L}_{q,\mu}(B_r)} + |(I_\alpha(f\chi_{2r}))_{B_1}| \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}. \tag{4.5}$$

Using the inequality

$$\left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}}\right| \lesssim \frac{|x|}{|y|^{n-\alpha+1}} \leq \frac{r}{|y|^{n-\alpha+1}}$$

for $x \in B_r$ and $y \in \mathbb{R}^n \setminus B_{2r}$, we have

$$|J_\alpha(f(1-\chi_{2r}))(x)| \leq r \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha+1}}\, dy.$$

By Lemma 4.1 we have

$$|J_\alpha(f(1-\chi_{2r}))(x)| \lesssim r^{\alpha+\lambda+\sigma}\|f\|_{B^\sigma(L_{p,\lambda})} = r^{\mu+\sigma}\|f\|_{B^\sigma(L_{p,\lambda})}. \tag{4.6}$$

Then $J_\alpha(f(1-\chi_{2r}))(x)$ is well defined for all $x \in B_r$. If $\mu \leq 0$, then we have

$$\begin{aligned}\|J_\alpha(f(1-\chi_{2r}))\|_{\mathcal{L}_{q,\mu}(B_r)} &\lesssim \|J_\alpha(f(1-\chi_{2r}))\|_{L_{q,\mu}(B_r)} \\ &\leq r^{-\mu}\|J_\alpha(f(1-\chi_{2r}))\|_{L^\infty(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}.\end{aligned}$$

If $\mu > 0$, then, for any $x, z \in B_r$, we have by Lemma 4.1

$$\begin{aligned}&|J_\alpha(f(1-\chi_{2r}))(x) - J_\alpha(f(1-\chi_{2r}))(z)| \\ &= \left|\int_{\mathbb{R}^n \setminus B_{2r}} f(y)\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}}\Big)\, dy\right|\end{aligned}$$

$$\lesssim |x-z| \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha+1}}\, dy$$
$$\lesssim |x-z|\, r^{\alpha-1+\lambda+\sigma} \|f\|_{B^\sigma(L_{p,\lambda})} = |x-z|\, r^{\mu-1+\sigma} \|f\|_{B^\sigma(L_{p,\lambda})}$$

and

$$\frac{|J_\alpha(f(1-\chi_{2r}))(x) - J_\alpha(f(1-\chi_{2r}))(z)|}{|x-z|^\mu} \lesssim \Big(\frac{|x-z|}{r}\Big)^{1-\mu} r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}$$
$$\lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}.$$

By Theorem 3.2 we have

$$\|J_\alpha(f(1-\chi_{2r}))\|_{\mathcal{L}_{q,\mu}(B_r)} \sim \|J_\alpha(f(1-\chi_{2r}))\|_{\mathrm{Lip}_\mu(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}.$$

Therefore,

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}(B_r)} = \|I_\alpha(f\chi_{2r}) + J_\alpha(f(1-\chi_{2r})) + C_\alpha(f(\chi_1 - \chi_{2r}))\|_{\mathcal{L}_{q,\mu}(B_r)}$$
$$\lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}$$

for any ball B_r, that is,

$$\|\tilde{I}_\alpha f\|_{B^\sigma(\mathcal{L}_{q,\mu})} \lesssim \|f\|_{B^\sigma(L_{p,\lambda})}.$$

Finally, by (4.4), (4.5) and (4.6) with $r=1$, we have

$$|(\tilde{I}_\alpha f)_{B_1}| \le |(I_\alpha(f\chi_2))_{B_1}| + |(J_\alpha(f(1-\chi_2)))_{B_1}| + |C_\alpha(f(\chi_1 - \chi_2))|$$
$$\lesssim \|f\|_{B^\sigma(L_{p,\lambda})}.$$

The proof of the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$ is the same as above. ∎

Proof of Theorem 2.6. Let $f \in B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$. We first prove that $\tilde{I}_\alpha f$ is well defined and that

$$\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}$$

for any ball B_r with $r \ge 1$. Next we prove that $|(\tilde{I}_\alpha f)_{B_1}| \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})} + |f_{B_1}|$.

Let h be defined by (3.2). For $x \in B_r$, let $\tilde{f} = f - f_{B_{4r}}$ and

$$\tilde{I}_\alpha f(x) = \tilde{I}_\alpha \tilde{f}(x) + \tilde{I}_\alpha(f_{B_{4r}})(x)$$
$$= I_\alpha(\tilde{f}h_{2r})(x) + J_\alpha(\tilde{f}(1-h_{2r}))(x) + C_\alpha(\tilde{f}(\chi_1 - h_{2r})) + f_{B_{4r}}(\tilde{I}_\alpha 1)(x),$$

where J_α and C_α are defined by (4.2) and (4.3), respectively. By Remark 2.1, $\tilde{I}_\alpha 1$ is a constant function. By the same observation as in the proof of Theorem 2.3, we see that $I_\alpha(\tilde{f}h_{2r})$ and $C_\alpha(\tilde{f}(\chi_1 - h_{2r}))$ are well defined and, if $r=1$, then by (2.1),

$$\big|C_\alpha(\tilde{f}(\chi_1 - h_2))\big| \le \Big\|\frac{\chi_1 - h_2}{|\cdot|^{n-\alpha}}\Big\|_{L^{p'}} \|\tilde{f}\|_{L^p(B_2)} \lesssim \|\tilde{f}\|_{L^p(B_2)} \lesssim \|f\|_{B^\sigma(\mathcal{L}_{p,\lambda})}. \tag{4.7}$$

We show later that $J_\alpha(\tilde{f}(1-h_{2r}))(x)$ is well defined for all $x \in B_r$. Then $\tilde{I}_\alpha f$ is well defined on $\mathbb{R}^n$ by the same reason as the proof of Theorem 2.1.

Now, by the boundedness of $\tilde{I}_\alpha$ from $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)/\mathcal{C}$ (Theorem 3.8) and (ii) of Lemma 3.5, we have

$$\|I_\alpha(\tilde{f}h_{2r})\|_{\mathcal{L}_{q,\mu}(B_r)} \le \|I_\alpha(\tilde{f}h_{2r})\|_{\mathcal{L}_{q,\mu}} \lesssim \|\tilde{f}h_{2r}\|_{\mathcal{L}_{p,\lambda}}$$

$$\lesssim \|\tilde{f}\|_{\mathcal{L}_{p,\lambda}(B_{6r})} = \|f\|_{\mathcal{L}_{p,\lambda}(B_{6r})} \lesssim r^{\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}. \quad (4.8)$$

In the same way, by the boundedness of $\tilde{I}_\alpha$ from $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}_{q,\mu}(\mathbb{R}^n)$ and Lemma 3.5, we have

$$\begin{aligned}\|I_\alpha(\tilde{f}h_{2r})\|_{\mathcal{L}_{q,\mu}(B_r)} + |(I_\alpha(\tilde{f}h_{2r}))_{B_1}| &\le \|I_\alpha(\tilde{f}h_{2r})\|_{\mathcal{L}_{q,\mu}} + |(I_\alpha(\tilde{f}h_{2r}))_{B_1}| \\ &\lesssim \|\tilde{f}h_{2r}\|_{\mathcal{L}_{p,\lambda}} + |(\tilde{f}h_{2r})_{B_1}| \lesssim r^{\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})} + |(\tilde{f}h_{2r})_{B_1}|. \quad (4.9)\end{aligned}$$

By Lemma 4.2 we have

$$\begin{aligned}\left|J_\alpha(\tilde{f}(1-h_{2r}))(x)\right| &\le r\int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y)-f_{B_{4r}}|}{|y|^{n-\alpha+1}}\,dy \\ &\lesssim r^{\alpha+\lambda+\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})} = r^{\mu+\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}. \quad (4.10)\end{aligned}$$

Then $J_\alpha(\tilde{f}(1-h_{2r}))(x)$ is well defined for all $x \in B_r$. For each case $\mu \le 0$ or $\mu > 0$, using the same way as in the proof of Theorem 2.3, we have

$$\|J_\alpha(\tilde{f}(1-h_{2r}))\|_{\mathcal{L}_{q,\mu}(B_r)} \lesssim r^{\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}.$$

Therefore,

$$\begin{aligned}\|\tilde{I}_\alpha f\|_{\mathcal{L}_{q,\mu}(B_r)} &= \|I_\alpha(\tilde{f}h_{2r}) + J_\alpha(\tilde{f}(1-h_{2r})) + C_\alpha(\tilde{f}(\chi_1-h_{2r})) + f_{B_{4r}}(\tilde{I}_\alpha 1)\|_{\mathcal{L}_{q,\mu}(B_r)} \\ &\lesssim r^{\sigma}\|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}\end{aligned}$$

for any ball B_r, and

$$\|\tilde{I}_\alpha f\|_{B^{\sigma}(\mathcal{L}_{q,\mu})} \lesssim \|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}.$$

Finally, we estimate each term of the right hand side in the following:

$$|(\tilde{I}_\alpha f)_{B_1}| \le |(I_\alpha(\tilde{f}h_2))_{B_1}| + |(J_\alpha(\tilde{f}(1-h_2)))_{B_1}| + |C_\alpha(\tilde{f}(\chi_1-h_2))| + |f_{B_4}(\tilde{I}_\alpha 1)|.$$

Taking $r=1$ in (4.9) and (4.10), we have

$$|(I_\alpha(\tilde{f}h_2))_{B_1}| \lesssim \|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})} + |(\tilde{f}h_2)_{B_1}| = \|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})} + |f_{B_1} - f_{B_4}|$$

and

$$|(J_\alpha(\tilde{f}(1-h_2)))_{B_1}| \le \|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})},$$

respectively. By (3.1) we have $|f_{B_1} - f_{B_4}| \lesssim \|f\|_{\mathcal{L}_{p,\lambda}(B_4)} \lesssim \|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})}$. Using these estimates and (4.7), we obtain

$$|(\tilde{I}_\alpha f)_{B_1}| \lesssim \|f\|_{B^{\sigma}(\mathcal{L}_{p,\lambda})} + |f_{B_1}|.$$

Then we have the conclusion.

The proof of the boundedness from $\dot{B}^{\sigma}(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)/\mathcal{C}$ to $\dot{B}^{\sigma}(\mathcal{L}_{q,\mu})(\mathbb{R}^n)/\mathcal{C}$ and from $\dot{B}^{\sigma}(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^{\sigma}(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$ is the same as above. ■

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