FRACTIONAL INTEGRAL OPERATORS ON $B^{p,\lambda}$ WITH MORREY-CAMPANATO NORMS

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Abstract. We introduce function spaces $B^{p,\lambda}$ with Morrey-Campanato norms, which unify $B^{p,\lambda}$, CMO$^{p,\lambda}$ and Morrey-Campanato spaces, and prove the boundedness of the fractional integral operator $I_\alpha$ on these spaces.

1. Introduction. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. It is known that the fractional integral operator $I_\alpha$ ($0 < \alpha < n$) is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $-n/p + \alpha = -n/q$, from $L^{n/\alpha}(\mathbb{R}^n)$ to BMO$(\mathbb{R}^n)$, from $L^p(\mathbb{R}^n)$ to Lipschitz space Lip$_{\alpha-n/p}(\mathbb{R}^n)$ for $0 < \alpha - n/p < 1$, from BMO$(\mathbb{R}^n)$ to Lip$_\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$, and from Lip$_\beta(\mathbb{R}^n)$ to Lip$_{\alpha+\beta}(\mathbb{R}^n)$ for $0 < \beta < \alpha + \beta < 1$. In this paper we introduce $B^{p,\lambda}(\mathbb{R}^n)$ with Morrey-Campanato norms and extend the boundedness of $I_\alpha$ to these function spaces.

The space $B^p(\mathbb{R}^n)$ is introduced by Beurling [3] together with its predual $A^p(\mathbb{R}^n)$. Feichtinger [5] gave an equivalent norm on $B^p(\mathbb{R}^n)$. The space $B^p(\mathbb{R}^n)$ is a special case of Herz spaces $K_{p,\alpha}^{\alpha,\gamma}(\mathbb{R}^n)$ introduced by Herz [8]. Lu and Yang [12] proved the boundedness of $I_\alpha$ on Herz spaces. However, it does not cover the space $B^p(\mathbb{R}^n)$ which is an end point

2010 Mathematics Subject Classification: Primary 42B35; Secondary 46E35, 46E30, 26A33.

Key words and phrases: fractional integral, $B^p$-space, BMO, CMO, Morrey space, Campanato space.

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The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc92-0-17 [249] © Instytut Matematyczny PAN, 2011
the sets of all functions, i.e. the homogeneous spaces, Alvarez, Guzmán-Partida and Lakey \[2\] introduced the spaces $B^{p,\lambda}(\mathbb{R}^n)$ and $\text{CMO}^{p,\lambda}(\mathbb{R}^n)$. Our spaces in this paper unify $B^{p,\lambda}(\mathbb{R}^n)$, $\text{CMO}^{p,\lambda}(\mathbb{R}^n)$, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ and Campanato spaces $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. Note that $B^{p,\lambda}$ with $\text{Lip}_\alpha$-norms is considered in Komori-Furuya and Matsuoka \[9\]. For Morrey-Campanato spaces and the boundedness of $I_\alpha$ on them, see Peetre \[22\]. For Herz spaces, see for example Lu, Yang and Hu \[13\].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \}$. We denote $B(0, r)$ by $B_r$ and the characteristic function of $B_r$ by $\chi_r$. For a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a ball $B$, let

$$f_B = \int_B f(y) \, dy = \frac{1}{|B|} \int_B f(y) \, dy,$$

where $|B|$ is the Lebesgue measure of $B$.

For $1 \leq p < \infty$, $-\infty < \lambda < \infty$ and $0 < \alpha \leq 1$, let $B^{p,\lambda}(\mathbb{R}^n)$, $\text{CMO}^{p,\lambda}(\mathbb{R}^n)$, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$, i.e. the homogeneous $B^{p,\lambda}$ space, $\text{CBMO}^{p,\lambda}(\mathbb{R}^n)$, $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\text{Lip}_\alpha(\mathbb{R}^n)$ be the sets of all functions $f$ such that the following functionals are finite, respectively:

$$\|f\|_{B^{p,\lambda}} = \sup_{r \geq 1} \frac{1}{r^\lambda} \left( \int_{B_r} |f(y)|^p \, dy \right)^{1/p},$$

$$\|f\|_{\text{CMO}^{p,\lambda}} = \sup_{r \geq 1} \frac{1}{r^\lambda} \left( \int_{B_r} |f(y) - f_{B_r}|^p \, dy \right)^{1/p},$$

$$\|f\|_{\dot{B}^{p,\lambda}} = \sup_{r > 0} \frac{1}{r^\lambda} \left( \int_{B_r} |f(y)|^p \, dy \right)^{1/p},$$

$$\|f\|_{\text{CBMO}^{p,\lambda}} = \sup_{r > 0} \frac{1}{r^\lambda} \left( \int_{B_r} |f(y) - f_{B_r}|^p \, dy \right)^{1/p},$$

$$\|f\|_{L_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \left( \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \left( \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \, dy \right)^{1/p}$$

and

$$\|f\|_{\text{Lip}_\alpha} = \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$
By the definition we have

\[ L_{p,\lambda}(\mathbb{R}^n) \subset B^{p,\lambda}(\mathbb{R}^n) \subset B^{p,\lambda}(\mathbb{R}^n), \quad \mathcal{L}_{p,\lambda}(\mathbb{R}^n) \subset \text{CMO}^{p,\lambda}(\mathbb{R}^n) \subset \text{CMO}^{p,\lambda}(\mathbb{R}^n), \]

and, for \( p < q, \)

\[ L_{q,\lambda}(\mathbb{R}^n) \subsetneq L_{p,\lambda}(\mathbb{R}^n), \quad B^{q,\lambda}(\mathbb{R}^n) \subsetneq B^{p,\lambda}(\mathbb{R}^n), \quad B^{q,\lambda}(\mathbb{R}^n) \subsetneq B^{p,\lambda}(\mathbb{R}^n), \]

\[ \mathcal{L}_{q,\lambda}(\mathbb{R}^n) \subsetneq \mathcal{L}_{p,\lambda}(\mathbb{R}^n), \quad \text{CMO}^{q,\lambda}(\mathbb{R}^n) \subsetneq \text{CMO}^{p,\lambda}(\mathbb{R}^n), \quad \text{CMO}^{q,\lambda}(\mathbb{R}^n) \subsetneq \text{CMO}^{p,\lambda}(\mathbb{R}^n). \]

Moreover,

\[ L_{p,\lambda}(\mathbb{R}^n) \subset \mathcal{L}_{p,\lambda}(\mathbb{R}^n) \quad \text{with} \quad \|f\|_{L_{p,\lambda}} + |f|_{B^1} \leq 3\|f\|_{L_{p,\lambda}}, \]

\[ B^{p,\lambda}(\mathbb{R}^n) \subset \text{CMO}^{p,\lambda}(\mathbb{R}^n) \quad \text{with} \quad \|f\|_{\text{CMO}^{p,\lambda}} + |f|_{B^1} \leq 3\|f\|_{B^{p,\lambda}}, \]

\[ \dot{B}^{p,\lambda}(\mathbb{R}^n) \subset \text{CMO}^{p,\lambda}(\mathbb{R}^n) \quad \text{with} \quad \|f\|_{\text{CMO}^{p,\lambda}} + |f|_{B^1} \leq 3\|f\|_{\dot{B}^{p,\lambda}}. \]

If \( \lambda < 0, \) then

\[ B^{p,\lambda}(\mathbb{R}^n) \cong \text{CMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}, \quad \dot{B}^{p,\lambda}(\mathbb{R}^n) \cong \text{CMO}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}, \quad L_{p,\lambda}(\mathbb{R}^n) \cong \mathcal{L}_{p,\lambda}(\mathbb{R}^n)/\mathcal{C}, \]

where \( A \cong B \) means that there exists a bijective and bicontinuous map from \( A \) to \( B, \)

\[ \dot{B}^{p,\lambda}(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \quad \text{for} \quad \lambda = -n/p, \]

\[ \dot{B}^{p,\lambda}(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) = \{0\} \quad \text{for} \quad \lambda < -n/p. \]

However, for \( \lambda \geq 0, \) \( B^{p,\lambda}(\mathbb{R}^n), \dot{B}^{p,\lambda}(\mathbb{R}^n), \text{CMO}^{p,\lambda}(\mathbb{R}^n) \) and \( \text{CMO}^{p,\lambda}(\mathbb{R}^n) \) are quite different from \( L_{p,\lambda}(\mathbb{R}^n) \) and \( \mathcal{L}_{p,\lambda}(\mathbb{R}^n). \) For \( \lambda = 0, \) we denote \( B^{p,\lambda}(\mathbb{R}^n), \text{CMO}^{p,\lambda}(\mathbb{R}^n), \dot{B}^{p,\lambda}(\mathbb{R}^n) \) and \( \text{CMO}^{p,\lambda}(\mathbb{R}^n) \) by \( B^p(\mathbb{R}^n), \text{CMO}^p(\mathbb{R}^n), \dot{B}^p(\mathbb{R}^n) \) and \( \text{CMO}^p(\mathbb{R}^n), \) respectively. Then

\[ L^\infty(\mathbb{R}^n) \subsetneq \bigcap_{p \geq 1} \dot{B}^p(\mathbb{R}^n) \subsetneq \bigcap_{p \geq 1} B^p(\mathbb{R}^n), \]

\[ \text{BMO}(\mathbb{R}^n) \subsetneq \bigcap_{p \geq 1} \text{CMO}^p(\mathbb{R}^n) \subsetneq \bigcap_{p \geq 1} \text{CMO}^p(\mathbb{R}^n). \]

On the other hand, for every \( p \geq 1, \)

\[ L_{p,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n), \quad \mathcal{L}_{p,0}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n). \]

The first equality follows from the Lebesgue differentiation theorem and the second follows from the John-Nirenberg theorem and Hölder’s inequality. Moreover, if \( \lambda > 0, \) then \( L_{p,\lambda}(\mathbb{R}^n) = \{0\}, \) while \( B^{p,\lambda}(\mathbb{R}^n) \) is a larger space than \( B^{p,\lambda}(\mathbb{R}^n). \) It is known that \( \mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \text{Lip}_\alpha(\mathbb{R}^n) \) modulo null-functions for \( 0 < \lambda = \alpha \leq 1, \) and \( \mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathcal{C} \) for \( \lambda > 1, \) while \( \text{CMO}^{p,\lambda}(\mathbb{R}^n) \) is a larger space than \( \text{CMO}^{p,\lambda}(\mathbb{R}^n). \)

In the next section we introduce \( B^{p,\lambda} \) with Morrey-Campanato norms which unify \( B^{p,\lambda}(\mathbb{R}^n), \text{CMO}^{p,\lambda}(\mathbb{R}^n), L_{p,\lambda}(\mathbb{R}^n) \) and \( \mathcal{L}_{p,\lambda}(\mathbb{R}^n) \) and state our main results. To prove the results we state several properties of functions in Morrey-Campanato spaces on balls and the whole space \( \mathbb{R}^n \) in Section 3. We give proofs of the main results in Section 4.

2. Definitions and main results. First we define Morrey-Campanato spaces on balls. For \( 1 \leq p < \infty, -\infty < \lambda < \infty, 0 < \alpha \leq 1 \) and the ball \( B_r, \) let \( L_{p,\lambda}(B_r), \mathcal{L}_{p,\lambda}(B_r), \text{Lip}_\alpha(B_r) \) and \( W^{p,q}(B_r) \) be the sets of all functions \( f \) such that the following functionals
are finite, respectively:

$$
\|f\|_{L^p\lambda(B_r)} = \sup_{B(x,s) \subseteq B_r} \frac{1}{s^\lambda} \left( \int_{B(x,s)} |f(y)|^p \, dy \right)^{1/p},
$$

$$
\|f\|_{\mathcal{L}^p\lambda(B_r)} = \sup_{B(x,s) \subseteq B_r} \frac{1}{s^\lambda} \left( \int_{B(x,s)} |f(y) - f_{B(x,s)}|^p \, dy \right)^{1/p},
$$

$$
\|f\|_{\text{Lip}_\alpha(B_r)} = \sup_{x,y \in B_r, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}
$$

and

$$
\|f\|_{WL^p\lambda(B_r)} = \inf_{t > 0} \left( \sup_{B_r} t \, m(B_r, f, t)^{1/p} = \sup_{t > 0} \left| \{ x \in B_r : |f(x)| > t \} \right|^{1/p} \right).
$$

Note that for any ball $B$,

$$
\inf_c \left( \int_B |f(y) - c|^p \, dy \right)^{1/p} \leq \left( \int_B |f(y) - f_{B_r}|^p \, dy \right)^{1/p} \leq 2 \inf_c \left( \int_B |f(y) - c|^p \, dy \right)^{1/p}. \tag{2.1}
$$

Now we introduce the spaces $B^{p,\lambda}$ with Morrey-Campanato norms.

**Definition 2.1.** For $1 \leq p < \infty$, $-\infty < \lambda < \infty$ and $0 \leq \sigma < \infty$, let $B^\sigma(L^p\lambda)(\mathbb{R}^n)$, $B^\sigma(\mathcal{L}^p\lambda)(\mathbb{R}^n)$, $\dot{B}^\sigma(L^p\lambda)(\mathbb{R}^n)$ and $\dot{B}^\sigma(\mathcal{L}^p\lambda)(\mathbb{R}^n)$ be the sets of all functions $f$ such that the following functionals are finite, respectively:

$$
\|f\|_{B^\sigma(L^p\lambda)} = \sup_{r \geq 1} \frac{1}{r^\sigma} \|f\|_{L^p\lambda(B_r)},
$$

$$
\|f\|_{B^\sigma(\mathcal{L}^p\lambda)} = \sup_{r \geq 1} \frac{1}{r^\sigma} \|f\|_{\mathcal{L}^p\lambda(B_r)},
$$

$$
\|f\|_{\dot{B}^\sigma(L^p\lambda)} = \sup_{r > 0} \frac{1}{r^\sigma} \|f\|_{L^p\lambda(B_r)}
$$

and

$$
\|f\|_{\dot{B}^\sigma(\mathcal{L}^p\lambda)} = \sup_{r > 0} \frac{1}{r^\sigma} \|f\|_{\mathcal{L}^p\lambda(B_r)}.
$$

In the same way we define $B^\sigma(L^p(\mathbb{R}^n))$, $B^\sigma(WL^p)(\mathbb{R}^n)$, $B^\sigma(\text{BMO})(\mathbb{R}^n)$, $B^\sigma(\text{Lip}_\alpha)(\mathbb{R}^n)$, and $\dot{B}^\sigma(L^p(\mathbb{R}^n))$, $\dot{B}^\sigma(WL^p)(\mathbb{R}^n)$, $\dot{B}^\sigma(\text{BMO})(\mathbb{R}^n)$, $\dot{B}^\sigma(\text{Lip}_\alpha)(\mathbb{R}^n)$.

Then $B^\sigma(L^p\lambda)(\mathbb{R}^n)$, $B^\sigma(\mathcal{L}^p\lambda)(\mathbb{R}^n)$, $\dot{B}^\sigma(\mathcal{L}^p\lambda)(\mathbb{R}^n)$, are Banach spaces equipped with the norm $\|f\|_{B^\sigma(L^p\lambda)}$, $\|f\|_{B^\sigma(\mathcal{L}^p\lambda)}$ and $\|f\|_{\dot{B}^\sigma(L^p\lambda)} + |f_{B_1}|$. The same thing can be said about $\dot{B}^\sigma(\mathcal{L}^p\lambda)(\mathbb{R}^n)$.

In the definition, we have equivalent norms if we replace balls $B_r$ by cubes $Q_r$ centered at the origin and of side-length $r$.

By the John-Nirenberg theorem, for each $1 \leq p < \infty$,

$$
B^\sigma(\mathcal{L},0)(\mathbb{R}^n) = B^\sigma(\text{BMO})(\mathbb{R}^n), \quad \dot{B}^\sigma(\mathcal{L},0)(\mathbb{R}^n) = \dot{B}^\sigma(\text{BMO})(\mathbb{R}^n)
$$

with equivalent norms. By Theorem 3.2 below, if $0 < \lambda = \alpha \leq 1$, then, for each $1 \leq p < \infty$,

$$
B^\sigma(\mathcal{L}\lambda)(\mathbb{R}^n) = B^\sigma(\text{Lip}_\alpha)(\mathbb{R}^n), \quad \dot{B}^\sigma(\mathcal{L}\lambda)(\mathbb{R}^n) = \dot{B}^\sigma(\text{Lip}_\alpha)(\mathbb{R}^n)
$$
with equivalent norms. We note that $B^\sigma(L_p,\lambda)(\mathbb{R}^n)$ unifies $L_p,\lambda(\mathbb{R}^n)$ and $B^{p,\lambda}(\mathbb{R}^n)$ and that $B^\sigma(L_p,\lambda)(\mathbb{R}^n)$ unifies $\mathcal{L}_p,\lambda(\mathbb{R}^n)$ and $\text{CMO}^{p,\lambda}(\mathbb{R}^n)$. Actually, we have the following relations:

$$
B^0(L_p,\lambda)(\mathbb{R}^n) = L_p,\lambda(\mathbb{R}^n), \quad B^0(\mathcal{L}_p,\lambda)(\mathbb{R}^n) = \mathcal{L}_p,\lambda(\mathbb{R}^n),
$$

$$
B^{\lambda+n/p}(L_{p,-n/p})(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n), \quad B^{\lambda+n/p}(\mathcal{L}_{p,-n/p})(\mathbb{R}^n) = \text{CMO}^{p,\lambda}(\mathbb{R}^n).
$$

In the above, the last equality follows from Theorem 3.3 below. We also have the same relations:

$$
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$$

2.1 **Remark** (see [16]). Therefore we define the modified version of $B^\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$.

Next we consider the fractional integral operator $I_\alpha (0 < \alpha < n)$ defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

It is known that $I_\alpha$ is bounded from $L_{p,\lambda}(B_\tau)$ to $L_{q,\mu}(B_\tau)$ with appropriate indices. However, we cannot use directly this boundedness to prove the boundedness on our function spaces, since $I_\alpha f \neq I_\alpha (f \chi_{B_\tau})$ on $B_\tau$ in general.

In general, $I_\alpha f$ is not necessarily well defined for functions $f$ in our spaces. Actually, $I_\alpha |f| \neq \infty$ is equivalent to

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n-\alpha}} \, dy < \infty$$

(see [16]). Therefore we define the modified version of $I_\alpha$ as follows;

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1 - \chi_1(y)}{|y|^{n-\alpha}} \right) \, dy.$$

If $I_\alpha f$ is well defined, then $\tilde{I}_\alpha f$ is also well defined and $I_\alpha f - \tilde{I}_\alpha f$ is a constant function.

**Remark 2.1.** For the constant function $1$, $I_\alpha 1 \equiv \infty$, while $\tilde{I}_\alpha 1$ is well defined and also a constant function. Actually,

$$\tilde{I}_\alpha 1(x) = \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1 - \chi_1(y)}{|y|^{n-\alpha}} \right) \, dy$$

$$= \int_{\mathbb{R}^n} \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right) \, dy + \int_{B_1} \frac{1}{|y|^{n-\alpha}} \, dy$$

$$= \int_{B_1} \frac{1}{|y|^{n-\alpha}} \, dy = C,$$

since

$$\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \quad \text{with} \quad 0 < \alpha < n$$

is integrable as a function with respect to $y$ and the value of its integral is zero independent of $x$. This property is important to define operators on function spaces modulo constants.

Our main results are the following.

**Theorem 2.1.** Let $0 < \alpha < n$, $1 < p < \infty$, $-n/p + \alpha \leq \lambda + \alpha = \mu < 0$ and $0 \leq \sigma < -\lambda - \alpha$. If $1 \leq q \leq (\lambda/\mu)p$, then $I_\alpha$ is bounded from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $B^\sigma(L_{q,\mu})(\mathbb{R}^n)$, that is, there exists a positive constant $C$ such that

$$\|I_\alpha f\|_{B^\sigma(L_{q,\mu})} \leq C\|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n).$$

The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(L_{q,\mu})(\mathbb{R}^n)$.
In the theorem above, if \( \sigma = 0 \), then \( I_\alpha \) is bounded from \( L_{p,\lambda}(\mathbb{R}^n) \) to \( L_{q,\mu}(\mathbb{R}^n) \). This is the result of Adams [1] (see Theorem 3.6 in the next section).

If \( \lambda = -n/p \), then \( \lambda/\mu = n/(n - p\alpha) \) in the theorem above. Hence, by (2.2), we have the following (Fu, Lin and Lu [6, Proposition 1.1]).

**Corollary 2.2** ([6]). Let \( 0 < \alpha < n, 1 < p < \infty \) and \(-n/p \leq \lambda < -\alpha \). If \( 1 \leq q \leq pn/(n - p\alpha) \) and \( \lambda + \alpha = \mu \), then \( I_\alpha \) is bounded from \( B^{p,\lambda}(\mathbb{R}^n) \) to \( B^{q,\mu}(\mathbb{R}^n) \), that is, there exists a positive constant \( C \) such that

\[
\|I_\alpha f\|_{B^{q,\mu}} \leq C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).
\]

The same conclusion holds for the boundedness from \( \dot{B}^{p,\lambda}(\mathbb{R}^n) \) to \( \dot{B}^{q,\mu}(\mathbb{R}^n) \).

**Theorem 2.3.** Let \( 0 < \alpha < n, 1 \leq p < \infty, -n/p + \alpha \leq \lambda + \alpha = \mu < 1 \) and \( 0 \leq \sigma < -\lambda - \alpha + 1 \). Assume that \( p \) and \( q \) satisfy one of the following conditions:

(i) \( p = 1 \) and \( 1 \leq q < n/(n - \alpha) \);
(ii) \( 1 < p < n/\alpha \) and \( 1 \leq q \leq pn/(n - p\alpha) \);
(iii) \( n/\alpha \leq p < \infty \) and \( 1 \leq q < \infty \) (in this case, \( 0 \leq \mu < 1 \)).

Then \( \tilde{I}_\alpha \) is bounded from \( B^\alpha(L_{p,\lambda})(\mathbb{R}^n) \) to \( B^\sigma(L_{q,\mu})(\mathbb{R}^n) \), that is, there exists a positive constant \( C \) such that

\[
\|\tilde{I}_\alpha f\|_{B^\sigma(L_{q,\mu})} + |(\tilde{I}_\alpha f)_{B_1}| \leq C\|f\|_{B^\alpha(L_{p,\lambda})}, \quad f \in B^\alpha(L_{p,\lambda})(\mathbb{R}^n).
\]

The same conclusion holds for the boundedness from \( \dot{B}^\alpha(L_{p,\lambda})(\mathbb{R}^n) \) to \( \dot{B}^\sigma(L_{q,\mu})(\mathbb{R}^n) \).

By (2.2) we have the following (cf. Komori-Furuya and Matsuoka [9]).

**Corollary 2.4.** Let \( 0 < \alpha < n, 1 < p < \infty \) and \(-n/p \leq \lambda < -\alpha + 1 \). If \( 1 \leq q \leq pn/(n - p\alpha) \) and \( \lambda + \alpha = \mu \), then \( \tilde{I}_\alpha \) is bounded from \( B^{p,\lambda}(\mathbb{R}^n) \) to \( \text{CMO}^{q,\mu}(\mathbb{R}^n) \), that is, there exists a positive constant \( C \) such that

\[
\|\tilde{I}_\alpha f\|_{\text{CMO}^{q,\mu}} + |(\tilde{I}_\alpha f)_{B_1}| \leq C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).
\]

The same conclusion holds for the boundedness from \( \dot{B}^{p,\lambda}(\mathbb{R}^n) \) to \( \text{CBMO}^{q,\mu}(\mathbb{R}^n) \).

By Theorem 3.4 below we have the following.

**Corollary 2.5.** Let \( 0 < \alpha < n, 1 < p < \infty, 0 \leq -n/p + \alpha = \beta < 1 \) and \( 0 \leq \sigma < n/p - \alpha + 1 \). If \( \beta = 0 \), then \( \tilde{I}_\alpha \) is bounded from \( B^\alpha(WL^p)(\mathbb{R}^n) \) to \( B^\alpha(\text{BMO})(\mathbb{R}^n) \), and if \( \beta > 0 \), then \( \tilde{I}_\alpha \) is bounded from \( B^\alpha(WL^p)(\mathbb{R}^n) \) to \( B^\alpha(\text{Lip}_\beta)(\mathbb{R}^n) \), that is, there exists a positive constant \( C \) such that

\[
\|\tilde{I}_\alpha f\|_{B^\alpha(\text{BMO})} + |(\tilde{I}_\alpha f)_{B_1}| \leq C\|f\|_{B^\alpha(WL^p)}, \quad f \in B^\alpha(WL^p)(\mathbb{R}^n), \quad \text{if } \beta = 0,
\]

and

\[
\|\tilde{I}_\alpha f\|_{B^\alpha(\text{Lip}_\beta)} + |(\tilde{I}_\alpha f)_{B_1}| \leq C\|f\|_{B^\alpha(WL^p)}, \quad f \in B^\alpha(WL^p)(\mathbb{R}^n), \quad \text{if } \beta > 0,
\]

respectively. The same conclusion holds for the boundedness from \( \dot{B}^\alpha(WL^p)(\mathbb{R}^n) \) to \( \dot{B}^\alpha(\text{BMO})(\mathbb{R}^n) \) and to \( \dot{B}^\alpha(\text{Lip}_\beta)(\mathbb{R}^n) \).

**Theorem 2.6.** Let \( 0 < \alpha < 1, 1 \leq p < \infty, -n/p + \alpha \leq \lambda + \alpha = \mu < 1 \) and \( 0 \leq \sigma < -\lambda - \alpha + 1 \). Assume that \( p \) and \( q \) satisfy one of the following conditions:

(i) \( p = 1 \) and \( 1 \leq q < n/(n - \alpha) \);
(iii) $1 < p < n/\alpha$ and $1 \leq q \leq pn/(n - \alpha p)$;

(iii) $n/\alpha \leq p < \infty$ and $1 \leq q < \infty$ (in this case, $0 \leq \mu < 1$).

Then $\tilde{I}_\alpha$ is bounded from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C$ to $B^\sigma(L_{q,\mu})(\mathbb{R}^n)/C$ and from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $B^\sigma(L_{q,\mu})(\mathbb{R}^n)$, that is, there exist positive constants $C_1$ and $C_2$ such that

$$\|\tilde{I}_\alpha f\|_{B^\sigma(L_{q,\mu})} \leq C_1 \|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C,$$

and

$$\|\tilde{I}_\alpha f\|_{B^\sigma(L_{q,\mu})} + \|\tilde{I}_\alpha f\|_{B^\sigma(L_{p,\lambda})} \leq C_2 \left( \|f\|_{B^\sigma(L_{p,\lambda})} + \|f\|_{B^\sigma(L_{q,\mu})} \right), \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C$ to $\dot{B}^\sigma(L_{q,\mu})(\mathbb{R}^n)/C$ and from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(L_{q,\mu})(\mathbb{R}^n)$.

By (2.2) we have the following.

**Corollary 2.7.** Let $0 < \alpha < 1$, $1 < p < \infty$ and $-n/p \leq \lambda < -\alpha + 1$. If $1 \leq q \leq pn/(n - \alpha p)$ and $\lambda + \alpha = \mu$, then $\tilde{I}_\alpha$ is bounded from $\text{CMO}^{p,\lambda}(\mathbb{R}^n)/C$ to $\text{CMO}^{q,\mu}(\mathbb{R}^n)/C$ and from $\text{CMO}^{p,\lambda}(\mathbb{R}^n)$ to $\text{CMO}^{q,\mu}(\mathbb{R}^n)$, that is, there exist positive constants $C_1$ and $C_2$ such that

$$\|\tilde{I}_\alpha f\|_{\text{CMO}^{q,\mu}} \leq C_1 \|f\|_{\text{CMO}^{p,\lambda}}, \quad f \in \text{CMO}^{p,\lambda}(\mathbb{R}^n)/C,$$

and

$$\|\tilde{I}_\alpha f\|_{\text{CMO}^{q,\mu}} + \|\tilde{I}_\alpha f\|_{\text{CMO}^{p,\lambda}} \leq C_2 \left( \|f\|_{\text{CMO}^{p,\lambda}} + \|f\|_{\text{CMO}^{q,\mu}} \right), \quad f \in \text{CMO}^{p,\lambda}(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness from $\text{CBMO}^{p,\lambda}(\mathbb{R}^n)/C$ to $\text{CBMO}^{q,\mu}(\mathbb{R}^n)/C$ and from $\text{CBMO}^{p,\lambda}(\mathbb{R}^n)$ to $\text{CBMO}^{q,\mu}(\mathbb{R}^n)$.

By Theorem 3.2 below we have the following.

**Corollary 2.8.** Let $0 < \beta < \beta + \alpha = \gamma < 1$ and $0 \leq \sigma < -\beta - \alpha + 1$. Then $\tilde{I}_\alpha$ is bounded from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C$ to $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C$ and from $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $B^\sigma(L_{p,\lambda})(\mathbb{R}^n)$, that is, there exist positive constants $C_1$ and $C_2$ such that

$$\|\tilde{I}_\alpha f\|_{B^\sigma(L_{p,\lambda})} \leq C_1 \|f\|_{B^\sigma(L_{p,\lambda})}, \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C,$$

and

$$\|\tilde{I}_\alpha f\|_{B^\sigma(L_{p,\lambda})} + \|\tilde{I}_\alpha f\|_{B^\sigma(L_{p,\lambda})} \leq C_2 \left( \|f\|_{B^\sigma(L_{p,\lambda})} + \|f\|_{B^\sigma(L_{p,\lambda})} \right), \quad f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n),$$

respectively. The same conclusion holds for the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C$ to $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)/C$ and from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$.

3. Morrey-Campanato spaces on balls and $\mathbb{R}^n$. First we state a lemma. See [21, 23] for the proof.

**Lemma 3.1.** Let $1 \leq p < \infty$, $-n/p \leq \lambda \leq 1$ and $r > 0$. Then there exists a positive constant $C_{n,\lambda}$, dependent only on $n$ and $\lambda$, such that, for $B(x,s) \subset B(z,t) \subset B_r$ and $f \in L_{p,\lambda}(B_r)$,

$$|f_B(x,s) - f_B(z,t)| \leq C_{n,\lambda} \int_s^t u^{\lambda - 1} \, du \|f\|_{L_{p,\lambda}(B_r)}.$$
The lemma follows from an elementary inequality
\[ |f_B - f_{B'}| \leq \frac{|B'|}{|B|} \int_{B'} |f(y) - f_{B'}| \, dy, \quad B \subset B'. \] (3.1)

By the lemma above we can prove the next two theorems. For the proofs, see [14, 23] and [15, 19], respectively.

**Theorem 3.2.** If \( 1 \leq p < \infty, 0 < \lambda = \alpha \leq 1 \) and \( r > 0 \), then \( \mathcal{L}_{p,\lambda}(B_r) = \text{Lip}_\alpha(B_r) \) modulo null-functions and there exists a positive constant \( C \), dependent only on \( n \) and \( \lambda \), such that
\[ C^{-1} \|f\|_{\mathcal{L}_{p,\lambda}(B_r)} \leq \|f\|_{\text{Lip}_\alpha(B_r)} \leq C \|f\|_{\mathcal{L}_{p,\lambda}(B_r)}. \]

**Theorem 3.3.** If \( 1 \leq p < \infty, -n/p \leq \lambda < 0 \) and \( r > 0 \), then \( \mathcal{L}_{p,\lambda}(B_r)/\mathcal{C} \cong L_{p,\lambda}(B_r) \) and there exists a positive constant \( C \), dependent only on \( n \) and \( \lambda \), such that
\[ C^{-1} \|f\|_{\mathcal{L}_{p,\lambda}(B_r)} \leq \|f - f_{B_r}\|_{L_{p,\lambda}(B_r)} \leq C \|f\|_{\mathcal{L}_{p,\lambda}(B_r)}. \]

**Remark 3.1.** Theorems 3.2 and 3.3 are valid for Morrey-Campanato spaces on \( \mathbb{R}^n \).

For the following theorem, see also [18, Theorem 3.4] which deals with Orlicz spaces on \( \mathbb{R}^n \).

**Theorem 3.4.** If \( 1 < p < \infty, -n/p = \lambda \) and \( r > 0 \), then \( WL^p(B_r) \subset L_{1,\lambda}(B_r) \) and there exists a positive constant \( C \), dependent only on \( n \) and \( p \), such that
\[ \|f\|_{L_{1,\lambda}(B_r)} \leq C \|f\|_{WL^p(B_r)}, \quad f \in WL^p(B_r). \]

**Proof.** Let \( f \in WL^p(B_r) \). We may assume that \( \|f\|_{WL^p(B_r)} = 1 \). Then \( m(B_r, f, t) \leq t^{-p} \). For any ball \( B(z, s) \subset B_r \), let \( \eta = s^\lambda = s^{-n/p} \) and
\[ f = f^n + f_\eta, \quad f^n(x) = \begin{cases} f(x), & |f(x)| > \eta, \\ 0, & |f(x)| \leq \eta. \end{cases} \]

Then
\[ \frac{1}{s^\lambda} \int_{B(z, s)} |f^n(x)| \, dx \leq \frac{1}{v_n s^{\lambda+n}} \int_0^\infty m(B_r, f^n, t) \, dt \]
\[ \leq \frac{1}{v_n s^{\lambda+n}} \left( \int_0^\eta m(B_r, f, \eta) \, dt + \int_\eta^\infty t^{-p} \, dt \right) \]
\[ \leq \frac{1}{v_n s^{\lambda+n}} \frac{p}{p-1} \eta^{-1} = \frac{p}{v_n(p-1)}, \]
where \( v_n \) is the volume of the unit ball in \( \mathbb{R}^n \). By Hölder’s inequality we have
\[ \frac{1}{s^\lambda} \int_{B(z, s)} |f_\eta(x)| \, dx \leq \frac{1}{s^\lambda} \left( \int_{B(z, s)} |f_\eta(x)|^{2p} \, dx \right)^{1/(2p)} \]
\[ = \frac{1}{v_n^{1/(2p)} s^{\lambda+n/(2p)}} \left( \int_0^\eta m(B_r, f, t)(2p)t^{2p-1} \, dt \right)^{1/(2p)} \]
\[ \leq \frac{1}{v_n^{1/(2p)} s^{\lambda+n/(2p)}} (2\eta^p)^{1/(2p)} = (2/v_n)^{1/(2p)}. \]
So we get the conclusion.  

Next we prove the following lemma (see also [17, Lemma 4.2] for the first part).
Lemma 3.5. Let $1 \leq p < \infty$. For $r > 0$, let $\chi_r$ be the characteristic function of $B_r$ and

$$h_r(x) = h(x/r), \quad h(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2, \end{cases} \quad \|h\|_{\text{Lip}_1(\mathbb{R}^n)} \leq 1. \quad (3.2)$$

(i) If $\lambda < 0$, then

$$\|f \chi_r\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}(B_{3r})}$$

for all $f \in L_{p,\text{loc}}^p(\mathbb{R}^n)$ with $\|f\|_{L_{p,\lambda}(B_{3r})} < \infty$.

(ii) If $0 \leq \lambda \leq 1$, then there exists a positive constant $C$, dependent only on $n$ and $\lambda$, such that

$$\|(f - f_{B_{2r}})h_r\|_{L_{p,\lambda}} \leq C\|f\|_{L_{p,\lambda}(B_{3r})}$$

for all $f \in L_{p,\text{loc}}^p(\mathbb{R}^n)$ with $\|f\|_{L_{p,\lambda}(B_{3r})} < \infty$.

Proof. (i) We show that, for all balls $B(x,s) \subset \mathbb{R}^n$,

$$\frac{1}{s^{p\lambda}} \int_{B(x,s)} |f(y)\chi_r(y)|^p \, dy \leq \|f\|_{L_{p,\lambda}(B_{3r})}^p. \quad (3.3)$$

We may assume that $B_r \cap B(x,s) \neq \emptyset$. If $s < r$, then $B(x,s) \subset B_{3r}$ and (3.3) holds. If $s \geq r$, then

$$\frac{1}{s^{p\lambda}} \int_{B(x,s)} |f(y)\chi_r(y)|^p \, dy \leq \frac{r^{p\lambda}}{s^{p\lambda}} \frac{|B_r|}{|B(x,s)|} \frac{1}{r^{p\lambda}} \int_{B_r} |f(y)|^p \, dy \leq \|f\|_{L_{p,\lambda}(B_{3r})}^p.$$ 

(ii) Let $\tilde{f} = f - f_{B_{2r}}$. We may assume that $B_{2r} \cap B(x,s) \neq \emptyset$, since supp $h_r \subset B_{2r}$. If $s < r/2$, then $B(x,s) \subset B_{2r}$. By (2.1) it is enough to show

$$\left( \int_{B(x,s)} |\tilde{f}(y)h_r(y) - \tilde{f}_{B(x,s)}(h_r)(B_{r})|^p \, dy \right)^{1/p} \leq \left( \int_{B(x,s)} |(\tilde{f}(y) - \tilde{f}_{B(x,s)})(h_r(y)|^p \, dy \right)^{1/p} + \left( \int_{B(x,s)} |\tilde{f}_{B(x,s)}(h_r(y) - (h_r)(B_{r}))|^p \, dy \right)^{1/p} \equiv I_1 + I_2 \leq Cs^\lambda\|f\|_{L_{p,\lambda}(B_{3r})}.$$ 

From $0 \leq h \leq 1$ it follows that $I_1 \leq s^\lambda\|f\|_{L_{p,\lambda}(B_{3r})}$. By Lemma 3.1 we get

$$|\tilde{f}_{B(x,s)}| = |f_{B(x,s)} - f_{B_{2r}}| \leq |f_{B(x,s)} - f_{B_{3r}}| + |f_{B_{2r}} - f_{B_{3r}}| \leq 2C_{n,\lambda}((6r)^\lambda/\lambda)\|f\|_{L_{p,\lambda}(B_{3r})}.$$ 

From $\|h_r\|_{\text{Lip}_1(\mathbb{R}^n)} \leq 1$ it follows that

$$|h_r(y) - (h_r)(B_{r})| = \int_{B(x,s)} |h_r(y) - h_r(z)| \, dz \leq 2s/r < 2(s/r)^\lambda.$$ 

Then $I_2 \leq 4C_{n,\lambda}(6^\lambda/\lambda)s^\lambda\|f\|_{L_{p,\lambda}(B_{3r})}$. If $s \geq r/2$, then

$$\frac{1}{s^\lambda}\left( \int_{B(x,s)} |\tilde{f}(y)h_r(y)|^p \, dy \right)^{1/p} \leq \frac{(2r)^\lambda}{s^\lambda}\left( \frac{|B_{2r}|}{|B(x,s)|} \int_{B_{2r}} |\tilde{f}(y)|^p \, dy \right)^{1/p} \leq \frac{4^{n+\lambda}}{s^{n+\lambda}}\|f\|_{L_{p,\lambda}(B_{3r})}.$$ 

By (2.1) we get the conclusion.
At the end of this section we recall the results on boundedness of $I_\alpha$ on Morrey-Campanato spaces.

**Theorem 3.6** ([1]). Let $0 < \alpha < n$, $1 < p < \infty$ and $-n/p + \alpha \leq \lambda + \alpha = \mu < 0$. If $q = (\lambda/\mu)p$, then $I_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant $C$ such that

$$\|I_\alpha f\|_{L_{q,\mu}} \leq C\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n).$$

**Remark 3.2.** If $q_1 < q_2$, then $L_{q_1,\mu}(\mathbb{R}^n) \supset L_{q_2,\mu}(\mathbb{R}^n)$ and $\|f\|_{L_{q_1,\mu}} \leq \|f\|_{L_{q_2,\mu}}$. Therefore, if $1 \leq q \leq (\lambda/\mu)p$, then $I_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$.

**Theorem 3.7** ([20]). Let $0 < \alpha < n$, $1 \leq p < \infty$ and $-n/p + \alpha \leq \lambda + \alpha = \mu < 1$. Assume that $p$ and $q$ satisfy one of the following conditions:

(i) $p = 1$ and $1 \leq q < n/(n - \alpha)$;
(ii) $1 < p < n/\alpha$ and $1 \leq q \leq p/(n - \alpha\alpha)$;
(iii) $n/\alpha \leq p < \infty$ and $1 \leq q < \infty$ (in this case, $0 \leq \mu < 1$).

Then $\tilde{I}_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, that is, there exists a positive constant $C$ such that

$$\|\tilde{I}_\alpha f\|_{L_{q,\mu}} + |(\tilde{I}_\alpha f)_{B_1}| \leq C\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n).$$

**Theorem 3.8** ([20]). Let $0 < \alpha < 1$, $1 \leq p < \infty$ and $-n/p + \alpha \leq \lambda + \alpha = \mu < 1$. Assume that $p$ and $q$ satisfy one of the following conditions:

(i) $p = 1$ and $1 \leq q < n/(n - \alpha)$;
(ii) $1 < p < n/\alpha$ and $1 \leq q \leq p/(n - \alpha\alpha)$;
(iii) $n/\alpha \leq p < \infty$ and $1 \leq q < \infty$ (in this case, $0 \leq \mu < 1$).

Then $\tilde{I}_\alpha$ is bounded from $L_{p,\lambda}(\mathbb{R}^n)/C$ to $L_{q,\mu}(\mathbb{R}^n)/C$ and from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$, that is, there exist positive constants $C_1$ and $C_2$ such that

$$\|\tilde{I}_\alpha f\|_{L_{q,\mu}} \leq C_1\|f\|_{L_{p,\lambda}}, \quad f \in L_{p,\lambda}(\mathbb{R}^n)/C,$$

and

$$\|\tilde{I}_\alpha f\|_{L_{q,\mu}} + |(\tilde{I}_\alpha f)_{B_1}| \leq C_2\left(\|f\|_{L_{p,\lambda}} + |f_{B_1}|\right), \quad f \in L_{p,\lambda}(\mathbb{R}^n),$$

respectively.

**4. Proofs.** In this section, we use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$.

**Lemma 4.1.** Let $1 \leq p < \infty$ and $\beta, \lambda, \sigma \in \mathbb{R}$. If $\beta + \lambda + \sigma < 0$, then there exists a positive constant $C$ such that

$$\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}} dy \leq C r^{\beta+\lambda+\sigma}\|f\|_{B^\sigma(L_{p,\lambda})} \quad \text{for all } f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \text{ and } r \geq 1,$$

and

$$\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}} dy \leq C r^{\beta+\lambda+\sigma}\|f\|_{B^\sigma(L_{p,\lambda})} \quad \text{for all } f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \text{ and } r > 0.$$
Proof. We prove only the case \( f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \) and \( r \geq 1 \).

\[
\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}} \, dy = \sum_{j=0}^{\infty} \int_{B_{2j+1,r} \setminus B_{2jr}} \frac{|f(y)|}{|y|^{n-\beta}} \, dy \\
\lesssim \sum_{j=0}^{\infty} \frac{1}{(2jr)^{n-\beta}} \int_{B_{2j+1,r} \setminus B_{2jr}} |f(y)| \, dy \\
\lesssim r^\beta \sum_{j=0}^{\infty} (2^\beta j)^j \int_{B_{2j+1,r}} |f(y)| \, dy \\
\leq r^\beta \sum_{j=0}^{\infty} (2^\beta j)^j \left( \int_{B_{2j+1,r}} |f(y)|^p \, dy \right)^{1/p} \\
\lesssim r^\beta + \lambda \sum_{j=0}^{\infty} (2^\beta + \lambda)^j \|f\|_{L_{p,\lambda}(B_{2j+1,r})} \\
\lesssim r^\beta + \lambda + \sigma \sum_{j=0}^{\infty} (2^\beta + \lambda + \sigma)^j \|f\|_{B^\sigma(L_{p,\lambda})} \\
\sim r^\beta + \lambda + \sigma \|f\|_{B^\sigma(L_{p,\lambda})}.
\]

The proof for \( f \in \dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n) \) and \( r > 0 \) is the same as above. \( \blacksquare \)

Lemma 4.2. Let \( 1 \leq p < \infty \) and \( \lambda, \sigma \in \mathbb{R} \). If \( \beta < 0 \) and \( \beta + \lambda + \sigma < 0 \), then there exists a positive constant \( C \) such that

\[
\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}} \, dy \leq C r^{\beta + \lambda + \sigma} \|f\|_{B^\sigma(L_{p,\lambda})} \quad \text{for all } f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \text{ and } r \geq 1,
\]

and

\[
\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}} \, dy \leq C r^{\beta + \lambda + \sigma} \|f\|_{\dot{B}^\sigma(L_{p,\lambda})} \quad \text{for all } f \in \dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n) \text{ and } r > 0.
\]

Proof. We prove only the case \( f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \) and \( r \geq 1 \).

\[
\int_{\mathbb{R}^n \setminus B_r} \frac{|f(y) - f_{B_{2r}}|}{|y|^{n-\beta}} \, dy = \sum_{j=0}^{\infty} \int_{B_{2j+1,r} \setminus B_{2jr}} \frac{|f(y) - f_{B_{2r}}|}{|y|^{n-\beta}} \, dy \\
\lesssim \sum_{j=0}^{\infty} \frac{1}{(2jr)^{n-\beta}} \int_{B_{2j+1,r} \setminus B_{2jr}} |f(y) - f_{B_{2r}}| \, dy \\
\lesssim r^\beta \sum_{j=0}^{\infty} (2^\beta j)^j \int_{B_{2j+1,r}} |f(y) - f_{B_{2r}}| \, dy \\
\leq r^\beta \sum_{j=0}^{\infty} (2^\beta j)^j \left( \int_{B_{2j+1,r}} |f(y) - f_{B_{2j+1,r}}| \, dy + \sum_{k=1}^{j} |f_{B_{2k+1,r}} - f_{B_{2k,r}}| \right) \\
\lesssim r^\beta + \lambda \sum_{j=0}^{\infty} (2^\beta j)^j \left( (2^\lambda j)^{j+1} \|f\|_{L_{p,\lambda}(B_{2j+1,r})} + \sum_{k=1}^{j} (2^\lambda k+1) \|f\|_{L_{p,\lambda}(B_{2k+1,r})} \right)
\]
for any ball $B$.

Therefore, $x$.

Then

\[ I_{\alpha}(f\chi_{2r})(x) + I_{\alpha}(f(1 - \chi_{2r}))(x) = I_{\alpha}(f\chi_{2r})(x) + I_{\alpha}(f(1 - \chi_{2r}))(x). \]

Therefore, $I_{\alpha}f$ is well defined on $\mathbb{R}^n$.

Now, by the boundedness of $I_{\alpha}$ from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ (Theorem 3.6) and (i) of Lemma 3.5, we have

\[ \|I_{\alpha}(f\chi_{2r})\|_{L_{q,\mu}(B_r)} \leq \|I_{\alpha}(f\chi_{2r})\|_{L_{q,\mu}} \lesssim \|f\chi_{2r}\|_{L_{p,\lambda}} \leq \|f\|_{L_{p,\lambda}(B_{6r})} \lesssim r^{\sigma}\|f\|_{B^\sigma(L_{p,\lambda})}. \] (4.1)

Since

\[ |I_{\alpha}(f(1 - \chi_{2r}))(x)| \lesssim \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}} \, dy \]

for $x \in B_r$, using Lemma 4.1, we have

\[ |I_{\alpha}(f(1 - \chi_{2r}))(x)| \lesssim r^{\alpha + \lambda + \sigma}\|f\|_{B^\sigma(L_{p,\lambda})} = r^{\mu + \sigma}\|f\|_{B^\sigma(L_{p,\lambda})}. \]

Then $I_{\alpha}(f(1 - \chi_{2r}))(x)$ is well defined for all $x \in B_r$ and

\[ \|I_{\alpha}(f(1 - \chi_{2r}))\|_{L_{q,\mu}(B_r)} \leq r^{-\mu}\|I_{\alpha}(f(1 - \chi_{2r}))\|_{L^\infty(B_r)} \lesssim r^{\sigma}\|f\|_{B^\sigma(L_{p,\lambda})}, \]

since $\mu < 0$. Therefore, we have

\[ \|I_{\alpha}f\|_{L_{q,\mu}(B_r)} \lesssim r^{\sigma}\|f\|_{B^\sigma(L_{p,\lambda})} \]

for any ball $B_r$. This shows the conclusion.

The proof of the boundedness from $\dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}^\sigma(L_{q,\mu})(\mathbb{R}^n)$ is the same as above. \(\blacksquare\)
Proof of Theorem 2.3. Let \( f \in B^\sigma(L_{p,\lambda})(\mathbb{R}^n) \). We first prove that \( \tilde{I}_\alpha f \) is well defined and that
\[
\| \tilde{I}_\alpha f \|_{L_{q,\mu}(B_r)} \lesssim r^\sigma \| f \|_{B^\sigma(L_{p,\lambda})}
\]
for any ball \( B_r \) with \( r \geq 1 \). Next we prove that \( |(\tilde{I}_\alpha f)|_{B_1} \lesssim \| f \|_{B^\sigma(L_{p,\lambda})} \).

For \( x \in B_r \), let
\[
\tilde{I}_\alpha f(x) = I_\alpha(f\chi_{2r})(x) + J_\alpha(f(1-\chi_{2r}))(x) + C_\alpha(f(\chi_1-\chi_{2r})),
\]
where
\[
J_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right) dy
\]
and
\[
C_\alpha f = \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{n-\alpha}} dy.
\]
In the above, \( I_\alpha(f\chi_{2r}) \) is well defined, since \( f\chi_{2r} \) is in \( L^p(\mathbb{R}^n) \). \( C_\alpha(f(\chi_1-\chi_{2r})) \) is also well defined, since \( (\chi_1-\chi_{2r})/|\cdot|^{n-\alpha} \) is in \( L^{p'}(\mathbb{R}^n) \). Note that \( C_\alpha(f(\chi_1-\chi_{2r})) \) is a constant. Moreover, if \( r = 1 \), then
\[
|C_\alpha(f(\chi_1-\chi_2))| \leq \left\| \frac{\chi_1-\chi_2}{|\cdot|^{n-\alpha}} \right\|_{L^{p'}} \| f \|_{L^p(B_2)} \lesssim \| f \|_{L^p(B_2)} \lesssim \| f \|_{L^p(B_{r/2})},
\]
We show later that \( J_\alpha(f(1-\chi_{2r}))(x) \) is well defined for all \( x \in B_r \). Then \( \tilde{I}_\alpha f \) is well defined on \( \mathbb{R}^n \) by the same reason as in the proof of Theorem 2.1.

Now, in the same way as (4.1), by the boundedness of \( \tilde{I}_\alpha \) from \( L_{p,\lambda}(\mathbb{R}^n) \) to \( L_{q,\mu}(\mathbb{R}^n) \) (Theorem 3.7) and (i) of Lemma 3.5, we have
\[
\| I_\alpha(f\chi_{2r}) \|_{L_{q,\mu}(B_r)} + |(I_\alpha(f\chi_{2r}))|_{B_1} \lesssim r^\sigma \| f \|_{B^\sigma(L_{p,\lambda})}.
\]
Using the inequality
\[
\left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right| \lesssim \frac{|x|}{|y|^{n-\alpha+1}} \leq \frac{r}{|y|^{n-\alpha+1}}
\]
for \( x \in B_r \) and \( y \in \mathbb{R}^n \setminus B_{2r} \), we have
\[
|J_\alpha(f(1-\chi_{2r}))(x)| \leq r \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha+1}} dy.
\]
By Lemma 4.1 we have
\[
|J_\alpha(f(1-\chi_{2r}))(x)| \lesssim r^{\alpha+\lambda+\sigma} \| f \|_{B^\sigma(L_{p,\lambda})} = r^{\mu+\sigma} \| f \|_{B^\sigma(L_{p,\lambda})}.
\]
Then \( J_\alpha(f(1-\chi_{2r}))(x) \) is well defined for all \( x \in B_r \). If \( \mu \leq 0 \), then we have
\[
\| J_\alpha(f(1-\chi_{2r})) \|_{L_{q,\mu}(B_r)} \lesssim \| J_\alpha(f(1-\chi_{2r})) \|_{L_{q,\mu}(B_r)} \leq r^{-\mu} \| f \|_{L^\infty(B_r)} \lesssim r^\sigma \| f \|_{B^\sigma(L_{p,\lambda})}.
\]
If \( \mu > 0 \), then, for any \( x, z \in B_r \), we have by Lemma 4.1
\[
|J_\alpha(f(1-\chi_{2r}))(x) - J_\alpha(f(1-\chi_{2r}))(z)| = \left| \int_{\mathbb{R}^n \setminus B_{2r}} f(y) \left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|z-y|^{n-\alpha}} \right) dy \right|
\]
\[ \lesssim |x - z| \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha + 1}} \, dy \]
\[ \lesssim |x - z| r^{\alpha - 1 + \lambda + \sigma} \|f\|_{B^\sigma(L_{p,\lambda})} = |x - z| r^{\mu - 1 + \sigma} \|f\|_{B^\sigma(L_{p,\lambda})} \]
and
\[ |J_\alpha(f(1 - \chi_{2r}))(x) - J_\alpha(f(1 - \chi_{2r}))(z)| \lesssim \left( \frac{|x - z|}{r} \right)^{1-\mu} r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})} \]
\[ \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}. \]

By Theorem 3.2 we have
\[ \|J_\alpha(f(1 - \chi_{2r}))\|_{L^q,\mu(B_r)} \sim \|J_\alpha(f(1 - \chi_{2r}))\|_{\text{Lip}_\mu(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})}. \]

Therefore,
\[ \|\tilde{I}_\alpha f\|_{L^q,\mu(B_r)} = \|I_\alpha(f \chi_2) + J_\alpha(f(1 - \chi_{2r})) + C_\alpha(f(\chi_1 - \chi_{2r}))\|_{L^q,\mu(B_r)} \]
\[ \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})} \]
for any ball \( B_r \), that is,
\[ \|\tilde{I}_\alpha f\|_{B^\sigma(L^q,\mu)} \lesssim \|f\|_{B^\sigma(L_{p,\lambda})}. \]

Finally, by (4.4), (4.5) and (4.6) with \( r = 1 \), we have
\[ \|\tilde{I}_\alpha f\|_{B^\sigma(L^q,\mu)} \lesssim \|f\|_{B^\sigma(L_{p,\lambda})}. \]

The proof of the boundedness from \( \dot{B}^\sigma(L_{p,\lambda})(\mathbb{R}^n) \) to \( \dot{B}^\sigma(L^q,\mu)(\mathbb{R}^n) \) is the same as above. 

**Proof of Theorem 2.6.** Let \( f \in B^\sigma(L^q,\mu)(\mathbb{R}^n) \). We first prove that \( \tilde{I}_\alpha f \) is well defined and that
\[ \|\tilde{I}_\alpha f\|_{L^q,\mu(B_r)} \lesssim r^\sigma \|f\|_{B^\sigma(L_{p,\lambda})} \]
for any ball \( B_r \) with \( r \geq 1 \). Next we prove that \( |(\tilde{I}_\alpha f)_{B_1}| \lesssim \|f\|_{B^\sigma(L_{p,\lambda})} + |f_{B_1}|. \)

Let \( h \) be defined by (3.2). For \( x \in B_r \), let \( \tilde{f} = f - f_{B_{4r}} \) and
\[ \tilde{I}_\alpha f(x) = \tilde{I}_\alpha \tilde{f}(x) + \tilde{I}_\alpha(f_{B_{4r}})(x) \]
\[ = I_\alpha(\tilde{f} h_{2r})(x) + J_\alpha(\tilde{f}(1 - h_{2r}))(x) + C_\alpha(\tilde{f}(\chi_1 - h_{2r})) + f_{B_{4r}}(\tilde{I}_\alpha 1)(x), \]
where \( J_\alpha \) and \( C_\alpha \) are defined by (4.2) and (4.3), respectively. By Remark 2.1, \( \tilde{I}_\alpha 1 \) is a constant function. By the same observation as in the proof of Theorem 2.3, we see that \( I_\alpha(\tilde{f} h_{2r}) \) and \( C_\alpha(\tilde{f}(\chi_1 - h_{2r})) \) are well defined and, if \( r = 1 \), then by (2.1),
\[ \|C_\alpha(\tilde{f}(\chi_1 - h_{2r}))\| \lesssim \|\chi_1 - h_{2r}\|_{L^p(B_{2r})} \]
\[ \lesssim \|\tilde{f}\|_{L^p(B_{2r})} \lesssim \|f\|_{B^\sigma(L_{p,\lambda})}. \]

We show later that \( J_\alpha(\tilde{f}(1 - h_{2r}))(x) \) is well defined for all \( x \in B_r \). Then \( \tilde{I}_\alpha f \) is well defined on \( \mathbb{R}^n \) by the same reason as the proof of Theorem 2.1.

Now, by the boundedness of \( \tilde{I}_\alpha \) from \( L^q,\mu(\mathbb{R}^n)/\mathcal{C} \) to \( L^q,\mu(\mathbb{R}^n)/\mathcal{C} \) (Theorem 3.8) and (ii) of Lemma 3.5, we have
\[ \|I_\alpha(\tilde{f} h_{2r})\|_{L^q,\mu(B_r)} \leq \|I_\alpha(\tilde{f} h_{2r})\|_{L^q,\mu} \lesssim \tilde{f} h_{2r} \|_{L^q,\mu}. \]
Then we have the conclusion. By (3.1) we have
\[ \|\tilde{f}\|_{L_{p,\lambda}(B_{r})} = \|f\|_{L_{p,\lambda}(B_{6r})} \lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}. \] (4.8)

In the same way, by the boundedness of \( \tilde{f} \) from \( L_{p,\lambda}(\mathbb{R}^{n}) \) to \( L_{q,\mu}(\mathbb{R}^{n}) \) and Lemma 3.5, we have
\[
\|I_{\alpha}(\tilde{f}h_{2r})\|_{L_{q,\mu}(B_{r})} + \|I_{\alpha}(\tilde{f}h_{2r})B_{1}\| \leq \|I_{\alpha}(\tilde{f}h_{2r})\|_{L_{q,\mu}} + \|I_{\alpha}(\tilde{f}h_{2r})B_{1}\| \\
\lesssim \|\tilde{f}h_{2r}\|_{L_{p,\lambda}} + \|\tilde{f}h_{2r}B_{1}\| \lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})} + \|\tilde{f}h_{2r}B_{1}\|. \] (4.9)

By Lemma 4.2 we have
\[
|J_{\alpha}(\tilde{f}(1 - h_{2r}))(x)| \leq r \int_{\mathbb{R}^{n}\setminus B_{r}} \frac{|f(y) - f_{B_{r}}|}{|y|^{n-\alpha+1}} \, dy \\
\lesssim r^{\alpha+\lambda+\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})} = r^{\mu+\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}. \] (4.10)

Then \( J_{\alpha}(\tilde{f}(1 - h_{2r}))(x) \) is well defined for all \( x \in B_{r} \). For each case \( \mu \leq 0 \) or \( \mu > 0 \), using the same way as in the proof of Theorem 2.3, we have
\[
\|J_{\alpha}(\tilde{f}(1 - h_{2r}))\|_{L_{q,\mu}(B_{r})} \lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}.
\]

Therefore,
\[
\|\tilde{f}\|_{L_{q,\mu}(B_{r})} = \|I_{\alpha}(\tilde{f}h_{2r}) + J_{\alpha}(\tilde{f}(1 - h_{2r})) + C_{\alpha}(\tilde{f}(\chi_{1} - h_{2r})) + f_{B_{4r}}(\tilde{f}1)\|_{L_{q,\mu}(B_{r})} \\
\lesssim r^{\sigma}\|f\|_{B^{\sigma}(L_{p,\lambda})}
\]
for any ball \( B_{r} \), and
\[
\|\tilde{f}\|_{B^{\sigma}(L_{p,\lambda})} \lesssim \|f\|_{B^{\sigma}(L_{p,\lambda})}.
\]

Finally, we estimate each term of the right hand side in the following:
\[
|(\tilde{f} - f)_{B_{1}}| \leq |(I_{\alpha}(\tilde{f}h_{2r}))_{B_{1}}| + |(J_{\alpha}(\tilde{f}(1 - h_{2r})))_{B_{1}}| + |C_{\alpha}(\tilde{f}(\chi_{1} - h_{2r}))| + |f_{B_{4}}(\tilde{f}1)|.
\]
Taking \( r = 1 \) in (4.9) and (4.10), we have
\[
|(I_{\alpha}(\tilde{f}h_{2r}))_{B_{1}}| \lesssim \|f\|_{B^{\sigma}(L_{p,\lambda})} + \|\tilde{f}h_{2r}B_{1}\| = \|f\|_{B^{\sigma}(L_{p,\lambda})} + \|f_{B_{1}} - f_{B_{4}}\|
\]
and
\[
|(J_{\alpha}(\tilde{f}(1 - h_{2r}))B_{1}| \leq \|f\|_{B^{\sigma}(L_{p,\lambda})},
\]
respectively. By (3.1) we have \( \|f_{B_{1}} - f_{B_{4}}\| \lesssim \|f\|_{L_{p,\lambda}(B_{4})} \lesssim \|f\|_{B^{\sigma}(L_{p,\lambda})} \). Using these estimates and (4.7), we obtain
\[
|(\tilde{f} - f)_{B_{1}}| \lesssim \|f\|_{B^{\sigma}(L_{p,\lambda})} + \|f_{B_{1}}|.
\]

Then we have the conclusion.

The proof of the boundedness from \( \dot{B}^{\sigma}(L_{p,\lambda})(\mathbb{R}^{n})/\mathcal{C} \) to \( \dot{B}^{\sigma}(L_{q,\mu})(\mathbb{R}^{n})/\mathcal{C} \) and from \( \dot{B}^{\sigma}(L_{p,\lambda})(\mathbb{R}^{n}) \) to \( \dot{B}^{\sigma}(L_{q,\mu})(\mathbb{R}^{n}) \) is the same as above. ■

Acknowledgments. Research of E. Nakai was partially supported by Grant-in-Aid for Scientific Research (C), No. 20540167, Japan Society for the Promotion of Science.

References


