CONSTANTS OF STRONG UNIQUENESS OF MINIMAL NORM-ONE PROJECTIONS

AGNIESZKA MICEK

Department of Mathematics, Jagiellonian University
Łojasiewicza 6, 30-348 Kraków, Poland
E-mail: Agnieszka.Micek@im.uj.edu.pl

Abstract. In this paper we calculate the constants of strong uniqueness of minimal norm-one projections on subspaces of codimension \( k \) in the space \( l^{(n)}_\infty \). This generalizes a main result of W. Odyniec and M. P. Prophet [J. Approx. Theory 145 (2007), 111–121]. We applied in our proof Kolmogorov’s type theorem (see A. Wójcik [Approximation and Function Spaces (Gdańsk, 1979), PWN, Warszawa / North-Holland, Amsterdam, 1981, 854–866]) for strongly unique best approximation.

1. Introduction. Let \( X \) be a normed space and let \( Y \subset X \) be a nonempty subset. An element \( y \in Y \) is called a strongly unique best approximation (briefly SUBA) to \( x \in X \) if and only if there exists \( r > 0 \) such that for every \( v \in Y \)
\[
\|x - v\| \geq \|x - y\| + r\|v - y\|.
\]
(1)
The largest possible constant \( r \) satisfying the above inequality is called the strong unicity constant.

The concept of strong unicity was introduced by Newman and Shapiro in 1963 (see [20]). The main classical example of this notion are spaces \( P_n \) of polynomials of degree not greater than \( n \) treated as subspaces of \( C[0,1] \). More precisely, for any \( f \in C[0,1] \) there exists \( r_n(f) > 0 \) such that
\[
\|f - p\| \geq \|f - p_n(f)\| + r_n(f)\|p - p_n(f)\|,
\]
where \( p_n(f) \) denotes the best approximation of \( f \) in \( P_n \).

Generally, this property is stronger than uniqueness of the best approximation. Strong unicity implies local Lipschitz continuity of the best approximation operator (see e.g. [6]).

2010 Mathematics Subject Classification: Primary 41A35, Secondary 41A44.
Key words and phrases: minimal projection, strong unicity, strongly unique minimal projection constant, SUP-constant.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc92-0-18 [265] © Instytut Matematyczny PAN, 2011
Another important application of this notion is the estimate of the error of the Remez algorithm ([9, 25, 26]).

Strong unicity has been studied by many authors and there are a large number of papers devoted to it, for example [1, 2, 3, 5, 8, 15, 18, 19, 21, 22, 23, 29, 31]. Also many authors have described the asymptotic behaviour of \( r_n(f) \) as \( n \) tends to infinity for a given function \( f \in C[0,1] \) (see e.g. [10, 12, 24]). The Kolmogorov criterion for strongly unique best approximation has been introduced by Bartelt and McLaughlin in [3].

Let us denote by \( L(X,Y) \) the set of all linear, continuous operators from \( X \) into \( Y \). We use the notation \( L(X) \) as an abbreviation for the space \( L(X,X) \). By \( P(X,Y) \) let us denote the set of all linear, continuous projections going from \( X \) onto \( Y \), i.e.,

\[
P(X,Y) = \{ P \in L(X,Y) : Py = y \text{ for any } y \in Y \}.
\]

In the case of projections the notion of strong unicity reduces to

**Definition 1.1.** Let \( P_0 \in P(X,Y) \). Then \( P_0 \) is called a **strongly unique minimal projection** (we will write a SUM-projection for brevity) if and only if there exists \( r > 0 \) such that for any \( P \in P(X,Y) \)

\[
\|P\| \geq \|P_0\| + r\|P - P_0\|.
\]  

(2)

The largest possible constant for which the inequality in (2) holds is called a **strongly unique projection constant** (briefly SUP-constant).

It is clear that any SUM-projection is the unique minimal projection in \( P(X,Y) \).

Now, let us introduce some notation. By \( S_X \) we denote the unit sphere in a normed space \( X \) and by \( \text{ext} S_X \) the set of its extreme points. The symbol \( e_s, s = 1, \ldots, n \), stands for the linear functional on \( l_\infty^{(n)} \) such that \( e_s(x) = x_s \) for \( x \in l_\infty^{(n)} \). Let \( X \) be a Banach space and \( Y \subset X \) be its closed subspace. Set

\[
E(x) = \{ f \in \text{ext} S_{X^*} : f(x) = \|x\| \}
\]

and

\[
\mathcal{L}_Y = \{ L \in L(X,Y) : L|_Y = 0 \}.
\]

**Lemma 1.2** (See e.g. [4]). Assume \( X \) is a normed space and let \( Y \) be its subspace of codimension \( k \), \( Y = \bigcap_{i=1}^{k} \ker f^i \), where \( f^i \in X^* \) are linearly independent. Then there exist \( y^1, \ldots, y^k \in X \) satisfying \( f^i(y^j) = \delta_{ij} \) for \( i, j = 1, \ldots, k \) such that

\[
Px = x - \sum_{i=1}^{k} f^i(x)y^i \quad \text{for} \quad x \in X.
\]

**Theorem 1.3** (See e.g. [27], [7, Prop. 2.1, p. 55]). Let \( X \) be a finite-dimensional normed space. Then

\[
\text{ext} S_{\mathcal{L}^*(X)} = \text{ext} S_{X^*} \otimes \text{ext} S_X,
\]

where \( (x^* \otimes x)(L) = x^*(Lx) \) for \( x \in X, x^* \in X^* \) and \( L \in L(X) \).

Now we present Kolmogorov’s type criterion, concerning strong unicity.

**Theorem 1.4** (See e.g. [30, Th. 2.1, p. 855]). Let \( X \) be a normed space and let \( Y \subset X \) be one of its subspaces. Assume that \( x \in X \setminus Y \). Then \( y_0 \in Y \) is a strongly unique best
approximation to $x$ with a constant $r > 0$ if and only if for any $y \in Y$ there exists an $f \in E(x - y_0)$ such that $\text{Re} f(y) \leq -r\|y\|$.

As a consequence of Theorem 1.4 we obtain the following

**Lemma 1.5** (See [15, Th. 1.3, p. 84]). Let $V \subset X$, $\dim V = n$, $x_0 \in X \setminus V$, $v_0 \in V$. If $v_0$ is a SUBA to $x_0$ in $V$, then

$$r = \inf_{v \in S_V} \left( \sup_{g \in E(x_0 - v_0)} g(v) \right)$$

is the strong unicity constant.

Lemma 1.5 permits us to calculate the strong unicity constants of some subspaces of $\mathcal{L}(l_{\infty}^{(n)})$.

**2. The main result.** Let $n > k \geq 3$. Define

$$f^{(1)} = (f_1^1, \ldots, f_1^{n-k}, f_1^{n-k+1}, 0, \ldots, 0),$$

$$f^{(2)} = (f_2^2, \ldots, f_2^{n-k}, 0, f_2^{n-k+2}, 0, \ldots, 0),$$

$$\vdots$$

$$f^{(k)} = (f_k^1, \ldots, f_k^{n-k}, 0, \ldots, 0, f_n^k)$$

under the constraints

$$0 < f_i^j < \frac{1}{2} \text{ for } i = 1, \ldots, n-k, j = 1, \ldots, k,$$

$$\sum_{i=1}^{n-k} f_i^j + f_{n-k+j}^j = 1, \quad f_{n-k+j}^j \geq \frac{1}{2} \text{ for } j = 1, \ldots, k. \quad (5)$$

Let

$$H = \bigcap_{j=1}^{k} \ker f^{(j)}. \quad (6)$$

The aim of this paper is to calculate the strongly unique minimal projection constant of the space $\mathcal{P}(l_{\infty}^{(n)}, H)$.

W. Odyniec and M. P. Prophet have given a lower bound for the above strongly unique minimal projection constant in [22]. Using Lemma 1.2 and (8) they have directly calculated all norms occurring in the formula (2). Next, with the help of arduous estimations they have proved that the SUP-constant $s_0$ of the space $\mathcal{P}(l_{\infty}^{(n)}, H)$ satisfies the inequality

$$\min\{s_0^1, \ldots, s_0^{n-k}\} \leq s_0 < 1, \quad (7)$$

where

$$s_0^i = \min\left\{ \frac{f_1^{n-k+1} + f_1^1}{f_1^{n-k+1} + f_1^1}, \frac{f_1^{n-k+2} - f_1^2}{f_1^{n-k+2} + f_1^2}, \ldots, \frac{f_n^k - f_1^k}{f_n^k + f_1^k} \right\}, \quad i = 1, \ldots, n-k.$$

In our paper we improve their result and we calculate the SUP-constant of the space $\mathcal{P}(l_{\infty}^{(n)}, H)$. We use the Kolmogorov criterion for the strong unicity.
It is well known (see [14, Remark 1.12], [21, Th. III.3.1]) that there exists exactly one minimal projection \( P_0 \in \mathcal{P}(l_\infty^{(n)}, H) \), \( \|P_0\| = 1 \) and \( P_0 \) has the form

\[
P_0(\cdot) = \text{Id} - \sum_{j=1}^{k} f^{(j)}(\cdot)y^{(j)},
\]

where

\[
y^{(j)} = (0, \ldots, 0_{n-k+j-1}, 1/f_{n-k+j}^{j}, 0_{n-k+j+1}, \ldots, 0_n), \quad j = 1, \ldots, k.
\]

(8)

Now we find the set \( E(P_0) \) (see (3)).

**Lemma 2.1.** Let us define for \( s = 1, \ldots, n-k \) the sets

\[
A_s = \{(x_1, \ldots, x_{s-1}, x_s, x_{s+1}, \ldots, x_n) : x_i \in \{-1, 1\} \text{ for } i = 1, \ldots, n, \ i \neq s\}
\]

and the set

\[
B = \{(-1, \ldots, -1_{n-k}, x_{n-k+1}, \ldots, x_n) : x_i \in \{-1, 1\} \text{ for } i = n-k+1, \ldots, n\}.
\]

Let \( P_0 \in \mathcal{P}(l_\infty^{(n)}, H) \) be the minimal projection. Then the set \( E(P_0) \) has the form

\[
E(P_0) = F \cup G,
\]

\[
F = \{e^s \otimes x : x \in A_s, s = 1, \ldots, n-k\},
\]

\[
G = \{e^s \otimes x : x \in B, s \text{ satisfying } f_s^{s-(n-k)} = \frac{1}{2}\},
\]

where \( e^s \in (l_\infty^{(n)})^* \) is such that \( e^s(x) = x_s \) for \( x \in l_\infty^{(n)}, s = 1, \ldots, n \).

**Proof.** By Theorem 1.3

\[
E(P_0) \subseteq \text{ext } l_1^{(n)} \otimes \text{ext } l_\infty^{(n)} \subseteq \{e^s \otimes x : x \in \text{ext } l_\infty^{(n)}\}.
\]

Notice that by Lemma 1.2

\[
(e^s \otimes x)(P_0) = x_s - \sum_{j=1}^{k} f^{(j)}(x)y_s^{(j)}.
\]

(9)

The equality \((e^s \otimes x)(P_0) = \|P_0\| = 1\) for \( s \in \{1, \ldots, n\}\) is true if and only if

\[
x_s - \sum_{j=1}^{k} f^{(j)}(x)y_s^{(j)} = 1.
\]

(10)

Since \( y_s^{(j)} = 0 \) for \( j = 1, \ldots, k, s = 1, \ldots, n-k \) (see (8)), the equality (10) implies \( x_s = 1 \) for \( s \in \{1, \ldots, n-k\} \). Hence we get the form of the set \( F \).

Let us consider now \( s \in \{n-k+1, \ldots, n\}\). By (8) and (9)

\[
(e^s \otimes x)(P_0) = x_s - f^{(s-(n-k))}(x)y_s^{(s-(n-k))} = x_s - \frac{f^{(s-(n-k))}(x)}{f_s^{s-(n-k)}}.
\]

Hence \( e^s \otimes x \in E(P_0) \) if and only if

\[
x_s - 1 = \frac{f^{(s-(n-k))}(x)}{f_s^{s-(n-k)}} \quad \text{and} \quad x \in \text{ext } l_\infty^{(n)}.
\]

(11)

It is easy to see that if \( f_s^{s-(n-k)} > 1/2 \) then \( 0 < \frac{|f^{(s-(n-k))}(x)|}{f_s^{s-(n-k)}} < 2 \), so (11) cannot hold.
Assume now that $s \in \{n - k + 1, \ldots, n\}$ satisfies $f_s^{s-(n-k)} = \frac{1}{2}$. Then the equality (11) is equivalent to the following alternative

$$f^{(s-(n-k))}(x) = -1, \quad x_s = -1, \quad \text{and} \quad x_i \in \{-1, 1\} \quad \text{for} \quad i = 1, \ldots, n$$

or

$$f^{(s-(n-k))}(x) = 0, \quad x_s = 1, \quad \text{and} \quad x_i \in \{-1, 1\} \quad \text{for} \quad i = 1, \ldots, n.$$  

By (5)

$$x_i = -1 \quad \text{for} \quad i = 1, \ldots, n - k, \quad x_s = -1, \quad x_i \in \{-1, 1\} \quad \text{for} \quad i = n - k + 1, \ldots, n$$

or

$$x_i = -1 \quad \text{for} \quad i = 1, \ldots, n - k, \quad x_s = 1, \quad x_i \in \{-1, 1\} \quad \text{for} \quad i = n - k + 1, \ldots, n.$$

**Remark 2.2.** Let $L \in \mathcal{L}_H$ (see (4)). Then

$$L(\cdot) = \sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)} ,$$

where $z^{(j)} \in H$ for $j = 1, \ldots, k$. Since $H = \bigcap_{j=1}^{k} \ker f^{(j)}$, we additionally get that

$$z^{(j)}_{n-k+i} = \frac{(-n^k_s z^{(j)}_s)}{f^{(j)}_s} / f^{(j)}_{n-k+i} \quad \text{for all} \quad i, j = 1, \ldots, k.$$  

**Lemma 2.3.** Let $L \in \mathcal{L}_H$, $L(\cdot) = \sum_{j=1}^{k} f^{(j)}(\cdot)z^{(j)}$, where $z^{(j)} \in H$ for all $j = 1, \ldots, k$. Then

$$\|L\| = \max_{s=1, \ldots, n-k} \left( \sum_{i=1}^{k} \left| \sum_{p=1}^{n-k} f^{(i)} f^{(i)}_{p+s} \right| + \sum_{i=1}^{k} f^{(i)}_{n-k+i} \left| z^{(i)} \right| \right).$$

**Proof.** Let $x \in l^{(n)}$. By Remark 2.2 we get

$$L(x) = \sum_{i=1}^{k} f^{(i)}(x)z^{(i)} = \sum_{i=1}^{k} \left( z^{(i)}_{1}, \ldots, z^{(i)}_{n-k}, \frac{\sum_{p=1}^{n-k} f^{(i)} f^{(i)}_{p+s}}{f^{(i)}_{n-k+1}}, \ldots, \frac{\sum_{p=1}^{n-k} f^{(i)} f^{(i)}_{p+s}}{f^{(i)}_{n-k+1}} \right).$$

Let us define the constant $A := \max_{i=1, \ldots, n-k} \|e^i \circ L\|$. Now we show that $\|e^s \circ L\| \leq A$ for any $s \in \{n - k + 1, \ldots, n\}$. Let $s \in \{n - k + 1, \ldots, n\}$ and let $x \in l^{(n)}$, $\|x\| = 1$. We find

$$\|(e^s \circ L)(x)\| = \left| -\sum_{i=1}^{k} f^{(i)}(x) \sum_{p=1}^{n-k} \frac{f^{s-(n-k)} f^{(i)}_{p+s}}{f^{s-(n-k)}} z^{(i)}_{s} \right|$$

$$\leq \sum_{p=1}^{n-k} f^{s-(n-k)} \left| \sum_{i=1}^{k} f^{(i)}(x) z^{(i)}_{s} \right| = \sum_{p=1}^{n-k} f^{s-(n-k)} \left| (e^p \circ L)(x) \right|$$

$$\leq \sum_{p=1}^{n-k} f^{s-(n-k)} \|e^p \circ L\| \leq \sum_{p=1}^{n-k} f^{s-(n-k)} \cdot A = \frac{1 - f^{s-(n-k)}}{f^{s-(n-k)}} \cdot A.$$  

Taking into account that $f^{s-(n-k)} \geq 1/2$ for $s = n - k + 1, \ldots, n$ we find

$$\|(e^s \circ L)\| \leq \max_{i=1, \ldots, n-k} \|e^i \circ L\| \quad \text{for} \quad s = n - k + 1, \ldots, n.$$
Hence
\[ \|L\| = \max_{i=1,\ldots,n-k} \|e^i \circ L\|. \tag{13} \]

To calculate the norm of the operator \( L \) we find \( \|e^s \circ L\| \) for \( s = 1,\ldots,n-k \). Let \( x \in l_\infty^{(n)} \) and \( s \in \{1,\ldots,n-k\} \). Then
\[ |(e^s \circ L)(x)| = \left| \sum_{i=1}^{k} f^i(x)z_s^i \right| = \sum_{i=1}^{k} \left( \sum_{p=1}^{n-k} f^i_p x_p + f^i_{n-k+i} x_{n-k+i} \right) \cdot z_s^i \]
\[ = \left| \sum_{p=1}^{n-k} \left( \sum_{i=1}^{k} f^i_p z_s^i \right) x_p + \sum_{i=1}^{k} (f^i_{n-k+i} z_s^i) x_{n-k+i} \right|. \]

By the above formula for \( s \in \{1,\ldots,n-k\} \)
\[ \|e^s \circ L\| = \max_{x \in l_\infty^{(n)}} \left| \sum_{p=1}^{n-k} \left( \sum_{i=1}^{k} f^i_p z_s^i \right) x_p + \sum_{i=1}^{k} (f^i_{n-k+i} z_s^i) x_{n-k+i} \right|. \tag{14} \]

Let \( s_0 \in \{1,\ldots,n-k\} \) be such that the maximum (13) is attained on the coordinate \( s_0 \). Then by (14)
\[ \|L\| = \max_{x \in l_\infty^{(n)}} \left| \sum_{p=1}^{n-k} \left( \sum_{i=1}^{k} f^i_p z_{s_0}^i \right) x_p + \sum_{i=1}^{k} (f^i_{n-k+i} z_{s_0}^i) x_{n-k+i} \right|. \]

Taking
\[ x_p = \text{sgn} \left( \sum_{i=1}^{k} f^i_p z_{s_0}^i \right) \text{ for } p = 1,\ldots,n-k \text{ and } x_{n-k+i} = \text{sgn}(z_{s_0}^i) \text{ for } i = 1,\ldots,k \]
we get
\[ \|L\| = \sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f^i_p z_{s_0}^i \right| + \sum_{i=1}^{k} (f^i_{n-k+i} z_{s_0}^i). \]

This completes the proof. \( \blacksquare \)

**Lemma 2.4.** Let
\[ U = \left\{ (z^{(1)},\ldots,z^{(k)}) \in (l_\infty^{(n)})^k : z^{(j)} \in H, j = 1,\ldots,k, \right\} \]
\[ \max_{s=1,\ldots,n-k} \left( \sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f^i_p z_{s}^i \right| + \sum_{i=1}^{k} (f^i_{n-k+i} z_{s}^i) \right) = 1. \tag{15} \]

We define
\[ M^s_1(z^{(1)},\ldots,z^{(k)}) = \sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f^i_p z_{s}^i \right| + \sum_{i=1}^{k} (f^i_{n-k+i} z_{s}^i) \tag{16} \]

for \( (z^{(1)},\ldots,z^{(k)}) \in U, s = 1,\ldots,n-k \),
\[ M_1(z^{(1)},\ldots,z^{(k)}) = \max_{s=1,\ldots,n-k} M^s_1(z^{(1)},\ldots,z^{(k)}) \text{ for } (z^{(1)},\ldots,z^{(k)}) \in U. \]
and
\begin{equation}
M_2^s(z^{(1)}, \ldots, z^{(k)}) = \sum_{j=1}^{k} (-1 + f_n^{j}) z^{(j)} + \sum_{j=1}^{k} f_n^{j} |z^{(j)}| \quad (17)
\end{equation}

\text{for} \quad (z^{(1)}, \ldots, z^{(k)}) \in U, \quad s \in \{1, \ldots, n-k\} \text{ satisfying } f_n^{s-(n-k)} = 1/2,

\begin{align*}
M_2(z^{(1)}, \ldots, z^{(k)}) &= \max_{s=n-k+1, \ldots, n} M_2^s(z^{(1)}, \ldots, z^{(k)}) \text{ for } (z^{(1)}, \ldots, z^{(k)}) \in U.
\end{align*}

Then the SUP-constant of \( \mathcal{P}(l_1^{\infty}, H) \) is equal to
\begin{equation}
r = \min_{(z^{(1)}, \ldots, z^{(k)}) \in U} \max\{M_1(z^{(1)}, \ldots, z^{(k)}), M_2(z^{(1)}, \ldots, z^{(k)})\}. \quad (18)
\end{equation}

\textbf{Proof.} Let \( P_0 \in \mathcal{P}(l_1^{\infty}, H) \) be a minimal projection. Then \( 0 \in \mathcal{L}_H \) is a best approximation to \( P_0 \) in \( \mathcal{L}_H \). Assume that \( L \in \mathcal{L}_H \) and \( \|L\| = 1 \), i.e.,
\begin{equation}
L(\cdot) = \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)} \quad \text{and} \quad \max_{s=1, \ldots, n-k} \left( \sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f_p^{i} z^{(i)} \right| + \sum_{i=1}^{k} f_n^{j} |z^{(j)}| \right) = 1
\end{equation}

(see Lemma 2.3). Applying Lemma 1.5, Remark 2.2 and Lemma 2.1 we get
\begin{equation}
r = \min_{L \in \mathcal{L}_H} \left( \max_{g \in \mathcal{E}(P_0)} g(L) = \min_{\|L\| = 1} \left[ \max_{(z^{(1)}, \ldots, z^{(k)}) \in U} \left( \max_{s=1, \ldots, n-k} \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)} \right) \right) \right] \right). \quad (19)
\end{equation}

Our proof will be completed if we calculate the quantities
\begin{align*}
\max_{x \in A_s} (e^s \otimes x) \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)} \right) &\quad \text{for } s = 1, \ldots, n-k, \\
\max_{x \in B} (e^s \otimes x) \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)} \right) &\quad \text{for } s = n-k+1, \ldots, n \quad \text{satisfying } f_n^{s-(n-k)} = 1/2.
\end{align*}

For \( s \in \{1, \ldots, n-k\} \) and \( x \in A_s \) we obtain
\begin{align*}
(e^s \otimes x) \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)} \right) &= \sum_{j=1}^{k} f_j^j(x) z^{(j)} = \sum_{j=1}^{k} \left( \sum_{p=1}^{n-k} f_p^j x_p z^{(j)} + f_n^{j} x_{n-k+j} z^{(j)} \right) \\
&= \sum_{p=1}^{n-k} \left( \sum_{j=1}^{k} f_p^j z^{(j)} \right) x_p + \sum_{j=1}^{k} f_n^{j} z^{(j)} x_{n-k+j}.
\end{align*}

Taking \( x_p = \text{sgn}(\sum_{i=1}^{k} f_p^i z^{(j)}) \) for \( p \in \{1, \ldots, n-k\} \setminus \{s\} \) and \( x_{n-k+j} = \text{sgn}(z^{(j)}_{n-k+j}) \) for \( j = 1, \ldots, k \) we get
\begin{align*}
\max_{x \in A_s} (e^s \otimes x) \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z^{(j)} \right) &= \sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f_p^i z^{(j)} \right| + \sum_{i=1}^{k} f_n^{i} z^{(j)} + \sum_{i=1}^{k} f_s^{i} z^{(j)}.
\end{align*}
Let us denote 
\[ M_1^s(z^{(1)}, \ldots, z^{(k)}) = \sum_{p=1, p \neq s}^{n-k} \left| \sum_{i=1}^{k} f_p^i z_{s}^{(j)} \right| + \sum_{i=1}^{k} f_{n-k+i}^i z_{s}^{(j)} + \sum_{i=1}^{k} f_i^i z_{s}^{(j)}. \]

Let \( s \in \{ n-k+1, \ldots, n \} \) be such that \( f_s^{s-(n-k)} = 1/2 \) and let \( x \in B \). Then we calculate
\[
(e^s \otimes x) \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z_{s}^{(j)} \right) = \sum_{j=1}^{k} f_j^j(x) z_{s}^{(j)} = \sum_{j=1}^{k} \left( (\sum_{p=1}^{n-k} f_p^j) z_{s}^{(j)} + f_{n-k+j}^j x_{n-k+j} z_{s}^{(j)} \right)
\]
\[ = \sum_{j=1}^{k} (-1 + f_{n-k+j}^j) z_{s}^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j x_{n-k+j} z_{s}^{(j)}. \]

Taking \( x_{n-k+j} = \text{sgn}(z_{s}^{(j)}) \) for \( j = 1, \ldots, k \) we get
\[ \max_{x \in B} (e^s \otimes x) \left( \sum_{j=1}^{k} f^{(j)}(\cdot) z_{s}^{(j)} \right) = \sum_{j=1}^{k} (-1 + f_{n-k+j}^j) z_{s}^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j z_{s}^{(j)}, \]

hence
\[ M_2^s(z^{(1)}, \ldots, z^{(k)}) = \sum_{j=1}^{k} (-1 + f_{n-k+j}^j) z_{s}^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^j z_{s}^{(j)}. \]

Consequently, by (19) we obtain our result. \( \blacksquare \)

**Theorem 2.5.** The SUP-constant of the space \( \mathcal{P}(l_1^{(n)}, H) \) satisfies the inequality
\[ r \leq 1 - 2 \max_{j=1, \ldots, k} f_j^j. \]  \( \quad (20) \)

**Proof.** Let \( i_0 \in \{ 1, \ldots, n-k \} \) and \( j_0 \in \{ 1, \ldots, k \} \) be such that
\[ f_{i_0}^{j_0} = \max_{j=1, \ldots, k} f_j^j. \]  \( \quad (21) \)

Let us define
\[ z^{(j_0)} = (0_1, \ldots, 0_{i_0-1}, -1_{i_0}, 0_{i_0+1}, \ldots, 0_{n-k}, \frac{f_{i_0}^1}{f_{n-k+1}^1}, \ldots, \frac{f_{i_0}^k}{f_n^k}) \]
and
\[ z^{(j)} = (0_1, \ldots, 0_n) \quad \text{for} \quad j \neq j_0. \]

Now we show that \( (z^{(1)}, \ldots, z^{(k)}) \in U \) (see (15)). Since
\[ f^{(p)}(z^{(j_0)}) = -f_{i_0}^p + f_{n-k+p}^p \cdot \frac{f_{i_0}^p}{f_{n-k+p}^p} = -f_{i_0}^p + f_{i_0}^p = 0 \]
and by the definition of \( z^{(j)} \) for \( j \neq j_0 \) we obtain that \( z^{(j)} \in H \) for \( j = 1, \ldots, k \). Taking
into account the equalities
\[
\sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f_p^i z_{i_0}^i \right| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{i_0}^i| = \sum_{p=1}^{n-k} |f_p^{j_0} \cdot (-1)| + (f_{n-k+j_0}^{j_0}) \cdot | -1 | \\
= \left( \sum_{p=1}^{n-k} f_p^{j_0} \right) + \left( 1 - \sum_{j=1}^{n-k} f_j^{j_0} \right) = 1,
\]

\[
\sum_{p=1}^{n-k} \left| \sum_{i=1}^{k} f_p^i z_{s}^i \right| + \sum_{i=1}^{k} f_{n-k+i}^i |z_{s}^i| = 0 \quad \text{for} \quad s \in \{ 1, \ldots, n \} \setminus \{ i_0 \},
\]

we demonstrate that \((z^{(1)}, \ldots, z^{(k)}) \in U\). Now we calculate constants \(M_1(z^{(1)}, \ldots, z^{(k)})\) and \(M_2(z^{(1)}, \ldots, z^{(k)})\). First note that

\[
M_1^{j_0}(z^{(1)}, \ldots, z^{(k)}) = \sum_{p=1}^{n-k} \sum_{i=1}^{k} f_p^i z_{i_0}^i + \sum_{i=1}^{k} f_{n-k+i}^i |z_{i_0}^i| + \sum_{i=1}^{k} f_{i_0}^i z_{i_0}^i
\]

\[
= \sum_{p=1}^{n-k} f_p^{j_0} + f_{n-k+j_0}^{j_0} - f_{i_0}^{j_0} = 1 - 2f_{i_0}^{j_0}.
\]

For \(s \in \{ 1, \ldots, n \} \setminus \{ i_0 \}\) we get

\[
M_1^s(z^{(1)}, \ldots, z^{(k)}) = \sum_{p=1}^{n-k} \sum_{i=1}^{k} f_p^i z_{s}^i + \sum_{i=1}^{k} f_{n-k+i}^i |z_{s}^i| + \sum_{i=1}^{k} f_{i_0}^i z_{s}^i = 0.
\]

Hence

\[
M_1(z^{(1)}, \ldots, z^{(k)}) = 1 - 2f_{i_0}^{j_0}.
\]

(22)

Let us consider \(s \in \{ n-k+1, \ldots, n \}\) such that \(f_{s}^{s-(n-k)} = 1/2\). Then

\[
z_s^{(j_0)} = f_{i_0}^{s-(n-k)} \frac{f_{s}^{s-(n-k)}}{f_{s}^{s-(n-k)}} = 2f_{i_0}^{s-(n-k)}
\]

and

\[
M_2^s(z^{(1)}, \ldots, z^{(k)}) = \sum_{j=1}^{k} (-1 + f_{n-k+j}^{j}) z_{s}^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^{j} |z_{s}^{(j)}| \\
= (-1 + f_{n-k+j_0}^{j_0}) z_{s}^{(j_0)} + f_{n-k+j_0}^{j_0} |z_{s}^{(j_0)}| = (2f_{n-k+j_0}^{j_0} - 1) z_{s}^{(j_0)}
\]

\[
= 2(2f_{n-k+j_0}^{j_0} - 1) f_{i_0}^{s-(n-k)}.
\]

We show that

\[
M_2^s(z^{(1)}, \ldots, z^{(k)}) \leq 1 - 2f_{i_0}^{j_0} \quad \text{for} \quad s \in \{ n-k+1, \ldots, n \}\) such that \(f_{s}^{s-(n-k)} = 1/2\). (23)

By (5) and definitions of \(i_0\) and \(j_0\) (see (21)) we obtain

\[
2(2f_{n-k+j_0}^{j_0} - 1) f_{i_0}^{s-(n-k)} \leq 2(1 - f_{i_0}^{j_0}) - 1)f_{i_0}^{j_0} = 2(1 - 2f_{i_0}^{j_0})f_{i_0}^{j_0}.
\]
Hence, to demonstrate inequality (23) it is sufficient to verify that
\[2(1 - 2^i_0)f^j_0 \leq 1 - 2^j_0.\] (24)

Note that (24) follows immediately from the fact that \(f^j_0 < 1/2\). By Lemma 2.4 and conditions (22)–(23) we get
\[r \leq \max\{M_1(z^{(1)}, \ldots, z^{(k)}), M_2(z^{(1)}, \ldots, z^{(k)})\} = 1 - 2 \max_{i=1, \ldots, n-k} f^j_i. \quad \blacksquare \] (25)

Now we prove the main result of this paper.

**Theorem 2.6.** Let \(f^{(1)}, \ldots, f^{(k)}\) satisfy conditions (5) and \(n > k \geq 3\). Then the SUP-constant \(r\) of \(P(l^{(n)}_\infty, H)\) is equal to
\[1 - 2 \max_{j=1, \ldots, k} f^j_i.\]

**Proof.** Let \((z^{(1)}, \ldots, z^{(k)}) \in U\) realize minimum in (18). Then by (15) there exists \(i_0 \in \{1, \ldots, n-k\}\) such that
\[\sum_{p=1}^{n-k} \left| \sum_{j=1}^{k} f_p^j z^{(j)}_{i_0} \right| + \sum_{j=1}^{k} f_j^{n-k+j} |z^{(j)}_{i_0}| = 1. \quad \text{(26)}\]

By (18), (20) and (26) we get
\[\sum_{j=1}^{k} f^j_{i_0} z^{(j)}_{i_0} < 0. \quad \text{(27)}\]

We show that \(j\) satisfying \(z^{(j)}_{i_0} > 0\) does not exist. Assume to the contrary that for \(l_1, \ldots, l_k\) such that \(\{l_1, \ldots, l_k\} = \{1, \ldots, k\}\)
\[z^{(l_1)}_{i_0} > 0, \ldots, z^{(l_s)}_{i_0} > 0, z^{(l_{s+1})}_{i_0} \leq 0, \ldots, z^{(l_k)}_{i_0} \leq 0. \quad \text{(28)}\]

Taking into account (20) it is sufficient to verify that inequalities (28) imply
\[M_1(z^{(1)}, \ldots, z^{(k)}) > 1 - 2 \max_{j=1, \ldots, k} f^j_i. \quad \text{(29)}\]

If we show that the inequality
\[\sum_{p=1}^{n-k} \left| \sum_{j=1}^{k} f_p^j z^{(j)}_{i_0} \right| + \sum_{j=1}^{k} f_j^{n-k+j} |z^{(j)}_{i_0}| + \sum_{j=1}^{k} f^j_{i_0} z^{(j)}_{i_0} \leq 1 - 2 \max_{j=1, \ldots, k} f^j_{i_0} \quad \text{(30)}\]
cannot be true then this will establish
\[M^i_0(z^{(1)}, \ldots, z^{(k)}) > 1 - 2 \max_{j=1, \ldots, k} f^j_{i_0} \geq 1 - 2 \max_{j=1, \ldots, k} f^j_i, \quad \text{which implies (29). Assume to the contrary that for (z^{(1)}, \ldots, z^{(k)}) satisfying (28) the} \]
inequality (30) is true. Then

\[
\sum_{p=1}^{n-k} \sum_{j=1}^{k} |f_{p}^{j} z_{(i)}| + \sum_{i=1}^{k} f_{n-k+1}^{i} z_{i0}^{(i)} + \sum_{i=1}^{k} f_{i0}^{i} z_{i0}^{(i)} = 1 - |\sum_{i=1}^{k} f_{i0}^{i} z_{i0}^{(i)}| + \sum_{i=1}^{k} f_{i0}^{i} z_{i0}^{(i)} = 1 + 2 \left( \sum_{i=1}^{k} f_{i0}^{i} z_{i0}^{(i)} \right).
\]

Hence (30) is equivalent to the following inequalities

\[
1 + 2 \left( \sum_{j=1}^{k} f_{i0}^{j} z_{i0}^{(j)} \right) \leq 1 - 2 \max_{j=1,\ldots,k} f_{i0}^{j}
\]

and

\[
\max_{j=1,\ldots,k} f_{i0}^{j} + \sum_{j=1}^{k} f_{i0}^{j} z_{i0}^{(j)} \leq 0. \tag{31}
\]

By (26)

\[
1 = \sum_{p=1}^{n-k} \sum_{j=1}^{k} f_{p}^{j} z_{i0}^{(j)} + \sum_{j=1}^{k} f_{n-k+j}^{j} z_{i0}^{(j)} \\
\geq - \sum_{p=1}^{n-k} \sum_{j=1}^{k} f_{p}^{j} z_{i0}^{(j)} + \sum_{j=1}^{s} f_{n-k+j}^{l_j} z_{i0}^{(l_j)} - \sum_{j=s+1}^{k} f_{n-k+j}^{l_j} z_{i0}^{(l_j)} \\
= \sum_{j=1}^{s} \left( - \sum_{p=1}^{n-k} f_{p}^{j} + f_{n-k+l_j}^{l_j} \right) z_{i0}^{(j)} + \sum_{j=s+1}^{k} \left( - \sum_{p=1}^{n-k} f_{p}^{j} - f_{n-k+l_j}^{l_j} \right) z_{i0}^{(l_j)} \\
= \sum_{j=1}^{s} (2f_{n-k+l_j}^{l_j} - 1) z_{i0}^{(l_j)} - \sum_{j=s+1}^{k} z_{i0}^{(l_j)}.
\]

From the above calculations we get

\[
\sum_{j=s+1}^{k} z_{i0}^{(l_j)} \geq \sum_{j=1}^{s} (2f_{n-k+l_j}^{l_j} - 1) z_{i0}^{(l_j)} - 1. \tag{32}
\]

Applying (28) and (32) we can write

\[
\max_{j=1,\ldots,k} f_{i0}^{j} + \sum_{j=1}^{k} f_{i0}^{j} z_{i0}^{(j)} = \max_{j=1,\ldots,k} f_{i0}^{j} + \sum_{j=1}^{s} f_{i0}^{l_j} z_{i0}^{(l_j)} + \sum_{j=s+1}^{k} f_{i0}^{l_j} z_{i0}^{(l_j)} \\
\geq \max_{j=1,\ldots,k} f_{i0}^{j} + \sum_{j=1}^{s} f_{i0}^{l_j} z_{i0}^{(l_j)} + \max_{j=1,\ldots,k} f_{i0}^{j} \cdot \sum_{j=s+1}^{k} z_{i0}^{(l_j)} \\
\geq \max_{j=1,\ldots,k} f_{i0}^{j} + \sum_{j=1}^{s} f_{i0}^{l_j} z_{i0}^{(l_j)} + \max_{j=1,\ldots,k} f_{i0}^{j} \cdot \left( \sum_{j=1}^{s} (2f_{n-k+l_j}^{l_j} - 1) z_{i0}^{(l_j)} - 1 \right).
\]
\[ \frac{s}{j=1,\ldots,k} \sum_{i=1}^{\max_{j=1,\ldots,k}f_{i_0}^{l_j}}(2f_{n-k+l_j}^{l_j} - 1) + f_{i_0}^{l_j}z_{i_0}^{(l_j)} > 0; \]

a contradiction with (31). Hence by (27) for \( i_0 \in \{1,\ldots,k\} \) satisfying (26)

\[ z_{i_0}^{(j)} \leq 0 \quad \text{for} \quad j = 1,\ldots,k. \quad (33) \]

Now (26) is equivalent to

\[ \sum_{j=1}^{k} z_{i_0}^{(j)} = -1. \quad (34) \]

By (16), (33) and (34) we get

\[
\begin{align*}
M_{i_0}^{(1)}(z^{(1)},\ldots,z^{(k)}) &= \sum_{p=1}^{n-k} \sum_{i=1}^{k} f_i^p z_{i_0}^{(i)} + \sum_{i=1}^{k} f_{n-k+i}^i z_{i_0}^{(i)} + \sum_{i=1}^{k} f_i^i z_{i_0}^{(i)} \\
&= 1 + 2 \left( \sum_{i=1}^{k} f_i^i z_{i_0}^{(i)} \right) = 1 + 2 \max_{j=1,\ldots,k} f_j^j \cdot \left( \sum_{i=1}^{k} z_{i_0}^{(i)} \right) = 1 - 2 \max_{j=1,\ldots,k} f_j^j.
\end{align*}
\]

Hence

\[ r = \max\{M_1(z^{(1)},\ldots,z^{(k)}), M_2(z^{(1)},\ldots,z^{(k)})\} \geq M_{i_0}^{(1)}(z^{(1)},\ldots,z^{(k)}) \geq 1 - 2 \max_{j=1,\ldots,k} f_j^j. \]

By (20)

\[ r = 1 - 2 \max_{j=1,\ldots,k} f_j^j. \]

The proof is complete.  \[ \blacksquare \]

References


