FUNCTION SPACES IX
BANACH CENTER PUBLICATIONS, VOLUME 92
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

STRONGLY EXPOSED POINTS IN MUSIELAK-ORLICZ SEQUENCE SPACES ENDOWED WITH ORLICZ NORM

ZHONGRUI SHI and CHUNYAN LIU

Department of Mathematics, Shanghai University
Shanghai 200444, P. R. China
E-mail: zshi@shu.edu.cn, chyliu@shu.edu.cn

Abstract. A criterion for strongly exposed points of the unit ball $B(l_M)$ in Musielak-Orlicz sequence spaces l_M equipped with Orlicz norm is given.

1. Introduction. It is well known that the notion of strongly exposed point is a basic conception in the geometric theory of Banach spaces. It has numerous applications in separation theory and control theory. Criteria for strongly exposed points in all classical Orlicz spaces were given [LWW, WJS]. In [SL], we obtained a criterion for such points in arbitrary Musielak-Orlicz sequence spaces endowed with the Luxemburg norm. In this paper, we give a criterion for strongly exposed points in arbitrary Musielak-Orlicz sequence spaces equipped with Orlicz norm.

Let $[X, \|\cdot\|]$ be a Banach space; $S(X)$ and $B(X)$ be the unit sphere and unit ball of X, respectively; X^* be the dual space of X. For $x \in S(X)$, define $\mathrm{Grad}(x) = \{f \in S(X^*) : f(x) = 1\}$. A point $x \in S(X)$ is called an *exposed point* of $B(X)$ if there exists $f \in \mathrm{Grad}(x)$ such that $1 = f(x) > f(y)$ for all $y \in B(X) \setminus \{x\}$ [S]; moreover, if there exists $f \in \mathrm{Grad}(x)$ such that for any sequence $\{x_n\} \subset B(X)$ the condition $f(x_n) \to f(x)$ implies $x_n \to x$ $(n \to \infty)$, then x is called a *strongly exposed point* of $B(X)$ [L]. Then f is called an *exposed functional* and a *strongly exposed functional* for x, respectively.

Let $\mathbb{N}$ be the set of all natural numbers; $\mathbb{R}$ the set of all real numbers. By $M = \{M_i\}_{i=1}^{\infty}$ we denote a Musielak-Orlicz function provided that for each $i \in \mathbb{N}$, $M_i : (-\infty, +\infty) \to [0, +\infty]$ satisfies

1. $M_i(0) = 0$, $\lim_{u\to\infty} M_i(u) = \infty$ and $M_i(u_i) < \infty$ for some $u_i > 0$;
2. $M_i(u)$ is even convex and left continuous in $[0, +\infty)$.

2010 *Mathematics Subject Classification*: 46B20; 46E30.
Key words and phrases: Musielak-Orlicz sequence spaces, Orlicz norm, exposed points, strongly exposed points.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc92-0-22

A Musielak-Orlicz function is a sequence of Orlicz functions in fact. Let $p_i^-(u)$ and $p_i(u)$ denote the left-hand and the right-hand derivatives of $M_i(u)$, respectively. The sequence $N = \{N_i\}_{i=1}^\infty$, where $N_i(v) = \sup_{u>0}\{u|v| - M_i(u)\}$, which has the same property as $M_i(u)$, is called the *complementary function* of M. The functions $q_i^-(s) = \sup\{t : p_i(t) < s\}$ and $q_i(s) = \sup\{t : p_i(t) \le s\}$ are the left-hand and the right-hand derivatives of $N_i(u)$, respectively [C]. Let us set

$$\alpha_i = \sup\{u \ge 0 : M_i(u) = 0\}, \quad \beta_i = \sup\{u > 0 : M_i(u) < \infty\},$$
$$\tilde{\alpha}_i = \sup\{u \ge 0 : N_i(u) = 0\}, \quad \tilde{\beta}_i = \sup\{u > 0 : N_i(u) < \infty\}.$$

It is easy to verify that

$$\tilde{\alpha}_i = \lim_{s\to 0^+} \frac{M_i(s)}{s} = \lim_{s\to 0^+} p_i^-(s) = \lim_{s\to 0^+} p_i(s),$$
$$\tilde{\beta}_i = \lim_{s\to +\infty} \frac{M_i(s)}{s} = \lim_{s\to +\infty} p_i^-(s) = \lim_{s\to +\infty} p_i(s).$$

Let

$$SC_{M_i} = \Big\{u \in \mathbb{R} : \forall \varepsilon > 0 \ M_i(u) < \frac{M_i(u+\varepsilon) + M_i(u-\varepsilon)}{2}\Big\}.$$

Clearly, SC_{M_i} is the set of all strictly convex points of M_i. An interval $[a, b]$ is called a *structurally affine interval* of $M_i(u)$ (SAI(M_i) for short) provided that $M_i(u)$ is affine on $[a, b]$ and it is not affine either on $[a-\varepsilon, b]$ or on $[a, b+\varepsilon]$ for all $\varepsilon > 0$ [C]. It is obvious that

$$SC_{M_i} = \mathbb{R} \setminus \bigcup_n (a_n, b_n), \text{ where } [a_n, b_n] \in \mathrm{SAI}(M_i), \ n = 1, 2, \ldots.$$

We say that $M = \{M_i\}_{i=1}^\infty$ satisfies the δ_2^0-condition ($M \in \delta_2^0$ for short) if there exist $a > 0$, $K > 0$, $i_0 \in \mathbb{N}$ and $c_i \ge 0$ ($i > i_0$) with $\sum_{i>i_0} c_i < \infty$ such that $M_i(2u) \le KM_i(u) + c_i$ holds for all $i > i_0$ and all u with $M_i(u) \le a$. It is known that $h_M = l_M$ if and only if $M \in \delta_2^0$ [HY].

Let l^0 denote the space of all real sequences $u = \{u(i)\}_{i=1}^\infty$. As usual, for $u \in l^0$, let $\operatorname{supp} u = \{i \in \mathbb{N} : u(i) \ne 0\}$. For each $u = \{u(i)\}_{i=1}^\infty \in l^0$, we define the *modular* ρ_M of u by $\rho_M(u) = \sum_{i=1}^\infty M_i(u(i))$. The linear set $\{u \in l^0 : \rho_M(\lambda u) < \infty$ for some $\lambda > 0\}$ endowed with the Luxemburg norm

$$\|u\|_{(M)} = \inf\{\lambda > 0 : \rho_M(u/\lambda) \le 1\}$$

or the Orlicz norm

$$\|u\|_M = \sup\Big\{\sum_{i=1}^\infty u(i)v(i) : \rho_N(v) \le 1\Big\} = \inf_{k>0} \frac{1}{k}(1 + \rho_M(ku))$$

is a Banach space, denoted by $l_{(M)}$ or l_M (respectively), and it is called the Musielak-Orlicz sequence space [C, M, RR]. The subspace

$$\Big\{u \in l_M : \forall \lambda > 0 \ \exists i_\lambda \text{ such that } \sum_{i>i_\lambda} M_i(\lambda u(i)) < \infty\Big\}$$

equipped with the norm $\|\cdot\|_{(M)}$ (or $\|\cdot\|_M$), which is also a Banach space, is denoted by $h_{(M)}$ (or h_M, respectively). For any $u \in l_M$, $\|u\|_{(M)} \le \|u\|_M \le 2\|u\|_{(M)}$ [C, M, RR].

Let $\theta_M(u) = \inf\{\lambda > 0 : \sum_{i>i_\lambda} M_i(\frac{u(i)}{\lambda}) < \infty$ for some $i_\lambda\}$. It is known that $\theta_M(u) = \operatorname{dist}(u, h_{(M)}) = \operatorname{dist}(u, h_M)$ [SZ] and $(h_{(M)})^* = l_M$, $(h_M)^* = l_{(N)}$ [KR, C, M, RR].

We say that $\varphi \in (l_M)^*$ is a singular functional ($\varphi \in F$ for short) if $\varphi(u) = 0$ for all $u \in h_M$. The dual space of l_M is represented in the form $(l_M)^* = l_N \oplus F$, i.e., each $f \in (l_M)^*$ has the unique representation $f = v + \varphi$, where $\varphi \in F$ and $v \in l_N$, and v is called the regular functional with $\langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ for all $u = \{u(i)\}_{i=1}^{\infty} \in l_M$ [KR, C, M, RR]. It is well known that $\|f\| = \inf\{\lambda > 0 : \rho_N(\frac{v}{\lambda}) + \frac{\|\varphi\|}{\lambda} \le 1\}$ for every $f \in (l_M)^*$ [WH].

2. Main results. For the convenience of reading, we present some auxiliary lemmas.

LEMMA 2.1 (see [WS]). *If $u \in l_M \setminus \{0\}$, then $K_M(u) \neq \emptyset$ if and only if* $\sum_{i\in \operatorname{supp} u} N_i(\tilde{\beta}_i) > 1$ *or* $\sum_{i\in \operatorname{supp} u} N_i(\tilde{\beta}_i) = 1$ *and* $\sup_{i\in \operatorname{supp} u} \frac{q_i^-(\tilde{\beta}_i)}{|u(i)|} < \infty$, *where* $K_M(u) = [k_u^*, k_u^{**}]$ *with*

$$k_u^* = \inf\Big\{k > 0 : \rho_N(p(k|u|)) = \sum_{i=1}^{\infty} N_i(p_i(k|u(i)|)) \ge 1\Big\},$$
$$k_u^{**} = \sup\big\{k > 0 : \rho_N(p(k|u|)) \le 1\big\}.$$

LEMMA 2.2 (see [CW]). *Let $u \in l_M \setminus \{0\}$. If* $\sum_{i\in \operatorname{supp} u} N_i(\tilde{\beta}_i) > 1$, *then* $\|u\|_M = \frac{1}{k}(1 + \rho_M(ku))$ *if and only if $k \in K_M(u)$. If* $\sum_{i\in \operatorname{supp} u} N_i(\tilde{\beta}_i) \le 1$, *then* $\|u\|_M = \sum_{i\in \operatorname{supp} u} |u(i)|\tilde{\beta}_i$.

LEMMA 2.3. *If $u \in l_M$ and $K_M(u) \neq \emptyset$, then $f = v + \varphi$, where $v \in l_{(N)}$, $\varphi \in F$, is a support functional of u if and only if for $k \in K_M(u)$*

(1) $\rho_N(v) + \|\phi\| = 1$,
(2) $\varphi(ku) = \|\varphi\|$,
(3) $u(i)v(i) \ge 0$ *and* $p_i^-(k|u(i)|) \le |v(i)| \le p_i(k|u(i)|)$ *for all* $i \in \mathbb{N}$.

Proof. It can proceed in an analogous way as the proof of Theorem 1.77 in [C]. ∎

LEMMA 2.4. *Let $u_n, u \in l_M$, $\|u_n\|_M \to \|u\|_M$ and $u_n(i) \to u(i)$ as $n \to \infty$ for each $i \in \mathbb{N}$. Then*:

(i) *If $K_M(u) = \emptyset$, then $\|u_n - u\|_M \to 0$;*
(ii) *If $K_M(u) \neq \emptyset$ and $M \in \delta_2^0$, then $\|u_n - u\|_M \to 0$.*

Proof. (i) $K_M(u) = \emptyset$.

By Lemmas 2.1 and 2.2, $\|u\|_M = \sum_{i=1}^{\infty} |u(i)|\tilde{\beta}_i$. For any $\varepsilon > 0$, choose $i_0 \in \mathbb{N}$ such that $\big\|\sum_{i>i_0} u(i)e_i\big\|_M = \sum_{i>i_0} |u(i)|\tilde{\beta}_i < \frac{\varepsilon}{3}$, where

$$e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots).$$

Since $u_n(i) \to u(i)$ $(i = 1, 2, \dots)$, there exists $n_1 \in \mathbb{N}$ satisfying

$$\Big\|\sum_{i=1}^{i_0} (u_n(i) - u(i))e_i\Big\|_M < \frac{\varepsilon}{3}$$

for $n > n_1$.

Case I. There are an infinite number of n for which $K_M(u_n) = \emptyset$, i.e., $\|u_n\|_M = \sum_{i=1}^{\infty} |u_n(i)|\tilde{\beta}_i$. Since

$$\Big\|\sum_{i>i_0} u_n(i)e_i\Big\|_M = \sum_{i>i_0} |u_n(i)|\tilde{\beta}_i = \sum_{i=1}^{\infty} |u_n(i)|\tilde{\beta}_i - \sum_{i=1}^{i_0} |u_n(i)|\tilde{\beta}_i = \|u_n\|_M - \sum_{i=1}^{i_0} |u_n(i)|\tilde{\beta}_i$$
$$\longrightarrow \|u\|_M - \sum_{i=1}^{i_0} |u(i)|\tilde{\beta}_i = \Big\|\sum_{i>i_0} u(i)e_i\Big\|_M,$$

we can choose $n_0 \geq n_1$ satisfying $\big\|\sum_{i>i_0} u_n(i)e_i\big\|_M < \frac{\varepsilon}{3}$ $(n > n_0)$. Hence

$$\|u_n - u\|_M \leq \Big\|\sum_{i=1}^{i_0}(u_n(i) - u(i))e_i\Big\|_M + \Big\|\sum_{i>i_0} u_n(i)e_i\Big\|_M + \Big\|\sum_{i>i_0} u(i)e_i\Big\|_M$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

for $n > n_0$.

Case II. There are an infinite number of n for which $K_M(u_n) \neq \emptyset$, i.e., $\|u_n\|_M = \frac{1}{k_n}(1 + \rho_M(k_n u_n))$ for some $k_n \in (0, \infty)$. We will show that $k_n \to \infty$. Otherwise, without loss of generality, we may assume that $k_n \to k_0 < \infty$. By the Fatou Lemma, we get the contradiction:

$$\|u\|_M < \frac{1}{k_0}(1 + \rho_M(k_0 u)) \leq \lim_{n\to\infty} \frac{1}{k_n}(1 + \rho_M(k_n u_n)) = \lim_{n\to\infty} \|u_n\|_M = \|u\|_M.$$

Now, combining $k_n \to \infty$ with $u_n(i) \to u(i)$ for $i = 1, 2, \ldots$, we get

$$\frac{1}{k_n}\Big(1 + \sum_{i>i_0} M_i(k_n u_n(i))\Big) \geq \frac{1}{k_n}(1 + \rho_M(k_n u_n)) - \frac{1}{k_n}\sum_{i=1}^{i_0} M_i(k_n u_n(i))$$
$$= \|u_n\|_M - \sum_{i=1}^{i_0} \frac{M_i(k_n u_n(i))}{k_n|u_n(i)|}|u_n(i)| \longrightarrow \|u\|_M - \sum_{i=1}^{i_0} |u(i)|\tilde{\beta}_i = \sum_{i>i_0} |u(i)|\tilde{\beta}_i$$

for $n \to \infty$. Hence

$$\Big\|\sum_{i>i_0} u_n(i)e_i\Big\|_M \leq \frac{1}{k_n}\Big(1 + \sum_{i>i_0} M_i(k_n u_n(i))\Big) < \frac{\varepsilon}{3}$$

for n large enough. Therefore

$$\|u_n - u\|_M \leq \Big\|\sum_{i=1}^{i_0}(u_n(i) - u(i))e_i\Big\|_M + \Big\|\sum_{i>i_0} u_n(i)e_i\Big\|_M + \Big\|\sum_{i>i_0} u(i)e_i\Big\|_M$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

for n large enough.

(ii) $K_M(u) \neq \emptyset$ and $M \in \delta_2^0$.

Case I. There are an infinite number of n for which $K_M(u_n) = \emptyset$, i.e., $\|u_n\|_M = \sum_{i=1}^{\infty} |u_n(i)|\tilde{\beta}_i$. By the Fatou Lemma, we have

$$\|u\|_M \le \sum_{i=1}^{\infty} |u(i)|\tilde{\beta}_i \le \lim_{n\to\infty} \sum_{i=1}^{\infty} |u_n(i)|\tilde{\beta}_i = \lim_{n\to\infty} \|u_n\|_M = \|u\|_M,$$

and consequently, $\|u\|_M = \sum_{i=1}^{\infty} |u(i)|\tilde{\beta}_i$. Repeating the process of the proof in Case I of (i), we obtain that $\|u_n - u\|_M \to 0$.

Case II. There are an infinite number of n for which $K_M(u_n) \neq \emptyset$, i.e., $\|u_n\|_M = \frac{1}{k_n}(1 + \rho_M(k_n u_n))$ for some $k_n \in (0, \infty)$.

II-1. If $k_n \to \infty$ then, for any $i \in \mathbb{N}$, $\lim_{n\to\infty} \frac{M_i(k_n u_n(i))}{k_n} = |u(i)|\tilde{\beta}_i$. By the Fatou Lemma,

$$\|u\|_M \le \sum_{i=1}^{\infty} |u(i)|\tilde{\beta}_i \le \lim_{n\to\infty} \frac{1}{k_n}(1 + \rho_M(k_n u_n)) = \lim_{n\to\infty} \|u_n\|_M = \|u\|_M,$$

whence $\|u\|_M = \sum_{i=1}^{\infty} |u(i)|\tilde{\beta}_i$. Repeating the process of the proof in Case II of (i), we obtain $\|u_n - u\|_M \to 0$.

II-2. If $k_n \to k < \infty$ then, by the Fatou Lemma,

$$\|u\|_M \le \frac{1}{k}(1 + \rho_M(ku)) \le \lim_{n\to\infty} (1 + \rho_M(k_n u_n)) = \lim_{n\to\infty} \|u_n\|_M = \|u\|_M,$$

whence $\|u\|_M = \frac{1}{k}(1 + \rho_M(ku))$ and $\rho_M(k_n u_n) \to \rho_M(ku)$. Combining this with $M \in \delta_2^0$, by Lemma 4 in [ZCH] and $k_n \to k$, we obtain $\|u_n - u\|_M \to 0$. ∎

LEMMA 2.5. *Let $\|u\|_{(M)} = 1$, $\|v_n\|_N = 1$ ($n \in \mathbb{N}$) and $\langle u, v_n\rangle \to 1$ as $n \to \infty$. Then:*

(a) *$v_n(j) \to 0$ as $n \to \infty$ whenever $|u(j)| < \alpha_j$,*

(b) *$v_n(j) \to 0$ or $\liminf_{n\to\infty}\big(|v_n(i)|p_j(|u(j)|) - |v_n(j)|p_i^-(|u(i)|)\big) \ge 0$ as $n \to \infty$ whenever $\alpha_j \le |u(j)| < \beta_j$ and $|u(i)| > 0$.*

Proof. The proof of (a) is the same as the proof of Lemma 1.3(i) in [BHW].

If (b) is not true, there are $\alpha_{j_0} \le |u(j_0)| < \beta_{j_0}$, $|u(i_0)| > 0$ and $a > 0$ satisfying

$$\inf_n |v_n(j_0)| = d_{j_0} > 0$$

and

$$|v_n(i_0)|p_{j_0}(|u(j_0)|) < |v_n(j_0)|p_{i_0}^-(|u(i_0)|) - 2a \quad (n = 1, 2, \ldots). \tag{1}$$

Since $\rho_N(v_n) \le \|v_n\|_{(N)} \le \|v_n\|_N = 1$, we get $|v_n(i)| \le b_i$, where $b_i = N_i^{-1}(1)$ if $N_i(\tilde{\beta}_i) \ge 1$ and $b_i = \tilde{\beta}_i$ if $N_i(\tilde{\beta}_i) < 1$. By (1), we have $p_{i_0}^-(|u(i_0)|) > 0$.

1. If $p_{i_0}^-(|u(i_0)|) < \infty$, we can obtain a contradiction via repeating the process of Lemma 1.3 (ii) in [BHW].

2. If $p_{i_0}^-(|u(i_0)|) = \infty$, there exists $r_1 > 0$ such that

$$\infty > p_{i_0}^-(|u(i_0)| - r) > \frac{b_{i_0} p_{j_0}(|u(j_0)|) + 2a}{d_{j_0}} \quad (0 < r < r_1).$$

Then

$$|v_n(j_0)|p_{i_0}^-(|u(i_0)|-r) \ge d_{j_0}p_{i_0}^-(|u(i_0)|-r) \\ > b_{i_0}p_{j_0}(|u(j_0)|) + 2a \ge |v_n(i_0)|p_{j_0}(|u(j_0)|) + 2a.$$

Since p_{j_0} is right continuous, one can find a number $0 < r < r_1$ such that

$$p_{j_0}(|u(j_0)|+r) - p_{j_0}(|u(j_0)|) < \frac{a}{b_{i_0}}\,.$$

Then

$$\big(p_{j_0}(|u(j_0)|+r) - p_{j_0}(|u(j_0)|)\big)|v_n(i_0)| < a.$$

Thus, for $n \in \mathbb{N}$,

$$|v_n(i_0)|p_{j_0}(|u(j_0)|+r) < |v_n(i_0)|p_{j_0}(|u(j_0)|) + a < |v_n(j_0)|p_{i_0}^-(|u(i_0)|-r) - a.$$

Now, repeating the proof of Lemma 1.3(ii) in [BHW], we can also get a contradiction finishing the proof of Lemma 2.5(b). ■

REMARK 2.6. According to the proof of Lemma 2.5(b),

$$\liminf_{n\to\infty}\big(|v_n(i)|p_j(|u(j)|) - |v_n(j)|p_i^-(|u(i)|)\big) \ge 0$$

whenever $\alpha_j \le |u(j)| < \beta_j$ and $0 < p_i^-(|u(i)|) < \infty$.

LEMMA 2.7 (see [BHW]). *If $\|u\|_{(M)} = 1$, $\theta_M(u) < 1$, $v_n \in S(l_N)$ for any $n \in \mathbb{N}$ and $\langle u, v_n\rangle \to 1$ as $n \to \infty$, then $\lim_{i_0\to\infty}\sup_n \sum_{i>i_0} N_i(v_n(i)) = 0$.*

LEMMA 2.8 (see [ZW]). *$u \in S(l_M)$ is an exposed point of $B(l_M)$ if and only if*

1. *In the case $K_M(u) = \emptyset$,* $\operatorname{supp} u = \{i_0\}$;
2. *In the case $K_M(u) \ne \emptyset$, we have*
   (a) *if* $\operatorname{Card}(\operatorname{supp} u) = 1$, *then $u(i) = 0$ implies $\alpha_i = 0$,*
   (b) *if* $\operatorname{Card}(\operatorname{supp} u) > 1$, *then for any $k \in K_M(u)$ we have*
      i. $\{i \in \mathbb{N} : k|u(i)| \in \mathbb{R} \setminus SC_{M_i} \cup A_i' \cup B_i'\} = \emptyset$,
      ii. *if $\rho_N(p^-(k|u|)) = 1$, then $\{i \in \mathbb{N} : k|u(i)| \in B_i\} = \emptyset$,*
      iii. *if $\rho_N(p(k|u|)) = 1$ and $\theta_M(ku) < 1$, then $\{i \in \mathbb{N} : k|u(i)| \in A_i\} = \emptyset$,*

where $A_i' \cup A_i$, $B_i' \cup B_i$ are the sets of all those left endpoints and right endpoints of SAI(M_i), respectively, satisfying $p_i^-(s) = p_i(s)$ whenever $s \in A_i' \cup B_i'$ and $p_i^-(s) < p_i(s)$ whenever $s \in A_i \cup B_i$.

THEOREM 2.9. *Let $u \in S(l_M)$ and $K_M(u) = \emptyset$. Then the following are equivalent:*

(a) *u is a strongly exposed point of $B(l_M)$;*
(b) *u is an exposed point of $B(l_M)$;*
(c) $\operatorname{supp} u = \{i_0\}$, *a singleton.*

Proof. It is obvious by Lemma 2.8 that (a) implies (b), and (b) implies (c).

Suppose $\operatorname{supp} u = \{i_0\}$. Without loss of generality, we may assume that $u(1) > 0$ and $u(i) = 0$ $(i \ne 1)$. Since $K_M(u) = \emptyset$, by Lemmas 2.1 and 2.2, $N_1(\tilde{\beta_1}) \le 1$ and

$1 = \|u\|_M = u(1)\tilde{\beta}_1$. Then $v = (\tilde{\beta}_1, 0, \dots) \in \mathrm{Grad}(u)$. Now, we shall prove that v is a strongly exposed functional for u.

Let $\{u_n\}$ be a sequence in $S(l_M)$ satisfying $\langle u_n, v\rangle \to 1$ $(n \to \infty)$. In view of Lemma 2.4 to prove $\|u_n - u\|_M \to 0$, it suffices to show that $u_n(i) \to u(i)$ $(i = 1, 2, \dots)$.

From $1 \leftarrow \langle u_n, v\rangle = u_n(1)v(1) = u_n(1)\tilde{\beta}_1$, we have $u_n(1) \to 1/\tilde{\beta}_1 = u(1)$ as $n \to \infty$. In the following, we will consider two cases to prove that $u_n(i) \to u(i)$ for $i \neq 1$.

Case I. There are an infinite number of n for which $K_M(u_n) = \emptyset$, i.e., $\|u_n\|_M = \sum_{i=1}^{\infty} |u_n(i)|\tilde{\beta}_i$. Then

$$\sum_{i=2}^{\infty} |u_n|\tilde{\beta}_i = \sum_{i=1}^{\infty} |u_n(i)|\tilde{\beta}_i - u_n(1)\tilde{\beta}_1 = \|u_n\|_M - \langle u_n, v\rangle \to 1 - 1 = 0,$$

whence $u_n(i) \to 0 = u(i)$ for $i = 2, 3, \dots$.

Case II. There are an infinite number of n for which $K_M(u_n) \neq \emptyset$, i.e., $\|u_n\|_M = \frac{1}{k_n}(1 + \rho_M(k_n u_n))$ for some $k_n \in (0, \infty)$. From

$$\begin{aligned} 1 \leftarrow \langle u_n, v\rangle = u_n(1)\tilde{\beta}_1 &\leq \frac{1}{k_n}\big(M_1(k_n u_n(1)) + N_1(\tilde{\beta}_1)\big) \\ &\leq \frac{1}{k_n}\big(M_1(k_n u_n(1)) + 1\big) \leq \frac{1}{k_n}\Big(M_1(k_n u_n(1)) + \sum_{i=2}^{\infty} M_i(k_n u_n(i)) + 1\Big) \\ &= \frac{1}{k_n}\,(1 + \rho_M(k_n u_n)) = \|u_n\|_M = 1, \quad (2) \end{aligned}$$

we can get that $k_n \to \infty$. Indeed, suppose that $k_n \to k_0 < \infty$. By (2), we have the contradiction:

$$1 = \lim_{n\to\infty} \frac{1}{k_n}\big(M_1(k_n u_n(1)) + 1\big) = \frac{1}{k_0}\big(M_1(k_0 u(1)) + 1\big) > \|u\|_M = 1.$$

From (2), we also have

$$\lim_{n\to\infty} \frac{1}{k_n} \sum_{i=2}^{\infty} M_i(k_n u_n(i)) = 0.$$

If $|u_n(i)| \geq c > 0$ $(i \neq 1)$, we can also get the contradiction:

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{k_n} \sum_{i=2}^{\infty} M_i(k_n u_n(i)) \\ &\geq \liminf_{n\to\infty} \frac{1}{k_n} M_i(k_n u_n(i)) \\ &= \liminf_{n\to\infty} \frac{M_i(k_n u_n(i))}{k_n |u_n(i)|} |u_n(i)| \geq c\tilde{\beta}_i > 0. \end{aligned}$$

So $u_n(i) \to 0 = u(i)$ for $i = 2, 3, \dots$, and the proof is finished. ∎

THEOREM 2.10. *A point $u \in S(l_M)$ with $K_M(u) \neq \emptyset$ is a strongly exposed point of $B(l_M)$ if and only if $M \in \delta_2^0$ and*

1. *if* $\mathrm{Card}(\mathrm{supp}\, u) = 1$, *then* $u(i) = 0$ *implies* $\alpha_i = 0$;
2. *if* $\mathrm{Card}(\mathrm{supp}\, u) > 1$, *then for any* $k \in K_M(u)$

(a) $\{i \in \mathbb{N} : k|u(i)| \in \mathbb{R} \setminus SC_{M_i} \cup A'_i \cup B'_i\} = \emptyset$,

(b) *there exists* $w \in \mathrm{Grad}(u)$ *such that* $\theta_N(w) < 1$,

(c) *if* $\rho_N(p^-(k|u|)) = 1$, *then* $\{i \in \mathbb{N} : k|u(i)| \in B_i\} = \emptyset$; *if* $\rho_N(p(k|u|)) = 1$, *then* $\{i \in \mathbb{N} : k|u(i)| \in A_i\} = \emptyset$,

where A_i, A'_i, B_i *and* B'_i *are defined as in Lemma* 2.8.

Proof. Necessity.

Suppose that $M \notin \delta_2^0$. For any $k \in K_M(u)$, since $\|u\|_M = \frac{1}{k}(1+\rho_M(ku))$, $\theta_M(ku) \le 1$.

If $\theta_M(ku) = 1$, for any $\varepsilon > 0$, $j \in \mathbb{N}$, by the definition of $\theta_M(ku)$ we have $\sum_{i=j}^{\infty} M_i(\frac{ku(i)}{1-\varepsilon}) = \infty$. Take $0 = n_0 < n_1 < n_2 < \ldots$ such that

$$\sum_{i=n_{j-1}+1}^{n_j} M_i\Big(\frac{ku(i)}{1-1/j}\Big) > 1 \quad (j = 1, 2, \ldots).$$

Setting $u^j = u - [u]_{n_{j-1}}^{n_j}$, where $[u]_{n_{j-1}}^{n_j} = \sum_{i=n_{j-1}+1}^{n_j} u(i)e_i$, we have $u^j \in B(l_M)$. For $f = v + \varphi \in \mathrm{Grad}(u)$ $(v \in l_{(N)}, \varphi \in F)$,

$$\begin{aligned} 1 \ge f(u^j) &= \langle u - [u]_{n_{j-1}}^{n_j}, v\rangle + \varphi\big(u - [u]_{n_{j-1}}^{n_j}\big) \\ &= \sum_{i=1}^{n_{j-1}} u(i)v(i) + \sum_{i=n_j+1}^{\infty} u(i)v(i) + \varphi(u) \\ &\to \langle u, v\rangle + \varphi(u) = f(u) = 1 \quad (j \to \infty) \end{aligned}$$

and

$$\big\|u - u^j\big\|_M = \big\|[u]_{n_{j-1}}^{n_j}\big\|_M \ge \frac{1}{k}\Big(1 - \frac{1}{j}\Big) \to \frac{1}{k} \quad (j \to \infty).$$

This shows that u is not a strongly exposed point, a contradiction.

Let now $\theta_M(ku) < 1$. Take $w \in l_M$ such that $\rho_M(w) < \infty$ and $\theta_M(ku - w) \ne 0$ (indeed, if $\theta_M(ku) \ne 0$, we take $w = 0$; if $\theta_M(ku) = 0$, we take $w \in S(l_M)$ with $\theta_M(w) \ne 0$ by Theorem 5 in [K]). Then there exists $\varphi \in S(F)$ such that $\varphi(ku - w) \ne 0$.

Letting

$$u_n = \Big(u(1), \ldots, u(n), \frac{1}{k}w(n+1), \frac{1}{k}w(n+2), \ldots\Big),$$

we have

$$\begin{aligned} \|u_n\|_M &\le \frac{1}{k}\big(1 + \rho_M(ku_n)\big) = \frac{1}{k}\Big(1 + \sum_{i=1}^{n} M_i(ku(i)) + \sum_{i=n+1}^{\infty} M_i(w(i))\Big) \\ &\to \frac{1}{k}(1 + \rho_M(ku)) = \|u\|_M = 1. \end{aligned}$$

For any $f = v + \varphi \in \mathrm{Grad}(u)$, since $\theta_M(ku) < 1$, $\varphi = 0$ by Lemma 2.3, and

$$1 \ge f(u_n) = \langle v, u_n\rangle = \sum_{i=1}^{n} u(i)v(i) + \frac{1}{k}\sum_{i=n+1}^{\infty} w(i)v(i) \to \langle u, v\rangle = f(u) = 1.$$

But

$$\|u_n - u\|_M \ge |\varphi(u - u_n)| = \Big|\varphi\Big(u - \frac{1}{k}w\Big)\Big| > 0 \quad (\forall n \in \mathbb{N}).$$

This contradicts the fact that u is a strongly exposed point.

So, we proved the necessity of $M \in \delta_2^0$.

Since any strongly exposed point is also an exposed point, by Lemma 2.8, the conditions 1, 2a and 2c are necessary.

Suppose the condition 2b is not necessary, i.e., $\mathrm{Card}(\operatorname{supp} u) > 1$ and $\theta_N(v) = 1$ for any $v \in \mathrm{Grad}(u)$. Then $\lim_{n\to\infty} \|v - [v]_n\|_{(N)} = 1$, where $[v]_n = (v(1), v(2), \dots, v(n), 0, \dots)$. Take $w_n \in S(l_M)$ such that

$$\langle w_n, v - [v]_n\rangle = \sum_{i=n+1}^{\infty} w_n(i)v(i) > \|v - [v]_n\|_{(N)} - \frac{1}{n}, \quad n = 1, 2, \dots.$$

Without loss of generality, we may assume that $w_n = \sum_{i=n+1}^{\infty} w_n(i)e_i$. Putting

$$u_n = \frac{1}{2}\Big(\sum_{i=1}^{n} u(i)e_i + w_n\Big),$$

we have

$$\begin{aligned} 1 &\geq \frac{1}{2}\big(\|[u]_n\|_M + \|w_n\|_M\big) \geq \|u_n\|_M \geq \langle u_n, v\rangle \\ &= \frac{1}{2}\Big(\sum_{i=1}^{n} u(i)v(i) + \sum_{i=n+1}^{\infty} w_n(i)v(i)\Big) \to 1. \end{aligned}$$

So, $\|u_n\|_M \to 1$ and $\langle u_n, v\rangle \to 1$. Noticing that

$$\|u - u_n\|_M \geq \frac{1}{2}\big\|[u]_n\big\|_M \to \frac{1}{2}\|u\|_M = \frac{1}{2} \quad (n \to \infty),$$

we obtain that u is not a strongly exposed point.

Sufficiency. We will consider two cases.

Case 1. $\operatorname{supp} u = \{i_0\}$. Without loss of generality, we may assume that $u(1) > 0$ and $u(i) = 0$ for $i \neq 1$. Set $v = \frac{1}{u(1)}e_1$. Then $v \in \mathrm{Grad}(u)$ and $\rho_N(v) = 1$ by virtue of Lemma 2.3. Now, we shall prove that v is a strongly exposed functional for u.

Let $u_n \in S(l_M)$ $(n = 1, 2, \dots)$ with $\langle u_n, v\rangle \to 1$ $(n \to \infty)$. In order to prove that $\|u_n - u\|_M \to 0$, we only need to show that $u_n(i) \to u(i)$ for $i = 1, 2, \dots$, by Lemma 2.4.

From $1 \leftarrow \langle u_n, v\rangle = u_n(1)\frac{1}{u(1)}$, we get $u_n(1) \to u(1)$.

When $j \neq 1$, if $\tilde{\alpha}_j > 0$, then $u_n(j) \to 0$ by Lemma 2.5(a); if $\tilde{\alpha}_j = 0$, then $q_j(|v(j)|) = q_j(0) = 0$ by condition 1: $\alpha_j = 0$. In view of Lemma 2.5(b), we get

$$0 \leq \lim_{n\to\infty}\big(|u_n(1)|q_j(|v(j)|) - |u_n(j)|q_1^-(v(1))\big) = -\lim_{n\to\infty}|u_n(j)|q_1^-(v(1)) \leq 0,$$

therefore $u_n(j) \to 0$.

Summing up, we have $u_n(i) \to u(i)$ for $i = 1, 2, \dots$.

Case 2. $\mathrm{Card}(\operatorname{supp} u) > 1$.

Subcase 2-1. $\rho_N(p^-(k|u|)) = 1$. In this case, $v = \{p_i^-(k|u(i)|)\operatorname{sign} u(i)\}_{i=1}^{\infty}$ is the unique support functional for u. In view of conditions 2a and 2c, we have

$$q_i^-(|v(i)|) = q_i^-(p_i^-(k|u(i)|)) = q_i(p_i^-(k|u(i)|)) = q_i(|v(i)|) = k|u(i)|$$

for $i = 1, 2, \dots$. Now, we shall prove that v is a strongly exposed functional for u.

Let $u_n \in S(l_M)$, $n = 1, 2, \ldots$, with $\langle u_n, v\rangle \to 1$ as $n \to \infty$. Since $\theta_N(v) < 1$, by Lemma 2.7,

$$\lim_{i_0\to\infty} \sup_n \sum_{i>i_0} M_i(u_n(i)) = 0,$$

whence,

$$\lim_{i_0\to\infty} \sup_n \sum_{i>i_0} |u_n(i)v(i)| \le \lim_{i_0\to\infty} \Big(\sup_n \sum_{i>i_0} M_i(u_n(i)) + \sum_{i>i_0} N_i(v(i))\Big) = 0. \tag{3}$$

If $u(j) = 0$, taking $i_0 \in \operatorname{supp} u$, by Lemma 2.5(b), we have

$$\begin{aligned} 0 &\le \liminf_{n\to\infty} \big(|u_n(i_0)| q_j(|v(j)|) - |u_n(j)| q_{i_0}^-(|v(i_0)|)\big) \\ &= \liminf_{n\to\infty} \big(|u_n(i_0)| k |u(j)| - |u_n(j)| k |u(i_0)|\big) \\ &= \liminf_{n\to\infty} \big(-|u_n(j)| k |u(i_0)|\big) \le 0. \end{aligned}$$

Consequently, $u_n(j) \to 0$.

If $u(j) \neq 0$, we claim that there exists a constant c satisfying

$$\lim_{n\to\infty} \frac{|u_n(j)|}{|u(j)|} = c.$$

In fact, for any $i, j \in \operatorname{supp} u$, according to Lemma 2.5(b), we have

$$\begin{aligned} &\liminf_{n\to\infty} \big[|u_n(i)| q_j(|v(j)|) - |u_n(j)| q_i^-(|v(i)|)\big] \\ &= \liminf_{n\to\infty} \big[|u_n(i)| k |u(j)| - |u_n(j)| k |u(i)|\big] \ge 0 \end{aligned}$$

and

$$\begin{aligned} &\liminf_{n\to\infty} \big[|u_n(j)| q_i(|v(i)|) - |u_n(i)| q_j^-(|v(j)|)\big] \\ &= \liminf_{n\to\infty} \big[|u_n(j)| k |u(i)| - |u_n(i)| k |u(j)|\big] \ge 0, \end{aligned}$$

i.e., $\limsup_{n\to\infty} \big[|u_n(i)| k |u(j)| - |u_n(j)| k |u(i)|\big] \le 0$. Consequently,

$$\lim_{n\to\infty} \big[|u_n(i)| k |u(j)| - |u_n(j)| k |u(i)|\big] = 0.$$

So, we can get $\lim_{n\to\infty} \frac{|u_n(j)|}{|u(j)|} = c$ (a constant). From

$$1 \leftarrow \langle u_n, v\rangle = \sum_{i=1}^\infty u_n(i)v(i) \le \sum_{i=1}^\infty |u_n(i)||v(i)| \le \|u_n\|_M \|v\|_{(N)} = 1, \tag{4}$$

combining this with (3), we have $1 = \lim_{n\to\infty} \sum_{i=1}^\infty |u_n(i)||v(i)| = c \sum_{i=1}^\infty |u(i)||v(i)| = c$, whence $\lim_{n\to\infty} |u_n(i)| = |u(i)|$ for any $i \in \mathbb{N}$. In order to prove that $\|u_n - u\|_M \to 0$, by Lemma 2.4(ii), it is enough to verify that $\lim_{n\to\infty} u_n(i) = u(i)$ for any $i \in \mathbb{N}$. From (4), we also have $\lim_{n\to\infty} |u_n(i)v(i)| = \lim_{n\to\infty} u_n(i)v(i)$, i.e.,

$$\lim_{n\to\infty} |u_n(i)| p_i^-(k|u(i)|) = \lim_{n\to\infty} u_n(i) p_i^-(k|u(i)|) \operatorname{sign} u(i).$$

Therefore, if $p_i^-(k|u(i)|) \neq 0$, then $\lim_{n\to\infty} u_n(i) = u(i)$; if $p_i^-(k|u(i)|) = 0$, then $u(i) = 0$ by condition 2a and 2c, whence $\lim_{n\to\infty} u_n(i) = 0 = u(i)$. So, for any $i \in \mathbb{N}$, we get $\lim_{n\to\infty} u_n(i) = u(i)$.

Subcase 2-2. $\rho_N(p^-(k|u|)) < 1 < \rho_N(p(k|u|))$. By condition 2b, there is $w \in \mathrm{Grad}(u)$ satisfying $\theta_N(w) < 1$. Choose $\tau > 0$ and $i_0 \in \mathbb{N}$ such that $\sum_{i>i_0} N_i((1+\tau)w(i)) < \infty$ and $(1+\tau)|w(i)| < \tilde{\beta}_i$ $(i > i_0)$. Let us define

$$J = \{i \in \mathbb{N} : p_i^-(k|u(i)|) < p_i(k|u(i)|)\}.$$

In the following, we shall construct $v \in \mathrm{Grad}(u)$ with $p_i^-(k|u(j)|) < |v(j)| < p_j(k|u(j)|)$ for $j \in J$.

I. If $p_i^-(k|u(j)|) < |w(j)| < p_j(k|u(j)|)$ $(j \in J)$, then we put $v = w$.

II. If $|w(j)| = p_j^-(k|u(j)|)$ for some $j \in J$, then we define $J_0 = \{j \in J : |w(j)| = p_j^-(k|u(j)|)\}$; if $|w(j)| = p_j(k|u(j)|)$ for some $j \in J$, then we define $J_0 = \{j \in J : |w(j)| < p_j(k|u(j)|)\}$. It is obvious that $\mathrm{Card}(J \setminus J_0) > 0$.

For $i \in J_0$, $i > i_0$, take $r_i' > 0$ such that

$$\sum_{i>i_0, i\notin J_0} N_i\big((1+\tau)w(i)\big) + \sum_{i>i_0, i\in J_0} N_i\big((1+\tau)(|w(i)| + r_i')\big) < \infty.$$

For $i \in J_0$, take $r_i > 0$ $(r_i \le r_i')$ for $i > i_0$, such that $|w(i)| + r_i < p_i(k|u(i)|)$ and

$$\sum_{i\notin J_0} N_i(w(i)) + \sum_{i\in J_0} N_i(|w(i)| + r_i) < 1 + \Big(\sum_{i\notin J_0} N_i(w(i)) - \sum_{i\notin J_0} N_i\big(p_i^-(k|u(i)|)\big)\Big).$$

Let

$$r = \frac{\sum_{i\notin J_0} N_i(w(i)) + \sum_{i\in J_0} N_i(|w(i)| + r_i) - 1}{\sum_{i\notin J_0} N_i(w(i)) - \sum_{i\notin J_0} N_i(p_i^-(k|u(i)|))}.$$

Then $0 < r < 1$. Put

$$v(i) = \begin{cases} rp_i^-(k|u(i)|)\,\mathrm{sign}\,u(i) + (1-r)w(i) & i \notin J_0 \\ w(i) + r_i\,\mathrm{sign}\,u(i) & i \in J_0. \end{cases}$$

Then $\rho_N(v) = 1$, $\theta_N(v) < 1$ and $v \in \mathrm{Grad}(u)$. Moreover,

$$p_i^-(k|u(j)|) < |v(j)| < p_j(k|u(j)|) \quad (j \in J).$$

By virtue of condition 2a, we can deduce that $q_i^-(|v(i)|) = q_i(|v(i)|) = k|u(i)|$ for each $i \in \mathbb{N}$. The remaining part of the proof in this case is the same as the proof of Subcase 2-1 to get that v is a strongly exposed functional for u.

Subcase 2-3. $\rho_N(p(k|u|)) = 1$. In this case, $v = \{p_i(k|u(i)|)\,\mathrm{sign}\,u(i)\}_{i=1}^{\infty}$ is the unique support functional for u. By conditions 2a, 2c, we also have $q_i^-(|v(i)|) = q_i(|v(i)|) = k|u(i)|$ for each $i \in \mathbb{N}$. With the same method as in the proof of Subcase 2-1, we can derive that v is a strongly exposed functional for u. ■

Acknowledgments. This work was supported by the National Natural Science Foundation (10971129) and SH (J50101).

References

[BHW] S. R. Bian, H. Hudzik, T. F. Wang, *Smooth, very smooth and strongly smooth points in Musielak-Orlicz sequence spaces*, Bull. Austral. Math. Soc. 63 (2001), 441–457.

[CW] L. Y. Cao, T. F. Wang, *Some notes about $K(x) \neq \varnothing$ in Musielak-Orlicz spaces*, Natur. Sci. J. Harbin Normal Univ. 16 (2000), no. 4, 1–4.

[C] S. T. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. (Rozprawy Mat.) 356 (1996).

[HY] H. Hudzik, Y. N. Ye, *Support functional and smoothness in Musielak-Orlicz sequence spaces endowed with the Luxemburg norm*, Comment. Math. Univ. Carolin. 31 (1990), 661–684; correction: ibid. 35 (1994), 209.

[K] A. Kamińska, *Rotundity of Orlicz-Musielak sequence spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (1981), 137–144.

[KR] M. A. Krasnosel'skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

[LWW] M. L. Li, B. X. Wang, T. F. Wang, *Strongly exposed points of Orlicz sequence spaces*, Acta Math. Sin. (Chin. Ser.) 42 (1999), 645–648 (in Chinese).

[L] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. 1 (1963), 139–148.

[LW] X. B. Liu, T. F. Wang, *Extreme points and strongly extreme points of Musielak-Orlicz sequence spaces*, Acta Math. Sin. (Engl. Ser.) 21 (2005), 267–268.

[M] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, Berlin, 1983.

[RR] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*, Monogr. Textbooks Pure Appl. Math. 146, Marcel Dekker, New York, 1991.

[SL] Z. R. Shi, C. Y. Liu, *Exposed points and strongly exposed in Musielak-Orlicz sequence spaces*, Taiwanese J. Math., to appear; preprint.

[SZ] Z. R. Shi, R. J. Zhang, *On smoothness of Orlicz-Musielak sequence spaces*, J. Math. (Wuhan) 9 (1999), 354–360.

[S] S. Straszewicz, *Über exponierte Punkte abgeschlossener Punktmengen*, Fund. Math. 24 (1935), 139–143.

[WB] T. F. Wang, S. R. Bian, *Smooth points and strong (very) smooth points of Musielak-Orlicz spaces with Orlicz norm*, Acta Anal. Funct. Appl. 1 (1999), 61–68.

[WH] T. F. Wang, C. X. Hao, L. F. Liu, *Gateaux differentiable points and Fréchet differentiable points in Musielak-Orlicz sequence spaces endowed with the Orlicz norm*, Acta Math. Sci. Ser. A Chin. Ed. 23 (2003), 145–153 (in Chinese).

[WJS] T. F. Wang, D. H. Ji, Z. R. Shi, *The criteria of strongly exposed points in Orlicz spaces*, Comment. Math. Univ. Carolin. 35 (1994), 721–724.

[WS] C. X. Wu, H. Y. Sun, *Norm calculation and complex convexity of the Musielak-Orlicz sequence spaces*, Chinese Ann. Math. Ser. A 12 (1991) suppl., 98–102.

[ZC] L. Zhao, Y. A. Cui, *Exposed points in Musielak-Orlicz sequence space endowed with the Luxemburg norm*, Natur. Sci. J. Harbin Normal Univ. 21 (2005), no. 4, 3–6.

[ZW] L. Zhao, C. Wu, *Exposed points in Musielak-Orlicz sequence spaces endowed with the Orlicz norm*, Heilongjiang Daxue Ziran Kexue Xuebao 23 (2006), 184–187 (in Chinese).

[ZCH] M. X. Zuo, Y. A. Cui, H. Hudzik, K. J. Zhang, *On the points of local uniform rotundity and weak local uniform rotundity in Musielak-Orlicz sequence spaces equipped with the Orlicz norm*, Nonlinear Anal. 71 (2009), 4906–4915.