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ON THE UNIFORM CONVERGENCE OF WEIGHTED TRIGONOMETRIC SERIES

BOGDAN SZAL

University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a, Poland
E-mail: B.Szal@wmie.uz.zgora.pl

Abstract. In the present paper we consider a new class of sequences called $GM(\beta, r)$, which is the generalization of a class defined by Tikhonov in [15]. We obtain sufficient and necessary conditions for uniform convergence of weighted trigonometric series with (β, r)-general monotone coefficients.

1. Introduction. Chaundy and Jolliffe [1] proved the following classical result (see also [19]).

THEOREM 1. *Suppose that $b_n \geq b_{n+1}$ and $b_n \to 0$. Then a necessary and sufficient condition for the uniform convergence of the series*

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1.1}$$

is $nb_n \to 0$.

This result has been generalized by weakening the monotone conditions of the coefficient sequences. Generally speaking, it has become an important topic how to generalize monotonicity.

For $r \in \mathbb{N}$ and a sequence (c_n), let

$$\Delta_r c_n = c_n - c_{n+r} \quad \text{and} \quad \Delta c_n = \Delta_1 c_n.$$

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Recently, Leindler [4] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBVS$, i.e.,

$$RBVS = \Big\{(c_n) : \sum_{n=m}^{\infty} |\Delta c_n| \le K(c)|c_m| \text{ for all } m \in \mathbb{N}\Big\},$$

where here and throughout the paper $K(c)$ always denote a constant depending on indicated parameters, not necessarily the same in each occurrence.

Denote by MS the class of monotone decreasing sequences and by $CQMS$ the class of classic quasimonotone decreasing sequences ($c \in CQMS$ means that $c_n \in \mathbb{R}_+$ for all $n \in \mathbb{N}$ and there exists an $\alpha > 0$ such that (c_n/n^{α}) is decreasing). It is obvious that

$$MS \subset RBVS \cap CQMS.$$

Leindler [5] proved that the classes $CQMS$ and $RBVS$ are not comparable. In [8] Leindler considered the class

$$MRBVS = \Big\{(c_n) : c_n \in \mathbb{C} \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \sum_{n=m}^{\infty} |\Delta c_n| \le K(c)\frac{1}{m}\sum_{n\ge m/2}^{m} |c_n| \text{ for all } m \in \mathbb{N}\Big\}$$

of mean rest bounded variation sequences. Further, the class of general monotone coefficients, GM, is defined as follows (see [13]):

$$GM = \Big\{(c_n) : c_n \in \mathbb{C} \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \sum_{n=m}^{2m-1} |\Delta c_n| \le K(c)|c_m| \text{ for all } m \in \mathbb{N}\Big\}.$$

It is clear that

$$RBVS \subset MRBVS \quad \text{and} \quad RBVS \cup CQMS \subset GM.$$

Very recently, Le and Zhou [3] suggested the following new class of sequences which includes GM:

$$GBVS = \Big\{(c_n) : c_n \in \mathbb{C} \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \sum_{n=m}^{2m-1} |\Delta c_n| \le K(c) \max_{m\le n\le N+m} |c_n| \text{ for some integer } N \text{ and all } m \in \mathbb{N}\Big\}.$$

The generalization of the Chaundy-Jolliffe criteria (Theorem 1) was studied in many papers: [9] for $CQMS$, [4] for $RBVS$, [10] for $MRBVS$, [13] for GM and [3] for $GBVS$.

In [6, 13, 14, 15] the class of β-general monotone sequences was examined as follows:

Definition 1. Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $c := (c_n)$ is said to be β-*general monotone*, or $c \in GM(\beta)$, if the relation

$$\sum_{n=m}^{2m-1} |\Delta c_n| \le K(c)\beta_m$$

holds for all m.

In [15] and [17] Theorem 1 was generalized to the class $GM(\beta^*)$, with

$$\beta^* := (\beta_n^*), \quad \beta_n^* = \sum_{k=[n/c]}^{[cn]} \frac{|c_k|}{k} \quad \text{for some} \quad c > 1.$$

We can note that (see [15, Remark 2.1])

$$GBVS \subset GM(\beta^*).$$

The next generalization of Theorem 1 was proved by Tikhonov and Dyachenko in [2]. They considered a class $GM(\beta^{\#})$, with

$$\beta^{\#} := (\beta_n^{\#}), \quad \beta_n^{\#} = \frac{1}{n} \max_{k \geq [n/c]} \sum_{s=k}^{2k} |c_s|$$

and showed that

$$GM(\beta^*) \subset GM(\beta^{\#}). \tag{1.2}$$

Moreover, they proved the following theorem.

THEOREM 2. (A) *If*

$$\sum_{k=n}^{\infty} |\Delta b_k| = o(n^{-1})$$

as $n \to \infty$, *then the series* (1.1) *converges uniformly.*

(B) *Let a nonnegative sequence* (b_n) *satisfy*

$$b_n \leq K \cdot \frac{1}{n} \max_{k \geq [n/c]} \sum_{s=k}^{2k} b_s \quad \textit{for some} \quad c > 1.$$

Then the uniform convergence of the series (1.1) *implies* $nb_n = o(1)$ *as* $n \to \infty$.

In order to formulate our new results we define another class of sequences (see [12]).

DEFINITION 2. Let $\beta := (\beta_n)$ be a nonnegative sequence and r a natural number. The sequence of complex numbers $c := (c_n)$ is said to be (β, r)-*general monotone*, or $c \in GM(\beta, r)$, if the relation

$$\sum_{n=m}^{2m-1} |\Delta_r c_n| \leq K(c)\beta_m$$

holds for all m.

It is clear that $GM(\beta, 1) \equiv GM(\beta)$. Moreover, the embedding relation between $GM(\beta, r)$ $(r > 1)$ and $GM(\beta, 1)$ can be deduced from the following remark:

REMARK 1 ([12]). Let r be a natural number. If a nonnegative sequence $\beta := (\beta_n)$ is such that

$$\sum_{i=0}^{r-1} \beta_{n+i} \leq K \cdot \beta_n$$

for all n, then

$$GM(\beta, 1) \subseteq GM(\beta, r).$$

In [12] it was also showed that

$$GM(\beta^*,1)\subseteq GM(\beta^*,r) \quad \text{and} \quad GM(\beta^{\#},1)\subseteq GM(\beta^{\#},r) \tag{1.3}$$

for $r\geq 1$.

2. Statement of the results. We formulate our results in two subsections.

2.1. Uniform convergence of weighted trigonometric series. Let $r\in\mathbb{N}$ and $\alpha\in\mathbb{R}$. We define an even 2π-periodic function $\omega_{\alpha,r}$, given on the interval $[0,\pi]$ by the formula

$$\omega_{\alpha,r}(x):=\begin{cases}\left(x-\frac{2l\pi}{r}\right)^{\alpha} & \text{for } x\in\left(\frac{2l\pi}{r},\frac{(2l+1)\pi}{r}\right] \text{ and } l\in U_1,\\ \left(\frac{2(l+1)\pi}{r}-x\right)^{\alpha} & \text{for } x\in\left(\frac{(2l+1)\pi}{r},\frac{2(l+1)\pi}{r}\right) \text{ and } l\in U_2,\\ 0 & \text{for } x=\frac{2l\pi}{r} \text{ and } l\in U_3,\end{cases}$$

where $U_1=\{0,1,\dots,[r/2]\}$ if r is an odd number and $U_1=\{0,1,\dots,[r/2]-1\}$ if r is even, $U_2=\{0,1,\dots,[r/2]-1\}$ for $r\geq 2$, and $U_3=\{0,1,\dots,[r/2]\}$ for $r\geq 1$.

THEOREM 3. *Let $r\in\mathbb{N}$ and $\alpha\in(0;1]$. If*

$$\sum_{k=n}^{\infty}|\Delta_r b_k|=o(n^{\alpha-1}) \tag{2.1}$$

as $n\to\infty$, then the series

$$\sum_{k=1}^{\infty}b_k\omega_{\alpha,r}(x)\sin kx \tag{2.2}$$

converges uniformly, and if

$$\sum_{k=n}^{\infty}|\Delta_r a_k|=o(n^{\alpha-1})$$

as $n\to\infty$, then the series

$$\sum_{k=1}^{\infty}a_k\omega_{\alpha,r}(x)\cos kx \tag{2.3}$$

is also uniformly convergent.

THEOREM 4. *Let $r=1$ or 2 and $\alpha\in(-1;0]$. If (2.1) holds then the series (2.2) converges uniformly.*

THEOREM 5. *Let $r\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and a nonnegative sequence (b_n) satisfy*

$$b_n\leq K\sum_{k=[n/c]}^{[cn]}\frac{b_k}{k} \quad \textit{for some} \quad c>1. \tag{2.4}$$

Then the uniform convergence of the series (2.2) implies

$$n^{1-\alpha}b_n=o(1) \tag{2.5}$$

as $n\to\infty$.

Similarly we can show the following theorem.

THEOREM 5'. *Let $r \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and a nonnegative sequence (a_n) satisfy*

$$a_n \le K \sum_{k=[n/c]}^{[cn]} \frac{a_k}{k} \quad \text{for some} \quad c > 1.$$

Then the uniform convergence of the series (2.3) implies

$$n^{1-\alpha} a_n = o(1) \tag{2.6}$$

as $n \to \infty$.

THEOREM 6. *Let $r \in \mathbb{N}$, $\alpha \le 0$ and a nonnegative sequence (b_n) satisfy*

$$b_n \le K \cdot \frac{1}{n} \max_{k \ge [n/c]} \sum_{s=k}^{2k} b_s \quad \text{for some} \quad c > 1. \tag{2.7}$$

Then the uniform convergence of the series (2.2) implies (2.5).

PROPOSITION 1. (i) *If a nonnegative sequence $(c_n) \in GM(\beta^*, r)$, where $r \in \mathbb{N}$, $\alpha < 1$ and $n^{1-\alpha} c_n = o(1)$ as $n \to \infty$, then*

$$n^{1-\alpha} \sum_{k=n}^{\infty} |\Delta_r c_k| = o(1) \tag{2.8}$$

as $n \to \infty$.

(ii) *If a nonnegative sequence $(c_n) \in GM(\beta^{\#}, r)$, where $r \in \mathbb{N}$, $\alpha \le 0$ and $n^{1-\alpha} c_n = o(1)$ as $n \to \infty$, then (2.8) holds.*

We conclude this subsection with a few remarks and corollary.

REMARK 2. If we take $r = 1$ and $\alpha = 0$ in Theorems 4 and 6, then we obtain the result of M. Dyachenko and S. Tikhonov (Theorem 2).

Combining the above results we derive the following corollary.

COROLLARY 1. (i) *Let $r \in \mathbb{N}$, $\alpha \in (0;1)$ and a nonnegative sequence $(b_n) \in GM(\beta^*, r)$. Then the series (2.2) converges uniformly if and only if (2.5) holds.*

(ii) *Let $r \in \mathbb{N}$, $\alpha \in (0;1)$ and a nonnegative sequence $(a_n) \in GM(\beta^*, r)$. Then the series (2.3) converges uniformly if and only if (2.6) holds.*

(iii) *Let $r = 1$ or 2, $\alpha \in (-1;0]$ and a nonnegative sequence $(b_n) \in GM(\beta^{\#}, r)$. Then the series (2.2) converges uniformly if and only if (2.5) holds.*

REMARK 3. If we take $r = 1$ and $\alpha = 0$ in Corollary 1 (iii), then the result of M. Dyachenko and S. Tikhonov [2, Corollary 5.3] follows from our Corollary 1 (iii).

2.2. The weighted best approximation. Let $\gamma \in C$. Denote by $E_n(\varphi, \gamma)$ the best approximation of a function φ, where $\gamma \cdot \varphi \in C$, by trigonometric polynomials of degree at most n in the weighted C-norm, that is,

$$E_n(\varphi, \gamma) := \inf_{P_n \in \Pi_n} \|\gamma(\varphi - P_n)\| \quad (E_n(\varphi) := E_n(\varphi, 1)),$$

where Π_n denotes the set of all trigonometric polynomials of degree at most n.

Write

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx, \quad g(x) = \sum_{k=1}^{\infty} b_k \sin kx$$

for those x where the series converge. Denote by ϕ either f or g and let λ_n be its associated coefficients, i.e., λ_n is either a_n or b_n.

THEOREM 7. *Let $r \in \mathbb{N}$, $\alpha \in (0;1]$. If*

$$\sum_{k=n}^{\infty} |\Delta_r \lambda_k| = o(n^{\alpha-1})$$

as $n \to \infty$, then

$$E_n(\phi, \omega_{\alpha,r}) \leq K \max_{v \geq n} \Big(v^{1-\alpha} \sum_{k=v}^{\infty} |\Delta_r \lambda_k| \Big).$$

Analyzing the proofs of Theorem 4 and 7 we get the following corollary.

COROLLARY 2. *If*

$$\sum_{k=n}^{\infty} |\Delta_2 b_k| = o(n^{-1})$$

as $n \to \infty$, then

$$E_n(g) \leq K \max_{v \geq n} \Big(v \sum_{k=v}^{\infty} |\Delta_2 \lambda_k| \Big).$$

Using Proposition 1, Theorem 7 and Corollary 2 we can derive the next corollary.

COROLLARY 3. (i) *Let $r \in \mathbb{N}$, $\alpha \in (0;1)$ and a nonnegative sequence $(\lambda_n) \in GM(\beta^*, r)$. If $n^{1-\alpha}\lambda_n = o(1)$ as $n \to \infty$ then*

$$E_n(\phi, \omega_{\alpha,r}) \leq K \max_{v \geq n} (v^{1-\alpha} \lambda_v).$$

(ii) *If a nonnegative sequence $(b_n) \in GM(\beta^{\#}, 2)$ and $nb_n = o(1)$ as $n \to \infty$, then*

$$E_n(g) \leq K \max_{v \geq n} (v b_v).$$

Finally, we have the following remark.

REMARK 4. By the embedding relations (1.2) and (1.3) we can observe that the result of S. Tikhonov [15] follows from our Corollary 3 (ii).

3. Auxiliary results. Denote, for $r \in \mathbb{Z}$, by

$$D_{k,r}(x) = \frac{\sin(k + \frac{r}{2})x}{2 \sin \frac{rx}{2}}$$

and

$$\widetilde{D}_{k,r}(x) = \frac{\cos(k + \frac{r}{2})x}{2 \sin \frac{rx}{2}}$$

the Dirichlet type kernels.

LEMMA 1 ([11]). *Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$ and (c_n) be a sequence of complex numbers. If $x \neq \frac{2l\pi}{r}$, then for all $m \geq n$*

$$\sum_{k=n}^{m} c_k \cos kx = \sum_{k=n}^{m} \Delta_r c_k D_{k,r}(x) - \sum_{k=m+1}^{m+r} c_k D_{k,-r}(x) + \sum_{k=n}^{n+r-1} c_k D_{k,-r}(x) \tag{3.1}$$

and

$$\sum_{k=n}^{m} c_k \sin kx = -\sum_{k=n}^{m} \Delta_r c_k \widetilde{D}_{k,r}(x) + \sum_{k=m+1}^{m+r} c_k \widetilde{D}_{k,-r}(x) - \sum_{k=n}^{n+r-1} c_k \widetilde{D}_{k,-r}(x). \tag{3.2}$$

4. Proofs of the main results

4.1. Proof of Theorem 3. We prove the theorem for sine series, only. In the case of cosine series we can argue analogously.

Let

$$\varepsilon_n := \sup_{v\geq n}\Big(v^{1-\alpha}\sum_{k=v}^{\infty}|\Delta_r b_k|\Big)$$

and

$$R_n(x) := \sum_{k=n}^{\infty} b_k \omega_{\alpha,r}(x)\sin kx.$$

In view of the assumptions, we see that $\varepsilon_n \to 0$ as $n\to\infty$. Further, we shall show that

$$|R_n(x)| \leq K\varepsilon_n \tag{4.1}$$

for all $x \in \mathbb{R}$. Since $R_n(\frac{2l\pi}{r}) = 0$, where $l \in \mathbb{Z}$, it suffices to prove (4.1) for $x \in (\frac{2l\pi}{r}; \frac{2(l+1)\pi}{r})$, where $l = 0, 1, \ldots, [r/2]-1$ if r is even, and $l = 0, 1, \ldots, [r/2]$ if r is odd.

First, we show that (4.1) is valid for $x \in (\frac{2l\pi}{r}; \frac{2l\pi}{r} + \frac{\pi}{r}]$. Let $N := N(x)$ be the natural number such that

$$\frac{2l\pi}{r} + \frac{\pi}{N+1} < x \leq \frac{2l\pi}{r} + \frac{\pi}{N}\,. \tag{4.2}$$

Then

$$R_n(x) = \sum_{k=n}^{n+N-1} b_k\omega_{\alpha,r}(x)\sin kx + \sum_{k=n+N}^{\infty} b_k\omega_{\alpha,r}(x)\sin kx = R_n^{(1)}(x) + R_n^{(2)}(x). \tag{4.3}$$

Further, by (4.2) we obtain that for $\alpha \in (0;1]$

$$\begin{aligned}\big|R_n^{(1)}(x)\big| &\leq \Big(x - \frac{2l\pi}{r}\Big)^{\alpha}\sum_{k=n}^{n+N-1} b_k \leq KN^{-\alpha}\sum_{k=n}^{n+N-1}\sum_{v=k}^{\infty}|\Delta_r b_v| \\ &\leq K\varepsilon_n N^{-\alpha}\sum_{k=n}^{n+N-1} k^{\alpha-1} \leq K\varepsilon_n N^{-\alpha}\big((n+N)^{\alpha} - n^{\alpha}\big) \leq K\varepsilon_n. \end{aligned}\tag{4.4}$$

Using Lemma 1, the inequality

$$\frac{r}{\pi}x - 2l \leq \Big|\sin\frac{rx}{2}\Big| \quad \text{for } x \in \Big[\frac{2l\pi}{r}, \frac{2l\pi}{r} + \frac{\pi}{r}\Big] \tag{4.5}$$

and (4.2) we get for $\alpha \le 1$

$$\begin{aligned}\left|R_n^{(1)}(x)\right| &= \left(x-\frac{2l\pi}{r}\right)^{\alpha}\left|\sum_{k=n+N}^{n+N+r-1} b_k\widetilde{D}_{k,-r}(x)+\sum_{k=n+N}^{\infty}\Delta_r b_k\widetilde{D}_{k,r}(x)\right|\\ &\le K\frac{(x-\frac{2l\pi}{r})^{\alpha}}{2|\sin\frac{rx}{2}|}\Big\{\sum_{k=n+N}^{\infty}|\Delta_r b_k|+\sum_{k=n+N}^{n+N+r-1} b_k\Big\}\le K\Big(x-\frac{2l\pi}{r}\Big)^{\alpha-1}\sum_{k=n+N}^{\infty}|\Delta_r b_k|\\ &\le KN^{1-\alpha}\sum_{k=n+N}^{\infty}|\Delta_r b_k|\le K\varepsilon_n\Big(\frac{N}{n+N}\Big)^{1-\alpha}\le K\varepsilon_n. \qquad (4.6)\end{aligned}$$

Now, we prove that (4.1) is valid for $x\in[\frac{2l\pi}{r}+\frac{\eta}{r};\frac{2(l+1)\pi}{r})$. Let $M:=M(x)\ge r$ be the natural number such that

$$\frac{2(l+1)\pi}{r}-\frac{\pi}{M}\le x<\frac{2(l+1)\pi}{r}-\frac{\pi}{M+1}\,. \qquad (4.7)$$

Then

$$R_n(x)=\sum_{k=n}^{n+M-1} b_k\omega_{\alpha,r}(x)\sin kx+\sum_{k=n+M}^{\infty} b_k\omega_{\alpha,r}(x)\sin kx=R_n^{(3)}(x)+R_n^{(4)}(x). \qquad (4.8)$$

Using (4.2) we obtain that for $\alpha\in(0;1]$

$$\begin{aligned}\left|R_n^{(4)}(x)\right| &\le\Big(\frac{2(l+1)\pi}{r}-x\Big)^{\alpha}\sum_{k=n}^{n+M-1} b_k\le KM^{-\alpha}\sum_{k=n}^{n+M-1}\sum_{v=k}^{\infty}|\Delta_r b_v|\\ &\le K\varepsilon_n M^{-\alpha}\sum_{k=n}^{n+M-1}k^{\alpha-1}\le K\varepsilon_n. \qquad (4.9)\end{aligned}$$

Applying Lemma 1, the inequality

$$2(l+1)-\frac{r}{\pi}x\le\left|\sin\frac{rx}{2}\right|\quad\text{for } x\in\Big[\frac{2l\pi}{r}+\frac{\pi}{r},\frac{2(l+1)\pi}{r}\Big] \qquad (4.10)$$

and (4.2) we get for $\alpha\le 1$

$$\begin{aligned}\left|R_n^{(4)}(x)\right| &=\Big(\frac{2(l+1)\pi}{r}-x\Big)^{\alpha}\left|\sum_{k=n+M}^{n+M+r-1} b_k\widetilde{D}_{k,-r}(x)+\sum_{k=n+M}^{\infty}\Delta_r b_k\widetilde{D}_{k,r}(x)\right|\\ &\le K\frac{(\frac{2(l+1)\pi}{r}-x)^{\alpha}}{2|\sin\frac{rx}{2}|}\Big\{\sum_{k=n+M}^{\infty}|\Delta_r b_k|+\sum_{k=n+M}^{n+M+r-1} b_k\Big\}\\ &\le K\Big(\frac{2(l+1)\pi}{r}-x\Big)^{\alpha-1}\sum_{k=n+M}^{\infty}|\Delta_r b_k|\\ &\le KM^{1-\alpha}\sum_{k=n+M}^{\infty}|\Delta_r b_k|\le K\varepsilon_n\Big(\frac{M}{n+M}\Big)^{1-\alpha}\le K\varepsilon_n. \qquad (4.11)\end{aligned}$$

From the estimations (4.3), (4.4), (4.6), (4.8), (4.9), and (4.11) the uniform convergence of the series (2.2) follows and thus the proof is complete. ∎

4.2. Proof of Theorem 4. The proof of Theorem 4 goes analogously as the proof of Theorem 3. The only difference is that instead of (4.4) (for $r = 1, 2$) and (4.9) (for $r = 2$) we have to use the following considerations.

Applying the inequalities $|\sin kx| \le kx$ for $x \in (0, \pi)$, $|\sin kx| \le k(\pi - x)$ for $x \in (0, \pi)$ we obtain that for $\alpha \in (-1; 0]$

$$\left|R_n^{(1)}(x)\right| \le x^{\alpha+1} \sum_{k=n}^{n+N-1} kb_k \le KN^{-\alpha-1} \sum_{k=n}^{n+N-1} \sum_{v=k}^{\infty} |\Delta_r b_v|$$
$$\le K\varepsilon_n N^{-\alpha-1} \sum_{k=n}^{n+N-1} k^{\alpha} \le K\varepsilon_n N^{-\alpha-1}\big((n+N)^{\alpha+1} - n^{\alpha+1}\big) \le K\varepsilon_n$$

and

$$\left|R_n^{(4)}(x)\right| \le (\pi - x)^{\alpha+1} \sum_{k=n}^{n+M-1} kb_k \le KM^{-\alpha-1} \sum_{k=n}^{n+M-1} \sum_{v=k}^{\infty} |\Delta_r b_v|$$
$$\le K\varepsilon_n M^{-\alpha-1} \sum_{k=n}^{n+M-1} k^{\alpha-1} \le K\varepsilon_n.$$

These estimates complete the proof. ■

4.3. Proof of Theorem 5. Suppose that the series (2.2) converges uniformly. Setting $x = \frac{\pi}{4cm}$, where $c > 1$, $4cm \ge r$ and $r \in \mathbb{N}$, we get

$$\sum_{n=[m/c]}^{[cm]} b_n \omega_{\alpha,r}(x) \sin nx = \left(\frac{\pi}{4cm}\right)^{\alpha} \sum_{n=[m/c]}^{[cm]} b_n \sin \frac{n\pi}{4cm} \ge Km^{-\alpha} \sum_{n=[m/c]}^{[cm]} b_n.$$

Hence

$$m^{-\alpha} \sum_{n=[m/c]}^{[cm]} b_n = o(1) \quad \text{as} \quad m \to \infty.$$

If (b_n) satisfies (2.4) then

$$n^{1-\alpha} b_n \le Kn^{1-\alpha} \sum_{k=[n/c]}^{[cn]} \frac{b_k}{k} \le Kn^{-\alpha} \sum_{k=[n/c]}^{[cn]} \frac{b_k}{k}$$

and $n^{1-\alpha} b_n = o(1)$ as $n \to \infty$. This finishes the proof. ■

4.4. Proof of Theorem 6. Suppose that the series (2.2) converges uniformly. Setting $x = \frac{\pi}{4m}$, where $4m \ge r$ and $r \in \mathbb{N}$, we get

$$\sum_{n=m}^{2m} b_n \omega_{\alpha,r}(x) \sin nx = \left(\frac{\pi}{4m}\right)^{\alpha} \sum_{n=m}^{2m} b_n \sin \frac{n\pi}{4m} \ge \left(\frac{\pi}{4}\right)^{\alpha} \sin \frac{\pi}{4} m^{-\alpha} \sum_{n=m}^{2m} b_n.$$

Hence

$$m^{-\alpha} \sum_{n=m}^{2m} b_n = o(1) \quad \text{as} \quad m \to \infty.$$

If (b_n) satisfies (2.7), then for $c>1$ and $\alpha\le 0$

$$n^{1-\alpha}b_n\le Kn^{-\alpha}\max_{k\ge[n/c]}\Big(\sum_{s=k}^{2k}b_s\Big)\le K\max_{k\ge[n/c]}\Big(k^{-\alpha}\sum_{s=k}^{2k}b_s\Big).$$

Therefore $n^{1-\alpha}b_n=o(1)$ as $n\to\infty$ and it completes the proof. ∎

4.5. Proof of Proposition 1. (i) Let $n^{1-\alpha}c_n=o(1)$ as $n\to\infty$. Then

$$n^{-\alpha}\sum_{k=[n/c]}^{[cn]}c_k\le K\sum_{k=[n/c]}^{[cn]}k^{-\alpha}c_k\le K\sup_{k\ge[n/c]}(k^{1-\alpha}c_k)\sum_{k=[n/c]}^{[cn]}\frac{1}{k}\le K\sup_{k\ge[n/c]}(k^{1-\alpha}c_k).$$

Hence

$$n^{-\alpha}\sum_{k=[n/c]}^{[cn]}c_k=o(1)\quad\text{as}\quad n\to\infty.$$

Further, if $(c_n)\in GM(\beta^*,r)$, then for $\alpha<1$

$$\begin{aligned}n^{1-\alpha}\sum_{k=n}^{\infty}|\Delta_r c_k|&=n^{1-\alpha}\sum_{s=0}^{\infty}\sum_{k=2^s n}^{2^{s+1}n-1}|\Delta_r c_k|\le Kn^{1-\alpha}\sum_{s=0}^{\infty}\sum_{k=[2^s n/c]}^{[c2^s n]}\frac{c_k}{k}\\&\le Kn^{1-\alpha}\sum_{s=0}^{\infty}\frac{1}{(2^s n)^{1-\alpha}}\sup_{m\ge 2^s n}\Big(m^{-\alpha}\sum_{k=[m/c]}^{[cm]}c_k\Big)\\&\le K\sup_{m\ge n}\Big(m^{-\alpha}\sum_{k=[m/c]}^{[cm]}c_k\Big)\sum_{s=0}^{\infty}\Big(\frac{1}{2^{1-\alpha}}\Big)^s\le K\sup_{m\ge n}\Big(m^{-\alpha}\sum_{k=[m/c]}^{[cm]}c_k\Big)\end{aligned}$$

and (2.8) holds.

(ii) Suppose $n^{1-\alpha}c_n=o(1)$ as $n\to\infty$. Then

$$n^{-\alpha}\sum_{k=n}^{2n}c_k\le K\sup_{k\ge n}(k^{1-\alpha}c_k)\sum_{k=n}^{2n}\frac{1}{k}\le K\sup_{k\ge n}(k^{1-\alpha}c_k).$$

Hence

$$n^{-\alpha}\sum_{k=n}^{2n}c_k=o(1)\quad\text{as}\quad n\to\infty.$$

If $(c_n)\in GM(\beta^{\#},r)$, then for $\alpha\le 0$ we get

$$\begin{aligned}n^{1-\alpha}\sum_{k=n}^{\infty}|\Delta_r c_k|&=n^{1-\alpha}\sum_{s=0}^{\infty}\sum_{k=2^s n}^{2^{s+1}n-1}|\Delta_r c_k|\le Kn^{1-\alpha}\sum_{s=0}^{\infty}\frac{1}{2^s n}\max_{k\ge[2^s n/c]}\Big(\sum_{s=k}^{2k}c_s\Big)\\&\le Kn^{-\alpha}\max_{k\ge[n/c]}\Big(\sum_{s=k}^{2k}c_s\Big)\sum_{s=0}^{\infty}\frac{1}{2^s}\le K\max_{k\ge[n/c]}\Big(k^{-\alpha}\sum_{s=k}^{2k}c_s\Big).\end{aligned}$$

Thus (2.8) is also valid. ∎

4.6. Proof of Theorem 7. We prove the theorem for the case when $\phi(x) = g(x)$, only. The case when $\phi(x) = f(x)$ can be proved similarly.

First, using the usual argument

$$E_n(\phi, \omega_{\alpha,r}) = \inf_{P_n \in \Pi_n} \|\omega_{\alpha,r}(\phi - P_n)\| \leq \|\omega_{\alpha,r}(\phi - S_n)\| = \|R_{n+1}\|,$$

where P_n is a trigonometric polynomial of degree n,

$$S_n = \sum_{k=1}^{n} b_k \sin kx$$

and

$$R_{n+1}(x) = \sum_{k=n+1}^{\infty} b_k \omega_{\alpha,r}(x) \sin kx.$$

Further, we will show that

$$\|R_{n+1}\| \leq K \max_{v \geq n} \Big(v^{1-\alpha} \sum_{k=v}^{\infty} |\Delta_r b_k| \Big). \tag{4.12}$$

Since $R_{n+1}(\frac{2l\pi}{r}) = 0$, where $l \in \mathbb{Z}$, it suffices to prove (4.1) for $x \in (\frac{2l\pi}{r}; \frac{2(l+1)\pi}{r})$, where $l = 0, 1, \dots, [r/2] - 1$ if r is an even number, and $l = 0, 1, \dots, [r/2]$ if r is an odd number.

First, we show that (4.12) is valid for $x \in (\frac{2l\pi}{r}; \frac{2l\pi}{r} + \frac{\pi}{r}]$. Let $x \in (\frac{2l\pi}{r}; \frac{2l\pi}{r} + \frac{\pi}{r}]$ and $j := [\pi/(x - \frac{2l\pi}{r})]$. Then, using Lemma 1 and the inequality (4.5), we obtain that for $\alpha \in (0; 1]$

$$\begin{aligned} |R_{n+1}(x)| &\leq K \Big(j^{-\alpha} \sum_{k=n}^{n+j-1} b_k + j^{1-\alpha} \sum_{k=n+j}^{\infty} |\Delta_r b_k| \Big) \\ &\leq K \Big(\max_{n \leq k \leq n+j} (k^{1-\alpha} b_k) + \Big(\frac{j}{n+j}\Big)^{1-\alpha} \max_{v \geq n+j} \Big(v^{1-\alpha} \sum_{k=v}^{\infty} |\Delta_r b_k| \Big) \Big) \\ &\leq K \max_{v \geq n} \Big(v^{1-\alpha} \sum_{k=v}^{\infty} |\Delta_r b_k| \Big). \end{aligned}$$

Now, we prove that (4.12) is valid for $x \in [\frac{2l\pi}{r} + \frac{\pi}{r}; \frac{2(l+1)\pi}{r})$.

Let $x \in [\frac{2l\pi}{r} + \frac{\pi}{r}; \frac{2(l+1)\pi}{r})$ and $i = [\pi/(\frac{2(l+1)\pi}{r} - x)]$. Applying Lemma 1 and the inequality (4.10) we get, for $\alpha \in (0; 1]$,

$$|R_{n+1}(x)| \leq K \Big(i^{-\alpha} \sum_{k=n}^{n+i-1} b_k + i^{1-\alpha} \sum_{k=n+i}^{\infty} |\Delta_r b_k| \Big) \leq K \max_{v \geq n} \Big(v^{1-\alpha} \sum_{k=v}^{\infty} |\Delta_r b_k| \Big).$$

Collecting the above estimates, we arrive at (4.12). The proof is now complete. ∎

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