ON SOME PROPERTIES FOR DUAL SPACES OF MUSIELAK-ORLICZ FUNCTION SPACES

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Abstract. We will present relationships between the modular $\rho^*$ and the norm in the dual spaces $(L_\Phi)^*$ in the case when a Musielak-Orlicz space $L_\Phi$ is equipped with the Orlicz norm. Moreover, criteria for extreme points of the unit sphere of the dual space $(L_0^\Phi)^*$ will be presented.

1. Introduction. The triple $(T, \Sigma, \mu)$ stands for a positive, nonatomic, $\sigma$-finite and complete measure space. By $L^0 = L^0(\mu)$ we denote the space of all (equivalence classes of) $\Sigma$-measurable real functions $x$ defined on $T$. A mapping $\Phi : T \times \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be a Musielak-Orlicz function if it satisfies the Carathéodory conditions, i.e. for any $u \in \mathbb{R}$, the function $\Phi(\cdot, u)$ is $\Sigma$-measurable and there is a set $T_0 \in \Sigma$ with $\mu(T_0) = 0$ such that for any $t \in T \setminus T_0$ the function $\Phi(t, \cdot)$ is an Orlicz function, i.e. it is convex, even, vanishing at zero and satisfying $\Phi(t, u)/u \rightarrow +\infty$ as $u \rightarrow +\infty$.

For every Musielak-Orlicz function $\Phi$ we define its complementary function in the sense of Young $\Psi : T \times \mathbb{R} \rightarrow [0, \infty)$ by the formula

$$\Psi(t, v) = \sup_{u>0} \{u|v| - \Phi(t, u)\}$$

for every $v \in \mathbb{R}$ and $t \in T$. Given any Musielak-Orlicz function $\Phi$ define on $L^0$ a convex modular $I_\Phi$ by

$$I_\Phi(x) = \int_T \Phi(t, x(t)) \, d\mu$$

for every $x \in L^0$. Then the Musielak-Orlicz function space $L_\Phi$ and its subspace $E_\Phi$ are

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defined as follows:

\[ L_{\Phi} = \{ x \in L^0 : I_{\Phi}(\lambda x) < +\infty \text{ for some } \lambda > 0 \} , \]
\[ E_{\Phi} = \{ x \in L^0 : I_{\Phi}(\lambda x) < +\infty \text{ for any } \lambda > 0 \} . \]

It is easy to see that \( E_{\Phi} \) is the subspace of order continuous elements in \( L_{\Phi} \).

The spaces \( L_{\Phi} \) and \( E_{\Phi} \) coincide if and only if \( \Phi \) satisfies the so-called \( \Delta_2 \)-condition. Recall that \( \Phi \) satisfies the \( \Delta_2 \)-condition (\( \Phi \in \Delta_2 \) for short), if there are a set \( T_0 \) of measure zero, a constant \( K > 0 \) and a \( \Sigma \)-measurable nonnegative function \( h \) defined on \( T \) such that \( \int_T h(t) \, d\mu < +\infty \) and \( \Phi(t, 2u) \leq K \Phi(t, u) + h(t) \) for every \( t \in T \setminus T_0 \) and \( u \in \mathbb{R} \).

For any \( t \in T \setminus T_0 \), by \( p(t, u) \) and \( q(t, u) \) we denote the right derivatives of \( \Phi(t, \cdot) \) and \( \Psi(t, \cdot) \) at any fixed point \( u \in \mathbb{R} \), respectively. For every \( u, v \in \mathbb{R} \) and all \( t \in T \setminus T_0 \), we have the following Young inequality

\[ uv \leq \Phi(t, u) + \Psi(t, v), \]

and for a given \( t \in T \setminus T_0 \) the equality \( uv = \Phi(t, u) + \Psi(t, v) \) holds whenever \( u \in \mathbb{R} \) and \( v \in [q(t, u), p(t, u)] \).

For any \( x \in L_{\Phi} \) the Luxembury norm is defined by

\[ ||x|| = \inf \{ k > 0 : I_{\Phi}(x/k) \leq 1 \}, \]

and the Orlicz norm is defined by

\[ ||x||^0 = \sup \left\{ \int_T x(t) y(t) \, d\mu : I_{\Psi}(y) \leq 1 \right\}. \]

Let us note that the Orlicz norm on \( L_{\Phi} \) can be also defined by the very useful Amemiya formula [F]:

\[ ||x||^0 = \inf_{k > 0} \frac{1}{k} (1 + I_{\Phi}(kx)). \]

**Lemma 1.1** (see [K]). Let \( \Phi \) be a Musielak-Orlicz function. Then there exists an ascending sequence \( (T_n)_{n=1}^\infty \) of measurable sets with \( 0 < \mu(T_n) < +\infty \) such that \( \sup_{t \in T_n} \Phi(t, \lambda) < +\infty \) for every \( \lambda > 0 \), for any \( n \in \mathbb{N} \) and \( \mu(T \setminus \bigcup_{n=1}^\infty T_n) = 0 \).

This yields that \( \chi_{T_n} \) (the characteristic function of \( T_n \)) belongs to \( E_{\Phi} \) for any \( n \in \mathbb{N} \).

We denote by \( L_{\Phi}^* \) the dual space of \( L_{\Phi} \) and \( \varphi \in L_{\Phi}^* \) is called a singular functional (\( \varphi \in F \) for short), if \( \varphi(E_{\Phi}) = \{0\} \), that is, \( \varphi(x) = 0 \) for any \( x \in E_{\Phi} \).

Any functional \( f \in L_{\Phi}^* \) has the unique decomposition

\[ f = v + \varphi \quad (v \in L_{\Psi}, \, \varphi \in F), \]

where \( v \) means in fact the regular functional defined by the function \( v \) from \( L_{\Psi} \) by the formula \( \langle v, x \rangle = \int_T v(t) x(t) \, d\mu \) for any \( x \in L_{\Phi} \).

Let us define for each \( f \in L_{\Phi}^* : \)

\[ ||f||_\Psi = ||f|| = \sup \{ f(u) : ||u|| = 1 \}, \quad ||f||_\Psi^0 = ||f|| = \sup \{ f(u) : ||u||^0 = 1 \}. \]
The following results are due to H. Hudzik and Z. Zbąszyniak [HZ]:

**Lemma 1.2.** Let \( f \in L^*_\Phi \). Then
\[
\|f\|^o = \|v\|^o + \|\varphi\|^o,
\]
\[
\|f\| = \inf\{\lambda > 0 : I_\Phi(v/\lambda) + \|\varphi\|/\lambda \leq 1\}.
\]

**Lemma 1.3.** For any \( \varphi \in F \),
\[
\|\varphi\| = \|\varphi\|^o = \sup\{\varphi(u) : I_\Phi(u) < +\infty\}.
\]

### 2. Result

**Proposition 2.1.** Let \( f \in L^*_\Phi \). If \( \|f\|^o \leq 1 \) then

(i) \( I_\Phi(q(|v|)) \leq 1 \),

(ii) \( \rho^*(f) \leq \|f\|^o \).

**Proof.** By virtue of Lemma 1.1 we can repeat the proof of Proposition 2.2 in [CHL] with the sequence of sets \((T_n)_{n=1}^\infty\) from Lemma 1.1. 

**Proposition 2.2.** Let \( \Phi \) be a Musielak-Orlicz function satisfying condition \( \Phi(t,u)/u \to 0 \) as \( u \to 0 \) for \( \mu \)-a.e. \( t \in T \). Then the convergence to zero in the Orlicz norm \( \|\cdot\|^o \) and in the modular in \( L^*_\Phi \) are equivalent if and only if \( \Psi \in \Delta_2 \).

**Proof.** We can repeat here the proof of the necessity of Proposition 2.3 from [CHL].

**Sufficiency.** Let \( f_n \in L^*_\Phi \), where \( f_n = v_n + \varphi_n \) for any \( n \in \mathbb{N} \) and \( \rho^*(f_n) = I_\Psi(v_n) + \|\varphi_n\| \to 0 \). Then \( I_\Phi(v_n) \to 0 \) and \( \|\varphi_n\| \to 0 \). Since \( \Psi \in \Delta_2 \) and \( \Psi \) vanishes only at zero, we can deduce from Theorem 3.3 in [KH] that \( \|v_n\|^o \to 0 \), and consequently \( \|f_n\|^o = \|v_n\|^o + \|\varphi_n\|^o \to 0 \). 

The proofs of the next three propositions and of Proposition 2.6 (2)-(5) can proceed analogously as the respective proofs in [CHL].

**Proposition 2.3.** Let \( f \in L^*_\Phi \). If there exists \( k > 0 \) such that
\[
\int_T \Phi(t,q(t,k|v(t)|)) \, d\mu = 1,
\]
then \( \|f\|^o = \int_T |v(t)| q(t,k|v(t)|) \, d\mu + \|\varphi\| = \frac{1}{k}(1 + \rho^*(k)) \).

**Proposition 2.4.** If \( f \in L^*_\Phi \), then
\[
\|f\|^o = \inf_{k>0} \frac{1}{k}(1 + \rho^*(k)).
\]

**Proposition 2.5.** If \( f \in L^*_\Phi \), then

(1) \( \|f\| \leq 1 \implies \rho^*(f) \leq \|f\| \),

(2) \( \|f\| > 1 \implies \rho^*(f) > \|f\| \),

(3) \( \|f\| \leq \|f\|^o \leq 2\|f\| \).

**Proposition 2.6.** Suppose \( \Psi \in \Delta_2 \), \( \Phi(t,u)/u \to 0 \) as \( u \to 0 \) for \( \mu \)-a.e. \( t \in T \) and \( f_n, f \in L^*_\Phi \), where \( f_n = v_n + \varphi_n \), \( f = v + \varphi \), \( (v_n, v \in L_\Psi, \varphi_n, \varphi \in F) \). Then

(1) \( \rho^*(f_n) \to \infty \implies \|f_n\| \to \infty \),

(2) \( \|f\| = 1 \implies \rho^*(f) = 1 \),
\(\forall \varepsilon > 0 \exists \delta > 0 \ (\|f\| \geq \varepsilon \implies \rho^{*}(f) \geq \delta)\),

(4) \(\forall \varepsilon \in (0, 1) \exists \delta \in (0, 1) \ (\rho^{*}(f) \leq 1 - \varepsilon \implies \|f\| \leq 1 - \delta)\),

(5) \(\forall \varepsilon \in (0, 1) \exists \delta \in (0, 1) \ (\rho^{*}(f) \geq 1 + \varepsilon \implies \|f\| \geq 1 + \delta)\).

**Proof.** Let us prove implication (1). Suppose that \(\Psi \in \Delta_2\), where the function \(h\) is defined on \(T\) and \(\Psi(t, 2u) \leq K\Psi(t, u) + h(t)\) for every \(t \in T \setminus T_0\) with \(\mu(T_0) = 0\) and \(u \in \mathbb{R}\). We have for a constant \(L > 1\)

\[
\Psi(t, Lv) \leq \tilde{K}\Psi(t, v) + \tilde{h}(t)
\]

for all \(t \in T \setminus T_0\) and \(v \in \mathbb{R}\) with a positive constant \(\tilde{K}\) and a nonnegative integrable function \(\tilde{h}\).

Then \(\|f\| \leq L\) implies that \(\|f/L\| \leq 1\), whence

\[
\rho^{*}(f/L) = I_{\Psi}(v/L) + \|\varphi/L\| \leq 1
\]

and so

\[
I_{\Psi}(v/L) \leq 1 \text{ and } \|\varphi/L\| \leq 1.
\]

In consequence, setting \(M = \int_{T} \tilde{h}(t) \, d\mu\), we obtain

\[
\rho^{*}(f) = I_{\Psi}(v) + \|\varphi\| = \int_{T} \Psi(t, \frac{1}{L}Lv(t)) \, d\mu + \|\varphi\|
\]

\[
\leq \int_{T} \left[ \tilde{K}\Psi(t, \frac{1}{L}v(t)) + \tilde{h}(t) \right] \, d\mu + \|\varphi\|
\]

\[
= \tilde{K}I_{\Psi}(v/L) + \int_{T} \tilde{h}(t) \, d\mu + \|\varphi\| \leq \tilde{K} + M + L.
\]

By the transposition law this finishes the proof of (1).  

**Proposition 2.7.** If the Musielak-Orlicz function \(\Phi\) satisfies the condition \(\Phi(t, u)/u \to 0\) as \(u \to 0\) for \(\mu\)-a.e. \(t \in T\) and \(\Psi \in \Delta_2\), then for any \(L > 0\) and \(\varepsilon > 0\), there exists \(\delta > 0\) such that for all \(f, g \in L^*_\Phi\) with \(\rho^{*}(f) \leq L\) and \(\rho^{*}(g) \leq \delta\) the inequality \(|\rho^{*}(f + g) - \rho^{*}(f)| < \varepsilon\) is satisfied.

**Proof.** Since \(\Psi(t, u)\) vanishes only at zero for \(\mu\)-a.e. \(t \in T\), by virtue of Lemma 1.6 in [H] there exists a set \(A \in \Sigma\) with \(\mu(A) = 0\) such that for any \(\varepsilon > 0\) there exist a function \(h_{\varepsilon}(\cdot) \geq 0\), \(\int_{T} h_{\varepsilon}(t) \, d\mu \leq \varepsilon\) and a constant \(M_{\varepsilon} \geq 2\), such that

\[
\Psi(t, 2u) \leq M_{\varepsilon}\Psi(t, u) + h_{\varepsilon}(t)
\]

for any \(u \in \mathbb{R}\) and \(t \in T \setminus A\). Therefore

\[
\rho^{*}(2f) = I_{\Psi}(2v) + 2\|\varphi\| = \int_{T} \Psi(t, 2v(t)) \, d\mu + 2\|\varphi\|
\]

\[
\leq M_{\varepsilon}\int_{T} \Psi(t, v(t)) \, d\mu + \int_{T} h_{\varepsilon}(t) \, d\mu + 2\|\varphi\|
\]

\[
\leq M_{\varepsilon}I_{\Psi}(v) + \varepsilon + M_{\varepsilon}\|\varphi\|
\]

\[
= M_{\varepsilon}(I_{\Psi}(v) + \|\varphi\|) + \varepsilon = M_{\varepsilon}\rho^{*}(f) + \varepsilon,
\]

which means that the modular \(\rho^{*}\) satisfies in \(L^*_\Phi\) the condition \(\Delta_2\) defined in [CH]. By Lemma 2.1 in [CH] the proof is complete.  

Theorem 2.8. Let $\Phi$ be a Musielak-Orlicz function such that $\Phi(t,u)/u \to 0$ as $u \to 0$ for $\mu$-a.e. $t \in T$. Then a functional $\varphi \in S(F)$ is an extreme point of $B(L^\infty_\Phi)$ if and only if $\|\varphi\| = \|\varphi|_{T \setminus E}\| = 0$ for every $E \in \Sigma$.

Proof. The proof proceeds in the same way as the proof of Theorem 3.1 in [CHL].

Theorem 2.9. A functional $f = v + \varphi \in S(L^\infty_\Phi)$ is an extreme point of $B(L^\infty_\Phi)$ if and only if the following conditions are satisfied:

1. $\rho^*(f) = 1$,
2. $v(t)$ is a point of strict convexity of $\Psi$ for $\mu$-a.e. $t \in T$,
3. $\varphi/\|\varphi\|$ is an extreme point of $B(L^\infty_\Phi)$.

Proof. The sufficiency follows in the same way as the sufficiency of Theorem 3.2 in [CHL].

Necessity. Let $f = v + \varphi \in \text{Ext } B(L^\infty_\Phi)$ and let us assume that condition (1) is not satisfied. If $\varepsilon = 1 - \rho^*(f) > 0$, then we can choose $E \in \Sigma$ such that

$$0 < \int_E \Psi(t,2v(t)) \, d\mu \leq \varepsilon.$$

Define

$$v_1(t) = \begin{cases} v(t) & \text{for } t \in T \setminus E \\ 0 & \text{for } t \in E \end{cases} \quad \text{and} \quad v_2(t) = \begin{cases} v(t) & \text{for } t \in T \setminus E \\ 2v(t) & \text{for } t \in E. \end{cases}$$

Then $v_1 \neq v_2$ and $v_1 + v_2 = 2v$. Defining $f_1 = v_1 + \varphi$ and $f_2 = v_2 + \varphi$, we have

$$\rho^*(f_1) = I_\Psi(v_1) + \|\varphi\| < I_\Psi(v_2) + \|\varphi\| = \rho^*(f_2)$$

$$< I_\Psi(v) + \varepsilon + \|\varphi\| = \rho^*(f) + \varepsilon = 1,$$

whence $\|f_1\| \leq 1$ and analogously $\|f_2\| \leq 1$. We have $1 = \|f\| = \|(f_1 + f_2)/2\| \leq \frac{1}{2}(\|f_1\| + \|f_2\|) \leq 1$ and consequently $\|f_1\| = \|f_2\| = 1$, which contradicts the assumption that $f \in \text{Ext } B(L^\infty_\Phi)$.

Assume now that condition (2) is not satisfied. Then there exist a set $A \in \Sigma$ with $\mu(A) > 0$ and two numbers $a, b$ with $0 < a < b < \infty$ such that $a < v(t) < b$ and $\Phi(t,\cdot)$ is affine on $[a,b]$ for all $t \in A$. Further there exist $\varepsilon > 0$ and $K \in \Sigma$ with $K \subset A$ and $\mu(K) > 0$ such that $a + \varepsilon < v(t) < b - \varepsilon$ for all $t \in K$. Let us write $\Psi(t,u)$ for $(t,u) \in K \times [a,b]$ in the form $\Psi(t,u) = \alpha(t)u + \beta(t)$ with $\alpha(t) > 0$, $\beta(t) > 0$ for $\mu$-a.e. $t \in K$.

Next, let us define on $\Sigma \cap K$ the measure

$$\mu_\alpha(B) = \int_B \alpha(t) \, d\mu \quad (\forall B \in \Sigma \cap K).$$

This is an atomless measure, so there exist two sets $K_1, K_2 \in \Sigma \cap K$ such that $K_1 \cap K_2 = \emptyset$, $K_1 \cup K_2 = K$ and $\mu_\alpha(K_1) = \mu_\alpha(K_2)$. This means that

$$\int_{K_1} \alpha(t) \, d\mu = \int_{K_2} \alpha(t) \, d\mu.$$
Let us define two functions $v_1$ and $v_2$ by

$$v_1(t) = \begin{cases} v(t) & \text{for } t \in T \setminus (K_1 \cup K_2) \\ v(t) - \varepsilon & \text{for } t \in K_1 \\ v(t) + \varepsilon & \text{for } t \in K_2, \end{cases} \quad v_2(t) = \begin{cases} v(t) & \text{for } t \in T \setminus (K_1 \cup K_2) \\ v(t) + \varepsilon & \text{for } t \in K_1 \\ v(t) - \varepsilon & \text{for } t \in K_2. \end{cases}$$

Then $v_1 \neq v_2$ and $v_1 + v_2 = 2v$. Let us define $f_1 = v_1 + \varphi$, $f_2 = v_2 + \varphi$. Then

$$I_\Psi(v_1) = \int_{T \setminus K} \Psi(t, v(t)) \, d\mu + \int_{K_1} [\alpha(t)(v(t) - \varepsilon) + \beta(t)] \, d\mu$$

$$+ \int_{K_2} [\alpha(t)(v(t) + \varepsilon) + \beta(t)] \, d\mu$$

$$= \int_{T \setminus K} \Psi(t, v(t)) \, d\mu + \int_{K} [\alpha(t)v(t) + \beta(t)] \, d\mu = \int_{T} \Psi(t, v(t)) \, d\mu = I_\Psi(v),$$

which gives

$$\rho^*(f_1) = I_\Psi(v_1) + \|\varphi\| = I_\Psi(v) + \|\varphi\| = \rho^*(f) = 1.$$

Similarly we deduce that $\rho^*(f_2) = 1$, whence it follows that $\|f_1\| = \|f_2\| = 1$. Since $f_1 \neq f_2$, this contradicts the assumption that $f$ is an extreme point of $B(L^\Phi_\varphi)$.

If (3) does not hold, then there exist $\varphi_1, \varphi_2 \in S(F)$, $\varphi_1 \neq \varphi_2$ such that $2\|\varphi\| = \varphi_1 + \varphi_2$.

Let $\varphi'_1 = \|\varphi\| \varphi_1$, $\varphi'_2 = \|\varphi\| \varphi_2$. Then $\|\varphi'_1\| = \|\varphi'_2\| = \|\varphi\|$. Defining $f_1 = v + \varphi'_1$, $f_2 = v + \varphi'_2$, we have $\rho^*(f_1) = I_\Psi(v) + \|\varphi'_1\| = I_\Psi(v) + \|\varphi\| = \rho^*(f) = 1$ and similarly $\rho^*(f_2) = 1$, which contradicts the assumption that $f \in \text{Ext } B(L^\Phi_\varphi)$. ■

References


