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CYCLIC COHOMOLOGY OF (EXTENDED) HOPF ALGEBRAS

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Abstract. We review recent progress in the study of cyclic cohomology of Hopf algebras, extended Hopf algebras, invariant cyclic homology, and Hopf-cyclic homology with coefficients, starting with the pioneering work of Connes-Moscovici.

1. Introduction. It is well known that the theory of characteristic classes of vector bundles, more precisely the Chern character, can be extended to noncommutative geometry, thanks to the noncommutative Chern-Weil theory of Connes [4, 7, 5]. In order to have a similar theory for Hopf-Galois extensions (algebraic quantum principal bundles), one would like to have appropriate analogues of group and Lie algebra cohomology for Hopf algebras. The recent work of Connes-Moscovici [11, 9, 8] on the index theory of transversely elliptic operators, more precisely their definition of cyclic cohomology of Hopf algebras, provides one with such a theory.

It is the goal of the present article to review the developments in the study of cyclic cohomology of Hopf algebras, starting with the pioneering work of Connes-Moscovici [11, 9, 8]. We will present a dual cyclic theory for Hopf algebras, first defined in [23], and independently in [35]. One motivation to introduce this theory was that, as observed by M. Crainic [12], cyclic cohomology of cosemisimple Hopf algebras, e.g. the algebra of polynomial functions on a compact quantum group, due to the existence of a normalized Haar integral, is always trivial. In other words, cyclic cohomology of Hopf algebras, as originally defined in [11], behaves in much the same way as continuous group cohomology which is also trivial for compact topological groups.

Let $HP^\bullet$ and $\widetilde{HP}_\bullet$ denote the resulting periodic cyclic (co)homology groups in the sense of [11] and [23], respectively. We present two very general results: for any commutative Hopf algebra $\mathcal{H}$, $HP^\bullet(\mathcal{H})$ decomposes into direct sums of Hochschild cohomology groups of the coalgebra $\mathcal{H}$ with trivial coefficients, and for any cocommutative $\mathcal{H}$, $\widetilde{HP}_\bullet(\mathcal{H})$

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decomposes into direct sums of Hochschild homology groups of the algebra $\mathcal{H}$ with trivial coefficients. So far very few examples of computations of $HP^\bullet$ and $\widetilde{HP}_\bullet$ for quantum groups are known. We present what is known in Sections 3 and 4.

In Section 5 we review the main results on cyclic cohomology of extended Hopf algebras known so far, following [8, 22]. Extended Hopf algebras are closely related to Hopf algebroids. The problem is that although Hopf algebroids, as defined in [27], are generalizations of Hopf algebras, it seems to be impossible to define a cyclic module for them. Thus one should first define an appropriate variation of the notion of Hopf algebroids and then define a cyclic cohomology theory for them. This is achieved in [22] and the resulting class of algebras are called extended Hopf algebras to distinguish them from Hopf algebroids. It seems that now the question of finding an appropriate algebraic framework to define cyclic cohomology of Hopf algebroids is settled by [22].

In Section 6 we present some of the results obtained in [1] on cyclic cohomology of smash products.

Cyclic (co)homology of Hopf algebras can be understood from two distinct points of view. The first view, due to Connes and Moscovici [9, 10, 11], is based on the existence of a characteristic map for (co)actions of Hopf algebras on algebras (see the introductory remarks in Section 4 for more on this). In the second point of view, first advocated in [21], cyclic (co)homology of Hopf algebras appears as a special case of a more general theory called invariant cyclic homology. We review this theory in Section 7. It turns out that the invariant cyclic homology of Hopf algebra is isomorphic to its Hopf algebraic cyclic homology. This is remarkably similar to interpreting the cohomology of the Lie algebra of a Lie group as invariant de Rham cohomology of its Lie group as is done by Chevalley and Eilenberg [3].

An important question left open in our paper [21] was the issue of identifying the most general type of coefficients allowable in cyclic homology of Hopf algebras and invariant cyclic homology in general. This problem is now completely solved, among other things, in [18]. It is shown in this paper that the most general coefficients are the class of so called stable anti-Yetter-Drinfeld modules. In Section 7 we briefly report on this very recent development as well.

It was not our intention to cover all aspects of this new branch of noncommutative geometry in this paper. For applications to transverse index theory and for the whole theory one should consult the original Connes-Moscovici articles [11, 10, 8] as well as their review article [9]. We also recommend [36] for a general introduction to applications of Hopf algebras in noncommutative geometry.

Much remains to be done in this area. For example, the relation between cyclic homology of Hopf algebras and developments in Hopf-Galois theory (see e.g. Montgomery's book [30]) remain to be explored. In this regard we should mention the recent article [20] which deals with computing the relative cyclic homology of a Hopf-Galois extension in terms of cyclic homology of Hopf algebras. As far as computation of cyclic (co)homology of quantum groups is concerned what is missing is a general conjecture about the nature of Hopf-cyclic homology of the algebra of polynomial functions (or smooth functions, provided they are defined) on quantum groups and its relation with intrinsic invariants of quantum groups.

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2. Preliminaries on Hopf algebras. In this paper algebra means an associative, not necessarily commutative, unital algebra over a fixed commutative ground ring k. Similar convention applies to *coalgebras*, *bialgebras* and *Hopf algebras*. The undecorated tensor product $\otimes$ means the tensor product over k. If $\mathcal{H}$ is a Hopf algebra, we denote its coproduct by $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, its counit by $\epsilon : \mathcal{H} \to k$, its unit by $\eta : k \to \mathcal{H}$ and its antipode by $S : \mathcal{H} \to \mathcal{H}$. We will use Sweedler's notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$, $(\Delta \otimes id)\Delta(h) = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$, etc., where summation is understood.

If $\mathcal{H}$ is a Hopf algebra, the word $\mathcal{H}$-*module* means a module over the underlying algebra of $\mathcal{H}$. Similarly, an $\mathcal{H}$-*comodule* is a comodule over the underlying coalgebra of $\mathcal{H}$. For a left (resp. right) $\mathcal{H}$-comodule M we write $\rho(m) = m^{(-1)} \otimes m^{(0)}$ (resp. $\rho(m) = m^{(0)} \otimes m^{(1)}$), where summation is understood, to denote the coaction $\rho : M \to \mathcal{H} \otimes M$ (resp. $\rho : M \to M \otimes \mathcal{H}$). The same convention applies to $\mathcal{H}$-bimodules and $\mathcal{H}$-bicomodules. The category of (left) $\mathcal{H}$-modules has a tensor product defined via the coproduct of $\mathcal{H}$: if M and N are left $\mathcal{H}$-modules, their tensor product $M \otimes N$ is again an $\mathcal{H}$-module via

$$h(m \otimes n) = h^{(1)} m \otimes h^{(2)} n.$$

Similarly, if M and N are left $\mathcal{H}$-comodules, the tensor product $M \otimes N$ is again an $\mathcal{H}$-comodule via

$$\rho(m \otimes n) = m^{(-1)} n^{(-1)} \otimes m^{(0)} \otimes n^{(0)}.$$

We take the point of view, standard in noncommutative geometry, that a noncommutative space is encoded by an algebra or by a coalgebra. The idea of *symmetry*, i.e. action of a group on a space, can be expressed by the action/coaction of a Hopf algebra on an algebra/coalgebra. Thus four possibilities arise. Let $\mathcal{H}$ be a Hopf algebra. An algebra A is called a left $\mathcal{H}$-*module algebra* if it is a left $\mathcal{H}$-module and the multiplication map $A \otimes A \to A$ and the unit map are morphisms of $\mathcal{H}$-modules. That is,

$$h(ab) = h^{(1)}(a) h^{(2)}(b), \qquad h(1) = \epsilon(h)1,$$

for $h \in \mathcal{H}, a, b \in A$. Similarly an algebra A is called an $\mathcal{H}$-*comodule algebra* if A is a left $\mathcal{H}$-comodule and the multiplication and the unit maps are morphisms of $\mathcal{H}$-comodules. In a similar fashion an $\mathcal{H}$-*module coalgebra* is a coalgebra C which is a left $\mathcal{H}$-module, and the comultiplication $\Delta : C \to C \otimes C$ and the counit map are $\mathcal{H}$-module maps. Finally an $\mathcal{H}$-comodule coalgebra is a coalgebra C which is an $\mathcal{H}$-comodule and the coproduct and counit map are comodule maps.

The *smash product* $A \# \mathcal{H}$ of an $\mathcal{H}$-module algebra A with $\mathcal{H}$ is, as a k-module, $A \otimes \mathcal{H}$ with the product

$$(a \otimes g)(b \otimes h) = a(g^{(1)} b) \otimes g^{(2)} h.$$

It is an associative algebra under the above product.

EXAMPLES. 1. For $\mathcal{H} = U(\mathfrak{g})$, the enveloping algebra of a Lie algebra $\mathfrak{g}$, A is an $\mathcal{H}$-module algebra iff $\mathfrak{g}$ acts on A by derivations, i.e. we have a Lie algebra map $\mathfrak{g} \to Der(A)$.

2. For $\mathcal{H} = kG$, the group algebra of a (discrete) group G, A is an $\mathcal{H}$-module algebra iff G acts on A via automorphisms $G \to Aut(A)$. The smash product $A \# \mathcal{H}$ is then isomorphic to the crossed product algebra $A \rtimes G$.

3. For any Hopf algebra $\mathcal{H}$, the algebra $A = \mathcal{H}$ is an $\mathcal{H}$-comodule algebra where the coaction is afforded by the comultiplication map $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. Similarly, the coalgebra $\mathcal{H}$ is an $\mathcal{H}$-module coalgebra where the action is given by the multiplication map $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. These are analogues of the action of a group on itself by translations.

4. By a theorem of Kostant [34], any cocommutative Hopf algebra $\mathcal{H}$ over an algebraically closed field of characteristic zero is isomorphic (as a Hopf algebra) with a smash product $\mathcal{H} = U(P(\mathcal{H})) \# kG(\mathcal{H})$, where $P(\mathcal{H})$ is the Lie algebra of primitive elements of $\mathcal{H}$ and $G(\mathcal{H})$ is the group of all grouplike elements of $\mathcal{H}$ and $G(\mathcal{H})$ acts on $P(\mathcal{H})$ by inner automorphisms $(g, h) \mapsto ghg^{-1}$, for $g \in G(\mathcal{H})$ and $h \in P(\mathcal{H})$.

3. Cyclic modules. Cyclic (co)homology was first defined for (associative) algebras through explicit complexes or bicomplexes. Soon after, Connes introduced the notion of cyclic module and defined cyclic homology of cyclic modules [5]. The motivation was to define cyclic homology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not an additive category, the standard (abelian) homological algebra is not enough. In Connes' approach, the category of cyclic modules appears as "abelianization" of the category of algebras with the embedding defined by the functor $A \mapsto A^\natural$, explained below. For an alternative approach one can consult ([16]), where cyclic cohomology is shown to be the nonabelian derived functor of the functor of traces on A. It was soon realized that cyclic modules and the flexibility they afford are indispensable tools in the theory. A recent example is the cyclic homology of Hopf algebras which can not be defined as the cyclic homology of an algebra or coalgebra.

In this section we recall the theory of cyclic and paracyclic modules and their cyclic homologies. We also consider the doubly graded version, i.e., biparacyclic modules and the generalized Eilenberg-Zilber theorem [5, 16, 17].

For $r \geq 1$ an integer or $r = \infty$, let Λ^r denote the r-cyclic category. An *r-cyclic object* in a category $\mathcal{C}$ is a contravariant functor $\Lambda^r \to \mathcal{C}$. Equivalently, we have a sequence $X_n, n \geq 0$, of objects of $\mathcal{C}$ and morphisms called face, degeneracy and cyclic operators

$$\delta_i : X_n \to X_{n-1}, \quad \sigma_i : X_n \to X_{n+1}, \quad \tau : X_n \to X_n, \qquad 0 \leq i \leq n,$$

such that (X, δ_i, σ_i) is a simplicial object and the following extra relations are satisfied:

$$\begin{aligned}
\delta_i \tau &= \tau \delta_{i-1}, && 1 \leq i \leq n, \\
\delta_0 \tau &= \delta_n, \\
\sigma_i \tau &= \tau \sigma_{i-1}, && 1 \leq i \leq n, \\
\sigma_0 \tau &= \tau^2 \sigma_n, \\
\tau^{r(n+1)} &= \mathrm{id}_n.
\end{aligned}$$

For $r = \infty$, the last relation is replaced by the empty relation and we have a *paracyclic* object. For $r = 1$, a Λ^1 object is a *cyclic object*.

A cocyclic object is defined in a dual manner. Thus a cocyclic object in $\mathcal{C}$ is a covariant functor $\Lambda^1 \to \mathcal{C}$. Let k be a commutative ground ring. A cyclic module over k is a cyclic object in the category of k-modules. We denote the category of cyclic k-modules by Λ_k.

Next, let us recall that a *biparacyclic* object in a category $\mathcal{C}$ is a contravariant functor $\Lambda^\infty \times \Lambda^\infty \to \mathcal{C}$. Equivalently, we have a doubly graded set of objects $X_{n,m}$, $n, m \geq 0$ in $\mathcal{C}$ with horizontal and vertical face, degeneracy and cyclic operators $\delta_i, \sigma_i, \tau, d_i, s_i, t$ such that each row and each column is a paracyclic object in $\mathcal{C}$ and vertical and horizontal operators commute. A biparacyclic object X is called *cylindrical* if the operators $\tau^{m+1}, t^{n+1} : X_{m,n} \to X_{m,n}$ are inverse of each other. If X is cylindrical then it is easy to see that its *diagonal*, $d(X)$, defined by $d(X)_n = X_{n,n}$ with face, degeneracy and cyclic maps $\delta_i d_i$, $\sigma_i s_i$ and τt is a cyclic object.

We give a few examples of cyclic modules that will be used in this paper. The first example is the most fundamental example which motivated the whole theory.

1. Let A be an algebra. The cyclic module $A^\natural$ is defined by $A^\natural_n = A^{\otimes(n+1)}, n \geq 0$, with the face, degeneracy and cyclic operators defined by

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\
\delta_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}, \\
\sigma_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes \ldots \otimes a_n, \\
\tau(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_n \otimes a_0 \ldots \otimes a_{n-1}.
\end{aligned}$$

The underlying simplicial module of $A^\natural$ is a special case of the following simplicial module. Let M be an A-bimodule. Let $C_n(A, M) = M \otimes A^{\otimes n}, \quad n \geq 0$. For $n = 0$, we put $C_0(A, M) = M$. Then the following faces and degeneracies δ_i, σ_i define a simplicial module structure on $C_\bullet(A, M)$:

$$\begin{aligned}
\delta_0(m \otimes a_1 \otimes \ldots \otimes a_n) &= m a_1 \otimes a_2 \otimes \ldots \otimes a_n, \\
\delta_i(m \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\
\delta_n(m \otimes a_1 \otimes \ldots \otimes a_n) &= a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}, \\
\sigma_0(m \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes 1 \otimes a_1 \otimes \ldots \otimes a_n, \\
\sigma_i(m \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes a_1 \otimes \ldots \otimes a_i \otimes 1 \otimes \ldots \otimes a_n \quad 1 \leq i \leq n.
\end{aligned}$$

Obviously, for $M = A$ we obtain $A^\natural$. In general, there is no cyclic structure on $C_\bullet(A, M)$.

2. Let C be a coalgebra. The cocyclic module $C_\natural$ is defined by $C^n_\natural = C^{\otimes n+1}, \quad n \geq 0$, with coface, codegeneracy and cyclic operators:

$$\begin{aligned}
\delta_i(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_0 \otimes \ldots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_n, \quad 0 \leq i \leq n \\
\delta_{n+1}(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_0^{(2)} \otimes c_1 \otimes \ldots \otimes c_n \otimes c_0^{(1)}, \\
\sigma_i(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_0 \otimes \ldots c_i \otimes \varepsilon(c_{i+1}) \otimes \ldots \otimes c_n, \quad 0 \leq i \leq n-1, \\
\tau(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_1 \otimes c_2 \otimes \ldots \otimes c_n \otimes c_0,
\end{aligned}$$

where as usual $\Delta(c) = c^{(1)} \otimes c^{(2)}$ (Sweedler's notation). The underlying cosimplicial module for $C_\natural$ is a special case of the following cosimplicial module. Let M be a C-bicomodule and $C^n(C, M) = M \otimes C^{\otimes n}$. The following coface and codegeneracy operators define a cosimplicial module:

$$\begin{aligned}\delta_0(m \otimes c_1 \otimes \ldots \otimes c_n) &= m^{(0)} \otimes m^{(1)} \otimes c_1 \ldots \otimes c_n,\\ \delta_i(m \otimes c_1 \otimes \ldots \otimes c_n) &= m \otimes c_1 \otimes \ldots \otimes c_i^{(0)} \otimes c_i^{(1)} \otimes c_n, \quad 1 \leq i \leq n,\\ \delta_{n+1}(m \otimes c_1 \otimes \ldots \otimes c_n) &= m_{(0)} \otimes c_1 \otimes \ldots \otimes c_n \otimes m_{(-1)},\\ \sigma_i(m \otimes c_1 \otimes \ldots \otimes c_n) &= m \otimes c_1 \ldots \varepsilon(c_{i+1}) c_i \otimes \ldots \otimes c_n, \quad 0 \leq i \leq n-1,\end{aligned}$$

where we have denoted the left and right comodule maps by $\Delta_l(m) = m_{(-1)} \otimes m_{(0)}$ and $\Delta_r(m) = m^{(0)} \otimes m^{(1)}$. Let

$$d = \sum_{i=0}^{n+1} (-1)^i \delta_i : C^n(C, M) \to C^{n+1}(C, M).$$

Then $d^2 = 0$. The cohomology of the complex $(C^\bullet(C, M), d)$ is the Hochschild cohomology of the coalgebra C with coefficients in the bicomodule M. For $M = C$, we obtain the Hochschild complex of $C_\natural$. Another special case occurs with $M = k$ and $\Delta_r : k \to k \otimes C \cong C$ and $\Delta_l : k \to C \otimes k \cong C$, are given by $\Delta_r(1) = 1 \otimes g$ and $\Delta_l(1) = h \otimes 1$, where $g, h \in C$ are grouplike elements. The differential $d : C^n \to C^{n+1}$ in the latter case is given by

$$\begin{aligned}d(c_1 \otimes c_2 \otimes \ldots \otimes c_n) &= g \otimes c_1 \otimes \ldots \otimes c_n\\ &+ \sum_{i=1}^{n} (-1)^i c_1 \otimes \ldots \otimes \Delta(c_i) \otimes \ldots \otimes c_n + (-1)^{n+1} c_1 \otimes \ldots \otimes c_n \otimes h.\end{aligned}$$

3. Let $g : A \to A$ be an automorphism of an algebra A. The paracyclic module $A_g^\natural$ is defined by $A_{g,n}^\natural = A^{\otimes(n+1)}$ with the same cyclic structure as $A^\natural$, except the following changes:

$$\begin{aligned}\delta_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= g(a_n) a_0 \otimes \ldots \otimes a_{n-1},\\ \tau(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= g(a_n) \otimes a_0 \otimes \ldots \otimes a_{n-1}.\end{aligned}$$

One can check that $A_g^\natural$ is a Λ^∞-module and if $g^r = id$, then it is a Λ^r-module. For $g = id$, we obtain example 1.

Next, let us indicate how one defines the Hochschild, cyclic and periodic cyclic homology of a cyclic module. This is particularly important since the cyclic homology of Hopf algebras is naturally defined as the cyclic homology of some cyclic modules associated with them. Given a cyclic module $M \in \Lambda_k$, its *cyclic homology group* $HC_n(M)$, $n \geq 0$, is defined in ([5]) by

$$HC_n(M) := Tor_n^{\Lambda_k}(M, k^\natural),$$

and similarly the *cyclic cohomology groups* of M are defined by

$$HC^n(M) := Ext^n_{\Lambda_k}(M, k^\natural).$$

Using a specific projective resolution for $k^\natural$, namely $k^\natural \leftarrow k^\natural \leftarrow \cdots$ where the differentials are zero and identity, one obtains the following bicomplex to compute cyclic homology. Given a cyclic module M, consider the following first quadrant bicomplex, called the *cyclic bicomplex* of M:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
M_2 & \xleftarrow{1-\tau} & M_2 & \xleftarrow{N} & M_2 & \xleftarrow{1-\tau} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_1 & \xleftarrow{1+\tau} & M_1 & \xleftarrow{N} & M_1 & \xleftarrow{1+\tau} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_0 & \xleftarrow{1-\tau} & & \xleftarrow{N} & M_0 M_0 & \xleftarrow{1-\tau} & \cdots
\end{array}
$$

We denote this bicomplex by $CC^+(M)$. The operators b, b' and N are defined by

$$
\begin{aligned}
b &= \sum_{i=0}^{n} (-1)^i \delta_i, \\
b' &= \sum_{i=0}^{n-1} (-1)^i \delta_i, \\
N &= \sum_{i=0}^{n} (-1)^{ni} \tau^i.
\end{aligned}
$$

Using the simplicial and cyclic relations, one can check that $b^2 = b'^2 = 0$, $b(1-(-1)^n\tau) = (1-(-1)^{n-1}\tau)b'$ and $b'N = Nb'$. The *Hochschild homology* of M, denoted $H_\bullet(M)$, is the homology of the first column $(M_\bullet, b)$. The *cyclic homology* of M, denoted by $HC_\bullet(M)$ is the homology of the total complex $TotCC^+(M)$.

To define the *periodic cyclic* homology of M, we extend the first quadrant bicomplex $CC^+(M)$ to the left and denote it by $CC(M)$. Let $TotCC(M)$ denote the " total complex" where instead of direct sums we use direct product,

$$
TotCC(M)_n = \prod_{i=0}^{\infty} M_i.
$$

It is obviously a 2-periodic complex and its homology is called the periodic cyclic homology of M and denoted by $HP_\bullet(M)$.

The complex $(M_\bullet, b')$ is acyclic with contracting homotopy $\sigma_{-1} = \tau\sigma_n$. One can then show that $CC^+(M)$ is homotopy equivalent to Connes's (b, B) bicomplex

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\
\downarrow b & & \downarrow b & & \\
M_1 & \xleftarrow{B} & M_0 & & \\
\downarrow b & & & & \\
M_0 & & & &
\end{array}
$$

where $B : M_n \to M_{n+1}$ is Connes's boundary operator defined by $B = (1-(-1)^n\tau)\sigma_{-1}N$.

Finally we arrive at the 3rd definition of cyclic homology by noticing that if k is a field of characteristic zero, then the rows of $CC^+(M)$ are acyclic in positive degree and its homology in dimension zero is

$$C_n^\lambda(M) = \frac{M_n}{(1-(-1)^n\tau)M_n}.$$

It follows that the total homology, i.e. cyclic homology of M can be computed, if k is a field of characteristic zero, as the homology of Connes's cyclic complex $(C_\bullet^\lambda(M), b)$

Now, if A is an associative algebra, its Hochschild, cyclic and periodic cyclic homology, are defined as the corresponding homology of the cyclic module $A^\natural$. We denote these groups by $HH_\bullet(A)$, $HC_\bullet(A)$ and $HP_\bullet(A)$, respectively. More generally, we denote the Hochschild homology of A with coefficients in a bimodule M by $H_\bullet(A, M)$. Similarly, if C is a coalgebra, its Hochschild, cyclic and periodic cyclic cohomology are defined as the corresponding homology of the cocyclic module $C_\natural$.

Our next goal is to recall the generalized Eilenberg-Zilber theorem for cylindrical modules from [17, 24]. This is needed in Section 6 to derive a spectral sequence for cyclic homology of smash products.

A *parachain complex* $(M_\bullet, b, B)$ is a chain complex $(M_\bullet, b)$ endowed with a map $B : M_\bullet \to M_{\bullet+1}$ such that $B^2 = 0$ and $T = 1 - (bB + Bb)$ is an invertible operator. For example, a *mixed complex* is a parachain complex such that $bB + Bb = 0$. Given a mixed complex M one can define its (b, B)-bicomplex as the Connes' (b, B) bicomplex. One can thus define the Hochschild, cyclic and periodic cyclic homology of mixed complexes. The definition of a *bi-parachain complex* should be clear. It is a double complex where each row and each column is a parachain complex and all vertical operators commute with all horizontal operators. Given a bi-parachain complex $X_{p,q}$, one defines its total complex $TotX$ by

$$(TotX)_n = \bigoplus_{p+q=n} X_{p,q}, \quad b = b_v + b_h, \quad B = B_v + TB_h,$$

where v and h refer to the horizontal and vertical differentials, respectively. One can check that $TotX$ is a parachain complex [17].

Now if X is a cylindrical module and $C(X)$ is the bi-parachain complex obtained by forming the associated mixed complexes horizontally and vertically, then one can check that $Tot(C(X))$ is indeed a mixed complex. On the other hand we know that the diagonal $d(X)$ is a cyclic module and hence its associated chain complex $C(d(X))$ is a mixed complex.

The following theorem was first proved in [17] using topological arguments. A purely algebraic proof can be found in [23]. The operator f_1 is called the cyclic shuffle, and u is a formal variable to keep track of the degree of cochains in the total complex.

THEOREM 3.1 ([17, 23]). *Let X be a cylindrical module. There is a quasi-isomorphism of mixed complexes $f_0 + uf_1 : Tot(C(X)) \to C(d(X))$ such that f_0 is the shuffle map.*

4. Cyclic cohomology of Hopf algebras. Thanks to the recent work of Connes-Moscovici [11, 10, 8], the following principle has emerged. A reasonable co/homology theory for Hopf algebras and Hopf algebra like objects in noncommutative geometry should address the following two issues:

• It should reduce to group co/homology or Lie algebra co/homology for $\mathcal{H} = kG$, $k[G]$ or $U(\mathfrak{g})$; Hopf algebras naturally associated to (Lie) groups or Lie algebras.

• There should exist a characteristic map, connecting the cyclic cohomology of a Hopf algebra $\mathcal{H}$ to the cyclic cohomology of an algebra A on which it acts. For example, for any $\mathcal{H}$-module algebra A and an invariant trace $\tau : A \to \mathbb{C}$, there should exist a map

$$\gamma : HC^\bullet(\mathcal{H}) \to HC^\bullet(A).$$

Let us explain both points starting with the first. It might seem that given a Hopf algebra $\mathcal{H}$, the Hochschild homology of the algebra $\mathcal{H}$ might be a good candidate for a homology theory for $\mathcal{H}$ in noncommutative geometry. After all one knows that for a Lie algebra $\mathfrak{g}$ and a $U(\mathfrak{g})$-bimodule M,

$$H_\bullet(\mathfrak{g}, M^{ad}) \cong H_\bullet(U(\mathfrak{g}), M)$$

where the action of $\mathfrak{g}$ on M is given by $g \cdot m = gm - mg$ [26]. Thus Hochschild homology of $U(\mathfrak{g})$ can be recovered from the Lie algebra homology of $\mathfrak{g}$. Conversely, if M is a $\mathfrak{g}$-module we can turn it into a $U(\mathfrak{g})$-bimodule where the left action is induced by $\mathfrak{g}$-action and the right action is by augmentation : $mX = \epsilon(X)m$. It follows that $H_\bullet(\mathfrak{g}, M) \cong H_\bullet(U(\mathfrak{g}), M)$, which shows that the Lie algebra homology can also be recovered from Hochschild homology. In particular $H_\bullet(\mathfrak{g}, k) \cong H_\bullet(U(\mathfrak{g}), k)$. Similarly, if G is a (discrete) group and M is a kG-bimodule then $H_\bullet(G; M^{ad}) \cong H_\bullet(kG, M)$ where the action of G on $M^{ad} = M$ is given by $gm = gmg^{-1}$.

In [23] these type of results were extended to all Hopf algebras in the following way. Let $\mathcal{H}$ be a Hopf algebra and M a left $\mathcal{H}$-module. One defines groups $H_\bullet(\mathcal{H}, M)$ as the left derived functor of the functor of coinvariants from $\mathcal{H}$-mod→k-mod,

$$M \mapsto M_\mathcal{H} := M/ \text{ submodule generated by } \{hm - \epsilon(h)m \mid h \in \mathcal{H},\ m \in M\}.$$

Obviously, $M_\mathcal{H} = k \otimes_\mathcal{H} M$ which shows that $H_\bullet(\mathcal{H}, M) \cong Tor_\bullet^\mathcal{H}(k, M)$. For $\mathcal{H} = kG$ or $U(\mathfrak{g})$, one obtains group and Lie algebra homologies.

Now let $\mathcal{H}$ be a Hopf algebra and M be an $\mathcal{H}$-bimodule. We can convert M to a new left $\mathcal{H}$ -module $M^{ad} = M$, where the action of $\mathcal{H}$ is given by

$$h \cdot m = h^{(2)} m S(h^{(1)}).$$

Proposition 4.1 ([23]; Mac Lane isomorphism for Hopf algebras). *Under the above hypotheses there is a canonical isomorphism*

$$H_n(\mathcal{H}, M) \cong H_n(\mathcal{H}; M^{ad}) = Tor_n^\mathcal{H}(k, M^{ad}),$$

where the left hand side is Hochschild homology.

Note that the result is true for all Hopf algebras irrespective of being (co)commutative or not.

This suggests defining $H_\bullet(\mathcal{H}, k)$ by viewing k as an $\mathcal{H}$-bimodule via the augmentation map, in analogy with the group homology, as our sought after homology theory for Hopf algebras. This is not, however, a reasonable candidate as can be seen by considering

$\mathcal{H} = k[G]$, the coordinate ring of an affine algebraic group. Then by the Hochschild-Kostant-Rosenberg theorem $H_\bullet(k[G], k) \cong \wedge^\bullet(Lie(G))$ and hence is independent of the group structure.

Next we discuss the second point above. Some interesting cyclic cocycles were defined by Connes in the context of Lie algebra homology and group cohomology. For example let A be an algebra and $\delta_1, \delta_2 : A \to A$ two commuting derivations. Let $\tau : A \to \mathbb{C}$ be an *invariant trace* in the sense that τ is a trace and $\tau(\delta_1(a)) = \tau(\delta_2(a)) = 0$ for all $a \in A$. Then one can directly check that the following is a cyclic 2-cocycle on A [4]:

$$\varphi(a_0, a_1, a_2) = \frac{1}{2\pi i}\tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))).$$

This cocycle is non-trivial. For example, if $A = A_\theta$ is the algebra of smooth noncommutative torus and $e \in A_\theta$ is the smooth Rieffel projection, then $\varphi(e, e, e) = q$, where $\tau(e) = p - q\theta$ [4].

For a second example let G be a (discrete) group and c be a normalized group cocycle on G with trivial coefficients. Then one can easily check that the following is a cyclic cocycle on the group algebra $\mathbb{C}G$ [9]:

$$\varphi(g_0, g_1, \ldots, g_n) = \begin{cases} c(g_1, g_2, \ldots, g_n) & \text{if } g_0 g_1 \ldots g_n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is highly desirable to understand the origin of these formulas, put them in a conceptual context and generalize them. For example we need to know in the case where a Lie algebra $\mathfrak{g}$ acts by derivations on an algebra A, $\mathfrak{g} \to Der(A)$, if there is a map

$$\gamma : H_\bullet(\mathfrak{g}, \mathbb{C}) \to HC^\bullet(A).$$

Now let us indicate how the cohomology theory defined by Connes-Moscovici [11, 10] and its dual version in [23] resolve both issues. Let $\mathcal{H}$ be a Hopf algebra. Let δ be *character* and σ a *group like* element of $\mathcal{H}$, i.e. $\delta : \mathcal{H} \to k$ is an algebra map and $\sigma : k \to \mathcal{H}$ a coalgebra map. Following [11, 10], we say (δ, σ) is a *modular pair* if $\delta\sigma = id_k$ and a *modular pair in involution* if, in addition, $(\sigma^{-1}\widetilde{S})^2 = id_{\mathcal{H}}$ where the *twisted antipode* $\tilde{S}$ is defined by

$$\widetilde{S}(h) = \delta(h^{(1)})S(h^{(2)}).$$

Given $\mathcal{H}$, and (δ, σ), Connes-Moscovici define a cocyclic module $\mathcal{H}^\natural_{(\delta,\sigma)}$ as follows. Let $\mathcal{H}^{\natural,0}_{(\delta,\sigma)} = k$ and $\mathcal{H}^{\natural,n}_{(\delta,\sigma)} = \mathcal{H}^{\otimes n}$, $n \geq 1$. The coface, codegeneracy and cyclic operators δ_i, σ_i, τ are defined by

$$\begin{aligned} \delta_0(h_1 \otimes \ldots \otimes h_n) &= 1_{\mathcal{H}} \otimes h_1 \otimes \ldots \otimes h_n, \\ \delta_i(h_1 \otimes \ldots \otimes h_n) &= h_1 \otimes \ldots \otimes \Delta(h_i) \otimes \ldots \otimes h_n \quad \text{for } 1 \leq i \leq n, \\ \delta_{n+1}(h_1 \otimes \ldots \otimes h_n) &= h_1 \otimes \ldots \otimes h_n \otimes \sigma, \\ \sigma_i(h_1 \otimes \ldots \otimes h_n) &= h_1 \otimes \ldots \otimes \epsilon(h_{i+1}) \otimes \ldots \otimes h_n \quad \text{for } 0 \leq i \leq n, \\ \tau(h_1 \otimes \ldots \otimes h_n) &= \Delta^{n-1}\widetilde{S}(h_1) \cdot (h_2 \otimes \ldots \otimes h_n \otimes \sigma). \end{aligned}$$

These formulas were discovered in [11] and then proved in full generality in [10]. In [12], M. Crainic gave an alternative approach based on Cuntz-Quillen formalism of cyclic homology [14]. Note that the cosimplicial module $\mathcal{H}^\natural_{(\delta,\sigma)}$ is the cosimplicial module associated to the coalgebra $\mathcal{H}$ with coefficients in k via the unit map and σ. The passage from the cyclic homology of (co)algebras to the cyclic homology of Hopf algebras is remarkably similar to passage from de Rham cohomology to Lie algebra cohomology. The key idea in both cases is *invariant cohomology*.

It is not difficult to see that the above complex is an exact analogue of *invariant cohomology* in noncommutative geometry. In fact, under the multiplication map $\mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ the coalgebra $\mathcal{H}$ is an $\mathcal{H}$-module coalgebra. Let $\hat{\mathcal{H}}_\natural$ be the cocyclic module of the coalgebra $\mathcal{H}$. The cocyclic module $\hat{\mathcal{H}}_\natural$ becomes a cocyclic $\mathcal{H}$-module via the diagonal action $\mathcal{H} \otimes \hat{\mathcal{H}}_\natural \to \hat{\mathcal{H}}_\natural$. We have $\hat{\mathcal{H}}_\natural{}^{\delta} = \mathcal{H}^\natural_{(\delta,1)}$ where $\hat{\mathcal{H}}_\natural{}^{\delta}$ is the space of δ-coinvariants.

The cohomology groups $HP^\bullet_{(\delta,\sigma)}(\mathcal{H})$ are so far computed for the following Hopf algebras. For quantum universal enveloping algebras no examples are known except for $U_q(sl_2)$ that we recall below.

1. If $\mathcal{H} = \mathcal{H}_n$ is the Connes-Moscovici Hopf algebra, we have [11]

$$HP^n_{(\delta,1)}(\mathcal{H}) \cong \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(\mathfrak{a}_n, \mathbb{C})$$

where $\mathfrak{a}_n$ is the Lie algebra of formal vector fields on $\mathbb{R}^n$.

2. If $\mathcal{H} = U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$, we have [11]

$$HP^n_{(\delta,1)}(\mathcal{H}) \cong \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H_i(\mathfrak{g}, \mathbb{C}_\delta)$$

3. If $\mathcal{H} = \mathbb{C}[G]$ is the coordinate ring of a nilpotent affine algebraic group G, we have [11]

$$HP^n_{(\epsilon,1)}(\mathcal{H}) \cong \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(\mathfrak{g}, \mathbb{C}),$$

where $\mathfrak{g} = Lie(G)$.

4. If $\mathcal{H}$ admits a normalized left Haar integral, then [12]

$$HP^1_{(\delta,\sigma)}(\mathcal{H}) = 0, \qquad HP^0_{(\delta,\sigma)}(\mathcal{H}) = k.$$

Recall that a linear map $\int : \mathcal{H} \to k$ is called a *normalized left Haar integral* if for all $h \in \mathcal{H}$, $\int(h) = \int(h^{(1)})h^{(2)}$ and $\int(1) = 1$. It is known that a Hopf algebra defined over a field admits a normalized left Haar integral if and only if it is cosemisimple [34]. Compact quantum groups and group algebras are known to admit normalized Haar integral in the above sense. In the latter case $\int : kG \to k$ sending $g \mapsto 0$ for all $g \neq e$ and $e \mapsto 1$ is a Haar integral. Note that G need not be finite. In this regard, we should also mention that there are interesting examples of finite-dimensional non-cosemisimple Hopf algebras defined as quantum groups at roots of unity (cf. [15]). Nothing is known about the cyclic (co)homology of these Hopf algebras.

5. If $\mathcal{H} = U_q(sl_2(k))$ is the quantum universal algebra of $sl_2(k)$, we have [12]

$$HP^0_{(\epsilon,\sigma)}(\mathcal{H}) = 0, \quad HP^1_{(\epsilon,\sigma)}(\mathcal{H}) = k \oplus k.$$

6. Let $\mathcal{H}$ be a commutative Hopf algebra. The periodic cyclic cohomology of the cocyclic module $\mathcal{H}^\natural_{(\epsilon,1)}$ can be computed in terms of the Hochschild homology of the coalgebra $\mathcal{H}$ with trivial coefficients.

PROPOSITION 4.2 ([23]). *Let $\mathcal{H}$ be a commutative Hopf algebra. Its periodic cyclic cohomology in the sense of Connes-Moscovici is given by*

$$HP^n_{(\epsilon,1)}(\mathcal{H}) = \bigoplus_{i=n \pmod 2} H^i(\mathcal{H}, k).$$

For example, if $\mathcal{H} = k[G]$ is the algebra of regular functions on an affine algebraic group G, the coalgebra complex of $\mathcal{H} = k[G]$ is isomorphic to the group cohomology complex of G where instead of regular cochains one uses regular functions $G \times G \times \ldots \times G \to k$. Denote this cohomology by $H^i(G, k)$. It follows that

$$HP^n_{(\epsilon,1)}(k[G]) = \bigoplus_{i=n \pmod 2} H^i(G, k).$$

As is remarked in [9], if the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ is nilpotent, it follows from Van Est's theorem that $H^i(G, k) \cong H^i(\mathfrak{g}, k)$. This gives an alternative proof of Proposition 4 and Remark 5 in [9].

Let A be an $\mathcal{H}$-module algebra and $Tr : A \to \mathbb{C}$ a δ-*invariant* linear map, i.e., $Tr(h(a)) = \delta(h)Tr(a)$ for $h \in \mathcal{H},\ a \in A$. Equivalently, Tr satisfies the *integration by part* property:

$$Tr(h(a)b) = Tr(a\tilde{S}(h)(b)).$$

Indeed,

$$Tr(h(a)b) = Tr(h^{(1)}(aS(h^{(2)})(b))) = \delta(h^{(1)})Tr(aS(h^{(2)})) = Tr(a\tilde{S}(h)(b)).$$

In addition we assume $Tr(ab) = Tr(b\sigma(a))$. Given $(A, \mathcal{H}, Tr)$, Connes-Moscovici show that the following map, called *the characteristic map*, defines a morphism of cyclic modules $\gamma : \mathcal{H}^\natural_{\delta,\sigma} \to A^\natural$, where $A^\natural = \hom(A_\natural, k)$ is the cocyclic module associated to A:

$$\gamma(h_1 \otimes \ldots \otimes h_n)(a_0, a_1, \ldots, a_n) = Tr(a_0 h_1(a_1) \ldots h_n(h_n)).$$

We therefore have well-defined maps

$$\gamma : HC^\bullet_{(\delta,\sigma)}(\mathcal{H}) \to HC^\bullet(A), \qquad \gamma : HP^\bullet_{(\delta,\sigma)}(\mathcal{H}) \to HP^\bullet(A).$$

Examples show that, in general, this map is non-trivial. For example let $\mathfrak{g}$ be an abelian n-dimensional Lie algebra acting by derivations on an algebra A. Let $\delta_i \in Der(A)$ be the family of derivations corresponding to a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, and $Tr : A \to k$ an invariant trace on A, i.e. $Tr\delta_i(a) = 0$, $1 \le i \le n$. We have $H_i(\mathfrak{g}, k) \cong \wedge^i \mathfrak{g}$. In particular $H_n(\mathfrak{g}, k)$ is 1-dimensional. The inclusion

$$H_n(\mathfrak{g}, k) \hookrightarrow \bigoplus_{i=n \pmod 2} H_i(\mathfrak{g}, k) \cong HP^n_{(\epsilon,1)}(U(\mathfrak{g}))$$

combined with the characteristic map γ defines a map

$$\gamma : H_n(\mathfrak{g}, k) \cong k \to HC^n(A).$$

The image of $X_1 \wedge X_2 \wedge \ldots \wedge X_n$ under γ is the cyclic n-cocycle φ given by

$$\varphi(a_0, a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^\sigma Tr(a_0 \delta_{\sigma(1)}(a_1)\delta_{\sigma(2)}(a_2) \ldots \delta_{\sigma(n)}(a_n)).$$

The rest of this section is devoted to a dual cyclic theory for Hopf algebras which was defined, independently, in [23, 35]. There is a need for a dual theory to be developed.

This is needed, for example, when one studies coactions of Hopf algebras (or quantum groups) on noncommutative spaces, since the original Connes-Moscovici theory works for actions only. A more serious problem is the fact that if $\mathcal{H}$ has normalized left Haar integral then its cyclic cohomology in the sense of Connes-Moscovici is trivial in positive dimensions [12], but the dual theory is non-trivial.

In [23] we associated a cyclic module to any Hopf algebra $\mathcal{H}$ over k if $\mathcal{H}$ has a modular pair (δ,σ) such that $\widehat{S}^2 = id_{\mathcal{H}}$, where $\widehat{S}(h) = \delta(h^{(2)})\sigma S(h^{(1)})$. This cyclic module can be seen as the dual of the cocyclic module introduced in [10] by A. Connes and H. Moscovici. Using ϵ and δ one can endow k with an $\mathcal{H}$-bimodule structure, i.e.,

$$\delta \otimes id : \mathcal{H} \otimes k \to k \quad \text{and} \quad id \otimes \epsilon : k \otimes \mathcal{H} \to k.$$

Our cyclic module as a simplicial module is exactly the Hochschild complex of $\mathcal{H}$ with coefficients in k where k is an $\mathcal{H}$-bimodule as above. So if we denote our cyclic module by $\widetilde{\mathcal{H}}_{\natural}^{(\delta,\sigma)}$, we have $\widetilde{\mathcal{H}}_{\natural_n}^{(\delta,\sigma)} = \mathcal{H}^{\otimes n}$, for $n > 0$ and $\widetilde{\mathcal{H}}_{\natural_0}^{(\delta,\sigma)} = k$. Its faces and degeneracies are as follows:

$$\begin{aligned}
\delta_0(h_1 \otimes h_2 \otimes \ldots \otimes h_n) &= \epsilon(h_1)h_2 \otimes h_3 \otimes \ldots \otimes h_n,\\
\delta_i(h_1 \otimes h_2 \otimes \ldots \otimes h_n) &= h_1 \otimes h_2 \otimes \ldots \otimes h_i h_{i+1} \otimes \ldots \otimes h_n,\\
\delta_n(h_1 \otimes h_2 \otimes \ldots \otimes h_n) &= \delta(h_n)h_1 \otimes h_2 \otimes \ldots \otimes h_{n-1},\\
\sigma_0(h_1 \otimes h_2 \otimes \ldots \otimes h_n) &= 1 \otimes h_1 \otimes \ldots \otimes h_n,\\
\sigma_i(h_1 \otimes h_2 \otimes \ldots \otimes h_n) &= h_1 \otimes h_2 \ldots \otimes h_i \otimes 1 \otimes h_{i+1} \ldots \otimes h_n,\\
\sigma_n(h_1 \otimes h_2 \otimes \ldots \otimes h_n) &= h_1 \otimes h_2 \otimes \ldots \otimes 1.
\end{aligned}$$

To define a cyclic module it remains to introduce an action of cyclic group on our module. Our candidate is

$$\tau_n(h_1 \otimes h_2 \otimes \ldots \otimes h_n) = \delta(h_n^{(2)})\sigma S(h_1^{(1)} h_2^{(1)} \ldots h_{n-1}^{(1)} h_n^{(1)}) \otimes h_1^{(2)} \otimes \ldots \otimes h_{n-1}^{(2)}.$$

It is not difficult to check that $(\delta \circ S^{-1}, \sigma^{-1})$, is a modular pair in involution if and only if (δ,σ) is a modular pair and $\widehat{S}^2 = id_{\mathcal{H}}$. In other words (δ,σ) is a modular pair in involution in the sense of Connes and Moscovici [10] if and only if $(\delta \circ S, \sigma^{-1})$ is a modular pair in involution in the sense of [23].

Theorem 4.1 ([23]). *Let $\mathcal{H}$ be a Hopf algebra over k with a modular pair (δ,σ) such that $\widehat{S}^2 = id_{\mathcal{H}}$. Then $\widetilde{\mathcal{H}}_{\natural}^{(\delta,\sigma)}$ with operators given above defines a cyclic module. Conversely, if (δ,σ) is a modular pair such that $\widetilde{\mathcal{H}}_{\natural}^{(\delta,\sigma)}$ is a cyclic module, then $\widehat{S}^2 = id_{\mathcal{H}}$.*

Now let A be an $\mathcal{H}$-comodule algebra. To define the characteristic map we need an analogue of an invariant trace.

Definition 4.1. A linear map $Tr : A \to k$ is called a δ-*trace* if

$$Tr(ab) = Tr(b^{(0)}a)\delta(b^{(1)}) \qquad \forall a, b \in A.$$

It is called σ-*invariant* if for all $a \in A$,

$$Tr(a^{(0)})a^{(1)} = Tr(a)\sigma.$$

We show that Tr is σ-invariant if and only if for all $a, b \in A$

$$Tr(a^{(0)}b)a^{(1)} = Tr(ab^{(0)})\sigma S(b^{(1)}).$$

To see this, it is evident that if we consider $b=1$, then the above property of Tr implies that Tr is σ-invariant. On the other hand assume that Tr is σ-invariant. Then we have

$$Tr(ab^{(0)})\sigma S(b^{(1)}) = Tr(a^{(0)}b^{(0)})a^{(1)}b^{(1)}S(b^{(2)}) = Tr(a^{(0)}b)a^{(1)}.$$

Consider the map $\gamma : A_\natural \to \widetilde{\mathcal{H}}_\natural^{(\delta,\sigma)}$ defined by

$$\gamma(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = Tr(a_0 a_1^{(0)} \ldots a_n^{(0)}) a_1^{(1)} \otimes a_2^{(1)} \otimes \ldots a_n^{(1)}.$$

It is proved in [23] that γ is a morphism of cyclic modules.

COROLLARY 4.1. *Under the above conditions, γ induces the following canonical maps:*

$$\gamma :\ HC_\bullet(A) \to \widetilde{HC}_\bullet^{(\delta,\sigma)}(\mathcal{H}), \qquad \gamma :\ HP_\bullet(A) \to \widetilde{HP}_\bullet^{(\delta,\sigma)}(\mathcal{H}).$$

Next, we state a theorem which computes the cyclic homology of cocommutative Hopf algebras.

THEOREM 4.2 ([23]). *If $\mathcal{H}$ is a cocommutative Hopf algebra, then*

$$\widetilde{HC}_n^{(\delta,1)}(\mathcal{H}) = \bigoplus_{i\geq 0} H_{n-2i}(\mathcal{H}, k_\delta),$$

where k_δ is the one-dimensional module defined by δ.

EXAMPLE 4.1. Let $\mathfrak{g}$ be a Lie algebra over k and $U(\mathfrak{g})$ be its enveloping algebra. One knows that $H_n(U(\mathfrak{g}); k) = H_n(\mathfrak{g}; k)$ [26]. So by Theorem 4.2 we have

$$\widetilde{HC}_n^{(\delta,1)}(\mathfrak{g}) = \bigoplus_{i\geq 0} H_i(\mathfrak{g}; k_\delta).$$

EXAMPLE 4.2. Let G be a discrete group and $\mathcal{H} = kG$ its group algebra. Then from Theorem 4.2 we have

$$\widetilde{HC}_n^{(\epsilon,1)}(kG) \cong \bigoplus_{i\geq 0} H_{n-2i}(G,k), \qquad \widetilde{HP}_n^{(\epsilon,1)}(kG) \cong \bigoplus_{i=n \pmod 2} H_i(G,k).$$

EXAMPLE 4.3. Let G be a discrete group and $\mathcal{H} = \mathbb{C}G$. Then the algebra $\mathcal{H}$ is a comodule algebra for the Hopf algebra $\mathcal{H}$ via the coproduct map $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. The map $Tr : \mathbb{C}G \to \mathbb{C}$ defined by

$$Tr(g) = \begin{cases} 1, & g = e, \\ 0, & g \neq e. \end{cases}$$

is a δ-invariant σ-trace for $\delta = \epsilon$, $\sigma = 1$. The dual characteristic map $\gamma^* : \widetilde{HC}_{(\epsilon,1)}^n(\mathbb{C}G) \to HC^n(\mathbb{C}G)$ combined with the inclusion $H^n(G,\mathbb{C}) \hookrightarrow \widetilde{HC}_{(\epsilon,1)}^n(\mathbb{C}G)$ is exactly the map $H^n(G,\mathbb{C}) \to HC^n(\mathbb{C}G)$ described earlier in this section.

It would be very interesting to compute the Hopf-cyclic homology $\widetilde{HC}_\bullet$ for compact quantum groups. Of course, one should look at algebras of polynomials or smooth functions on compact quantum groups, the C^*-completion being uninteresting from cyclic theory point of view. In the following we recall two results that are known so far about quantum groups.

Let k be a field of characteristic zero and $q \in k$, $q \neq 0$ and q not a root of unity. The Hopf algebra $\mathcal{H} = A(SL_q(2,k))$ is defined as follows. As an algebra it is generated by symbols a, b, c, d, with the following relations:

$$ba = qab, \;\; ca = qac, \;\; db = qbd, \;\; dc = qcd,$$

$$bc = cb, \;\; ad - q^{-1}bc = da - qbc = 1.$$

The coproduct, counit and antipode of $\mathcal{H}$ are defined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes c,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$
$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$
$$S(a) = d, \;\; S(d) = a, \;\; S(b) = -qb, \;\; S(c) = -q^{-1}c.$$

For more details about $\mathcal{H}$ we refer to [25]. Because $S^2 \neq id$, to define our cyclic structure we need a modular pair (σ, δ) in involution. Let δ be as follows:

$$\delta(a) = q, \;\; \delta(b) = 0, \;\; \delta(c) = 0, \;\; \delta(d) = q^{-1}.$$

And $\sigma = 1$. Then we have $\widetilde{S}^2_{(1,\delta)} = id$.

For computing cyclic homology we should at first compute the Hochschild homology $H_*(\mathcal{H}, k)$ where k is an $\mathcal{H}$-bimodule via δ, ϵ for left and right action of $\mathcal{H}$, respectively. One knows $H_*(\mathcal{H}, k) = Tor_*^{\mathcal{H}^e}(\mathcal{H}, k)$, where $\mathcal{H}^e = \mathcal{H} \otimes \mathcal{H}^{op}$. So we need a resolution for k, or $\mathcal{H}$ as an $\mathcal{H}^e$-module. We take advantage of the free resolution for $\mathcal{H}$ given by Masuda etal. [23]. By a lengthy computation one can check that $H_0(\mathcal{H}, k) = 0$, $H_1(\mathcal{H}, k) = H_2(\mathcal{H}, k) = k \oplus k$, and $H_n(\mathcal{H}, k) = 0$ for all $n \geq 3$. Moreover we find that the operator $B = (1 - \tau)\sigma N : H_1(\mathcal{H}, k) \to H_2(\mathcal{H}, k)$ is bijective and we obtain:

Theorem 4.3 ([23]). *For any $q \in k$ which is not a root of unity, $\widetilde{HC}_1(A(SL_q(2,k))) = k \oplus k$ and $\widetilde{HC}_n(A(SL_q(2,k))) = 0$ for all $n \neq 1$. In particular, $\widetilde{HP}_0(A(SL_q(2,k))) = \widetilde{HP}_1(A(SL_q(2,k))) = 0$.*

The above theorem shows that Theorem 4.2 is not true for non-cocommutative Hopf algebras.

The quantum universal enveloping algebra $U_q(sl(2,k))$ is an k-Hopf algebra which is generated as an k- algebra by symbols σ, σ^{-1}, x, y subject to the following relations:

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1, \;\; \sigma x = q^2 x\sigma, \;\; \sigma y = q^{-2} y\sigma, \;\; xy - yx = \frac{\sigma - \sigma^{-1}}{q - q^{-1}}.$$

The coproduct, counit and antipode of $U_q(sl(2,k))$ are defined by:

$$\Delta(x) = x \otimes \sigma + 1 \otimes x, \;\; \Delta(y) = y \otimes 1 + \sigma^{-1} \otimes y, \;\; \Delta(\sigma) = \sigma \otimes \sigma,$$
$$S(\sigma) = \sigma^{-1}, \;\; S(x) = -x\sigma^{-1}, \;\; S(y) = -\sigma y,$$
$$\varepsilon(\sigma) = 1, \; \varepsilon(x) = \varepsilon(y) = 0.$$

It is easy to check that $S^2(a) = \sigma a \sigma^{-1}$, so that $(\sigma^{-1}, \varepsilon)$ is a modular pair in involution. As the first step to compute its cyclic homology we should find its Hochschild homology group with trivial coefficients. (The filed k is a $U_q(sl(2,k))$ bimodule via ε.) We define a free resolution for $\mathcal{H} = U_q(sl(2,k))$ as an $\mathcal{H}^e$-module as follows

$$(*) \qquad \mathcal{H} \overset{\mu}{\leftarrow} M_0 \overset{d_0}{\leftarrow} M_1 \overset{d_1}{\leftarrow} M_2 \overset{d_2}{\leftarrow} M_3 \dots$$

where M_0 is $\mathcal{H}^e$, M_1 is the free $\mathcal{H}^e$-module generated by symbols $1\otimes e_\sigma, 1\otimes e_x, 1\otimes e_y$, M_2 is the free $\mathcal{H}^e$-module generated by symbols $1\otimes e_x\wedge e_\sigma, 1\otimes e_y\wedge e_\sigma, 1\otimes e_x\wedge e_y$, and finally M_3 is generated by $1\otimes e_x\wedge e_y\wedge e_\sigma$ as a free $\mathcal{H}^e$-module. We let $M_n = 0$ for all $n \geq 4$. We claim that with the following boundary operators, $(*)$ is a free resolution for $\mathcal{H}$:

$$\begin{aligned}
&d_0(1\otimes e_x) = x\otimes 1 - 1\otimes x,\\
&d_0(1\otimes e_y) = y\otimes 1 - 1\otimes y,\\
&d_0(1\otimes e_\sigma) = \sigma\otimes 1 - 1\otimes \sigma,\\
&d_1(1\otimes e_x\wedge e_\sigma) = (\sigma\otimes 1 - 1\otimes q^2\sigma)\otimes e_\sigma - (q^2x\otimes 1 - 1\otimes x)\otimes e_x,\\
&d_1(1\otimes e_y\wedge e_\sigma) = (\sigma\otimes 1 - 1\otimes q^{-2}\sigma)\otimes e_\sigma - (q^{-2}y\otimes 1 - 1\otimes y)\otimes e_y,\\
&d_1(1\otimes e_x\wedge e_y) = (y\otimes 1 - 1\otimes y)\otimes e_x - (x\otimes 1 - 1\otimes x)\otimes e_y\\
&\qquad\qquad + \frac{1}{q-q^{-1}}(\sigma^{-1}\otimes\sigma^{-1} + 1\otimes 1)\otimes e_\sigma,\\
&d_2(1\otimes e_x\wedge e_y\wedge e_\sigma) = (y\otimes 1 - 1\otimes q^2y)\otimes e_x\wedge e_\sigma\\
&\qquad -q^2(q^2x\otimes 1 - 1\otimes x)\otimes e_y\wedge e_\sigma + q^2(\sigma\otimes 1 - 1\otimes\sigma)\otimes e_y\wedge e_x.
\end{aligned}$$

To show that this complex is a resolution, we need a homotopy map. First we recall that the set $\{\sigma^l x^m y^n \mid l\in\mathbb{Z}, m,n\in\mathbb{N}_0\}$ is a P.B.W. type basis for $\mathcal{H}$ [25]. Let

$$\phi(a,b,n) = (a^{n-1}\otimes 1 + a^{n-1}\otimes b\dots a\otimes b^{n-1} + 1\otimes b^{n-1})$$

where $n\in\mathbb{N}, a\in\mathcal{H}, b\in\mathcal{H}^{op}$, and $\phi(a,b,0)=0$, and $\omega(p)=1$ if $p\geq 0$ and 0 otherwise. The following maps define a homotopy map for $(*)$, i.e., $Sd+dS=1$:

$$\begin{aligned}
&S_{-1}:\mathcal{H}\to M_0,\\
&S_{-1}(a) = 1\otimes a,\\
&S_0: M_0\to M_1,\\
&S_0(\sigma^l x^m y^n\otimes b) = (1\otimes b)((\sigma^l x^m\otimes 1)\phi(y,y,n)\otimes e_y\\
&\qquad +(\sigma^l\otimes y^n)\phi(x,x,m)\otimes e_x) + \omega(l)(1\otimes x^m y^n)\phi(\sigma,\sigma,l)\otimes e_\sigma\\
&\qquad\qquad +(\omega(l)-1)(1\otimes x^m y^n)\phi(\sigma^{-1},\sigma^{-1},-l)(\sigma^{-1}\otimes\sigma^{-1}\otimes e_\sigma),
\end{aligned}$$

$$\begin{aligned}
&S_1: M_1\to M_2,\\
&S_1(\sigma^l x^m y^n\otimes b\otimes e_y) = 0,\\
&S_1(\sigma^l x^m y^n\otimes b\otimes e_x) = (1\otimes b)((\sigma^l x^m\otimes 1)\phi(y,y,n)\otimes e_x\wedge e_y\\
&+\frac{1-q^{2n}}{(q-q^{-1})(1-q^2)}(\sigma^l\otimes y^{n-1})\phi(x,x,m)(\sigma^{-1}\otimes\sigma^{-1} + q^{-2}\otimes 1)\otimes e_x\wedge e_\sigma\\
&\qquad +\frac{1}{q-q^{-1}}(\sigma^l x^m\otimes 1)\phi(y,y,n-1)(\sigma^{-1}\otimes\sigma{-1} + q^2\otimes 1)\otimes e_y\wedge e_\sigma),\\
&S_1(\sigma^l x^m y^n\otimes b\otimes e_\sigma) = (1\otimes b)(q^2(\sigma^l x^m\otimes 1)\phi(y,q^2y,n)\otimes e_y\wedge e_\sigma\\
&\qquad\qquad +q^{2(n-1)}(\sigma^l\otimes y^n)\phi(x,q^{-2}x,m)\otimes e_x\wedge e_\sigma),
\end{aligned}$$

$$S_2 : M_2 \to M_3,$$
$$S_2(a \otimes b \otimes e_x \wedge e_y) = 0,$$
$$S_2(a \otimes b \otimes e_y \wedge e_\sigma) = 0,$$
$$S_2(\sigma^l x^m y^n \otimes b \otimes e_x \wedge e_\sigma) = (1 \otimes b)(\sigma^l x^m \otimes 1)\phi(y, q^2 y, n) \otimes e_x \wedge e_y \wedge e_\sigma,$$
$$S_n = 0 : M_n \to M_{n+1} \quad \text{for} \quad n \geq 3.$$

Again, by a rather long, but straightforward computation, we can check that $dS + Sd = 1$. By using the definition of Hochschild homology as $Tor^{\mathcal{H}^e}(\mathcal{H}, k)$ we have the following theorem:

THEOREM 4.4 ([23]). *$H_0(U_q(sl(2,k)), k) = k$ and $H_n(U_q(sl(2,k)), k) = 0$ for all $n \neq 0$, where k is a $U_q(sl(2,k))$-bimodule via ε for both sides.*

COROLLARY 4.2. *$\widetilde{HC}_n(U_q(sl(2,k))) = k$ when n is even, and 0 otherwise.*

5. Cohomology of extended Hopf algebras. In their study of index theory for transversely elliptic operators and in order to treat the general non-flat case, Connes and Moscovici [8] had to replace their Hopf algebra $\mathcal{H}_n$ by a so-called " extended Hopf algebra" $\mathcal{H}_{FM}$. In fact $\mathcal{H}_{FM}$ is neither a Hopf algebra nor a Hopf algebroid in the sense of [27], but it has enough structure to define a cocyclic module similar to the cocyclic module of Hopf algebras [11, 10, 9]. Since Hopf algebraic structures like $\mathcal{H}_{FM}$, and those related to Lie groupoids and Lie algebroids appear frequently in noncommutative geometry, it is necessary to develop a general framework where one can define a cyclic cohomology theory for these objects along the lines of cyclic cohomology theories for Hopf algebras.

A natural starting point would be to define a cyclic cohomology theory for Hopf algebroids. In attempting to do this, one faces two problems: firstly, it is by no means clear how the cocyclic module of Hopf algebras can be extended to Hopf algebroids as they are defined in [27], and, secondly, the Connes-Moscovici algebra $\mathcal{H}_{FM}$ is not a Hopf algebroid in the sense of [27]. We were led instead to define a closely related concept that we call an extended Hopf algebra. This terminology is already used in [8]. All examples of interest, including the Connes-Moscovici algebra $\mathcal{H}_{FM}$ are extended Hopf algebras.

Our first goal in this section is to recall the definition of an *extended Hopf algebra* from [22]. This is closely related to, but different from, *Hopf algebroids* in [27, 37]. The reason we prefer this concept to Hopf algebroids is that it is not clear how to define cyclic homology of Hopf algebroids, but it can be defined for extended Hopf algebras as we will recall from [22]. The whole theory is motivated by [8].

Broadly speaking, extended Hopf algebras and Hopf algebroids are quantizations (i.e. not necessarily commutative or cocommutative analogues) of groupoids and Lie algebroids. This should be compared with the point of view that Hopf algebras are quantizations of groups and Lie algebras. Commutative Hopf algebroids were defined as cogroupoid objects in the category of commutative algebras in [32]. The main example being algebra of functions on a groupoid. The concept was later generalized to allow noncommutative total algebras. A decisive step was taken in [27] where both total and base algebra are allowed to be noncommutative.

To define a cocyclic module one needs an *antipode pair* $(S, \widetilde{S})$ as defined below. Motivated by this observation and also the fundamental work of [8], we were led to define extended Hopf algebras and their cocyclic modules.

Recall from [27, 37] that a *bialgebroid* $(H, R, \Delta, \varepsilon)$ consists of

1: An algebra H, an algebra R, an algebra homomorphism $\alpha : R \to H$, and an algebra anti-homomorphism $\beta : R \to H$ such that the images of α and β commute in H. It follows that H can be regarded as R-bimodule via $axb = \alpha(a)\beta(b)x$, $a, b \in R$ $x \in H$. Here H is called the *total algebra*, R the *base algebra*, α the *source map* and β the *target map*.

2: A coproduct, i.e. an (R, R)-bimodule map $\Delta : H \to H \otimes_R H$ with $\Delta(1) = 1 \otimes_R 1$ satisfying the following conditions:

i) Coassociativity:

$$(\Delta \otimes_R id_H)\Delta = (id_H \otimes_R \Delta)\Delta : H \to H \otimes_R H \otimes_R H.$$

ii) Compatibility with product:

$$\Delta(a)(\beta(r) \otimes 1 - 1 \otimes \alpha(r)) = 0 \text{ in } H \otimes_R H \text{ for any } r \in R, \quad a \in H,$$
$$\Delta(ab) = \Delta(a)\Delta(b) \text{ for any } a, b \in H.$$

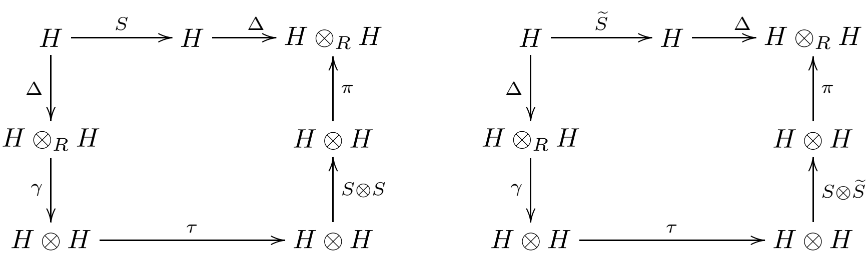
3: A counit, i.e., an (R, R)-bimodule map $\epsilon : H \to R$ satisfying $\epsilon(1_H) = 1_R$ and $(\epsilon \otimes_R id_H)\Delta = (id_H \otimes_R \epsilon)\Delta = id_H : H \to H$.

Definition 5.1. Let $(H, R, \alpha, \beta, \Delta, \varepsilon)$ be a k-bialgebroid. We call it a *Hopf algebroid* if there is a bijective map $S : H \to H$ which is an antialgebra map satisfying the following conditions:

i) $S\beta = \alpha$.
ii) $m_H(S \otimes id)\Delta = \beta\epsilon S : H \to H$.
iii) There exists a linear map $\gamma : H \otimes_R H \to H \otimes H$ satisfying $\pi \circ \gamma = id_{H \otimes_R H}$ and $m_H(id \otimes S)\gamma\Delta = \alpha\epsilon : H \to H$, where $\pi : H \otimes H \to H \otimes_R H$ is the natural projection.

Definition 5.2. Let (H, R) be a bialgebroid. An *antipode pair* $(S, \widetilde{S})$ consists of maps $S, \widetilde{S} : H \to H$ such that

(i) S and $\widetilde{S}$ are antialgebra maps.
(ii) $\widetilde{S}\beta = S\beta = \alpha$.
(iii) $m_H(S \otimes id)\Delta = \beta\epsilon S : H \to H$ and $m_H(\widetilde{S} \otimes id)\Delta = \beta\epsilon\widetilde{S} : H \to H$.
(iv) There exists a k-linear section $\gamma : H \otimes_R H \to H \otimes H$ for the natural projection $H \otimes H \to H \otimes_R H$ such that the map $\gamma \circ \Delta : H \to H \otimes H$ is coassociative and the following two diagrams are commutative:



In the above diagrams $\tau : H\otimes H \to H\otimes H$ is the "twisting map" defined by $\tau(h_1\otimes h_2) = h_2\otimes h_1$. Equivalently, and by abusing the language, we say S is an "anticoalgebra map" and $\widetilde{S}$ is a "twisted anticoalgebra map", i.e. for all $h\in H$

$$\Delta S(h) = S(\mathfrak{h}^{(2)}) \otimes_R S(\mathfrak{h}^{(1)}), \tag{1}$$

$$\Delta \widetilde{S}(h) = S(\mathfrak{h}^{(2)}) \otimes_R \tilde{S}(\mathfrak{h}^{(1)}), \tag{2}$$

where $\gamma(\Delta(h)) = \mathfrak{h}^{(1)}\otimes\mathfrak{h}^{(2)}$.

Definition 5.3. An *extended Hopf algebra* is a bialgebroid endowed with an antipode pair $(S,\widetilde{S})$ such that $\widetilde{S}^2 = id_H$.

Remark. The exchange operator $H\otimes_R H \to H\otimes_R H$, $x\otimes_R y \mapsto y\otimes_R x$, is not well-defined in general. A careful look at the proof of the cocyclic module property for the Connes-Moscovici cocyclic module $\mathcal{H}_{\natural}^{(\delta,1)}$ of a Hopf algebra $\mathcal{H}$ (cf. Theorem 2.1 in [23]) reveals that relations (1) and (2) (for $k = R$) play a fundamental role. The same is true for Theorem 5.1, but since R is noncommutative in general, these relations make sense only after we fix a section γ as in Definition 5.2. Coassociativity of the map $\gamma\circ\Delta : H \to H\otimes H$ is needed in the proof of Theorem 5.1. This motivates our definition of an extended Hopf algebra.

Recall the Connes-Moscovici algebra $(\mathcal{H}_{FM}, R)$ associated to a smooth manifold M [8]. It is shown in [8] that $\mathcal{H}_{FM}$ is a free $R\otimes R$-module where $R = C^\infty(FM)$ is the algebra of smooth functions on the frame bundle FM. In fact fixing a torsion free connection on FM, one obtains a Poincaré-Birkhoff-Witt type basis for $\mathcal{H}_{FM}$ over $R\otimes R$ consisting of differential operators $Z_I\cdot\delta_K$, where Z_I is a product of horizontal vector fields X_i, $1\le i\le n$ and vertical vector fields Y_j^i and δ_K is a product of vector fields δ. The coproduct Δ and the twisted antipode $\widetilde{S}$ are already defined in [8] and all the identities of a bialgebroid are verified. All we have to do is to define a section $\gamma : \mathcal{H}_{FM}\otimes_R\mathcal{H}_{FM} \to \mathcal{H}_{FM}\otimes\mathcal{H}_{FM}$, an antipode map $S : \mathcal{H}_{FM}\to\mathcal{H}_{FM}$ and verify the remaining conditions of Definition 5.2.

To this end, we first define S on the generations of $\mathcal{H}_{FM}$ by

$$\begin{array}{ll} S(\alpha(r)) = \beta(r), & S(\beta(r)) = \alpha(r), \\ S(Y_i^j) = -Y_i^j, & S(X_k) = -X_k + \delta_{kj}^i Y_i^j, \\ S(\delta_{jk}^i) = -\delta_{jk}^i. & \end{array} \tag{3}$$

We then extend S as an antialgebra map, using the Poincaré-Birkhoff-Witt basis of $\mathcal{H}_{FM}$.

We define a section $\gamma : \mathcal{H}_{FM}\otimes_R\mathcal{H}_{FM} \to \mathcal{H}_{FM}\otimes\mathcal{H}_{FM}$ by the formula

$$\gamma(\alpha(r)\otimes x\otimes\beta(s)\otimes_R\alpha(r')\otimes x'\otimes\beta(s')) = \alpha(r)\otimes x\otimes\beta(s)\alpha(r')\otimes 1\otimes x'\otimes\beta(s'),$$

where we use the fact that $\mathcal{H}_{FM}$ is a free $R\otimes R$-module. The following proposition is proved in [22].

Proposition 5.1. *The Connes-Moscovici algebra* $\mathcal{H}_{FM}$ *is an extended Hopf algebra.*

We give a few more examples of extended Hopf algebras.

Example 5.1. Let $\mathcal{H}$ be a k-Hopf algebra, $\delta : \mathcal{H}\to k$ a character, i.e. an algebra homomorphism and $\widetilde{S}_\delta = \delta * S$ the δ-twisted antipode defined by $\widetilde{S}_\delta(h) = \sum\delta(h^{(1)})S(h^{(2)})$, as in [11]. Assume that $\widetilde{S}_\delta^2 = id_{\mathcal{H}}$. Then $(\mathcal{H},\alpha,\beta,\Delta,\epsilon,S,\widetilde{S}_\delta)$ is an extended Hopf algebra, where $\alpha = \beta : k\to\mathcal{H}$ is the unit map. More generally, given any k-algebra R, let

$H = R \otimes \mathcal{H} \otimes R^{op}$, where R^{op} denotes the opposite algebra of R. With the following structure H is an extended Hopf algebra over R:

$$\begin{aligned}
\alpha(a) &= a \otimes 1 \otimes 1, \\
\beta(a) &= 1 \otimes 1 \otimes a, \\
\Delta(a \otimes h \otimes b) &= \sum a \otimes h^{(1)} \otimes 1 \otimes_R 1 \otimes h^{(2)} \otimes b, \\
\epsilon(a \otimes h \otimes b) &= \epsilon(h)ab, \\
S(a \otimes h \otimes b) &= (b \otimes S(h) \otimes a), \\
\widetilde{S}(a \otimes h \otimes b) &= (b \otimes \widetilde{S}_\delta(h) \otimes a),
\end{aligned}$$

and the section $\gamma : H \otimes_R H \to H \otimes H$ is defined by $\gamma(r \otimes h \otimes s \otimes_R r' \otimes h' \otimes s') = r \otimes h \otimes sr' \otimes 1 \otimes h' \otimes s'$. Then one can check that (H, R) is an extended Hopf algebra.

Example 5.2. Let $\mathcal{G}$ be a groupoid over a finite base (i.e., a category with a finite set of objects, such that each morphism is invertible). Then the groupoid algebra $H = k\mathcal{G}$ is generated by a morphism $g \in \mathcal{G}$ with unit $1 = \sum_{X \in \mathcal{O}bj(\mathcal{G})} id_X$, and the product of two morphisms is equal to their composition if the latter is defined and 0 otherwise. It becomes an extended Hopf algebra over $R = k\mathcal{S}$, where $\mathcal{S}$ is the subgroupoid of $\mathcal{G}$ whose objects are those of $\mathcal{G}$ and $\mathcal{M}or(X, Y) = id_X$ whenever $X = Y$ and $\emptyset$ otherwise. The relevant maps are defined for $g \in \mathcal{G}$ by $\alpha = \beta : R \hookrightarrow H$ is natural embedding, $\Delta(g) = g \otimes_R g$, by $\epsilon(g) = id_{target(g)}$, $S(g) = g^{-1}$. Note that $H \otimes H$ splits into the direct sum of vector spaces spanned by the tensor products of morphisms with the same targets and distinct targets, respectively. Since $H \otimes_R H$ can be identified with the quotient of $H \otimes H$ by the latter vector space, we can conclude that the simple tensors $h \otimes_R g$, $target(h) = target(g)$, form a basis of $H \otimes_R H$. Consequently, one can define a section $\gamma : H \otimes_R H \to H \otimes H$ by $\gamma(h \otimes_R g) = h \otimes g$, where $target(h) = target(g)$. It can easily be checked that H is both a Hopf algebroid and an extended Hopf algebra with $\widetilde{S} = S$.

Given an extended Hopf algebra (H, R) we define a cocyclic module $H_\natural$ as follows:

$$H_\natural^0 = R, \text{ and } H_\natural^n = H \otimes_R \otimes_R \dots \otimes_R H \qquad (n \text{ factors}), \ \ n \geq 1.$$

The coface, codegeneracy and cyclic actions δ_i, σ_i and τ are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes_R \dots \otimes_R h_n) &= 1_H \otimes_R h_1 \otimes_R \dots \otimes_R h_n, \\
\delta_i(h_1 \otimes_R \dots \otimes_R h_n) &= h_1 \otimes_R \dots \otimes_R \Delta(h_i) \otimes_R \dots \otimes_R h_n \ \text{ for } \ 1 \leq i \leq n, \\
\delta_{n+1}(h_1 \otimes_R \dots \otimes_R h_n) &= h_1 \otimes_R \dots \otimes_R h_n \otimes_R 1_H, \\
\sigma_i(h_1 \otimes_R \dots \otimes_R h_n) &= h_1 \otimes_R \dots \otimes_R \epsilon(h_{i+1}) \otimes_R \dots \otimes_R h_n \ \text{ for } \ 0 \leq i \leq n, \\
\tau(h_1 \otimes_R \dots \otimes_R h_n) &= \Delta^{n-1}\widetilde{S}(h_1) \cdot (h_2 \otimes \dots \otimes h_n \otimes 1_H).
\end{aligned}$$

These formulas were obtained in [8] by transporting a cocyclic submodule of $A_\natural$ via a faithful trace to $\mathcal{H}_{FM\natural}$, where A is an algebra on which $\mathcal{H}_{FM}$ acts. In [22] we proved directly that these formulas define a cocyclic modules for any extended Hopf algebra.

Theorem 5.1 ([22]). *For any extended Hopf algebra (H, R), the above formulas define a cocyclic module structure on $H_\natural$.*

In the following all algebras are unital, and all modules are unitary, i.e., the unit of the algebra acts as the identity on the module. Let k be a commutative ring. A *Lie-Rinehart* algebra over k is a pair (L, R) where R is a commutative k-algebra, L is a k-Lie algebra and a left R- module, L acts on R by derivations $\rho : L \to \mathcal{D}er_k(R)$ such that $\rho[X, Y] = [\rho(X), \rho(Y)]$ for all X, Y in L and the action is R-linear, and the Leibniz property holds:

$$[X, aY] = a[X, Y] + \rho(X)(a)Y \;\text{ for all }\; X, Y \in L \text{ and } a \in R.$$

Instead of $\rho(X)(a)$ we simply write $X(a)$.

EXAMPLE 5.3. Let $R = C^\infty(M)$ be the algebra of smooth functions on a manifold M and $L = C^\infty(TM) = \mathcal{D}er_{\mathbb{R}}(C^\infty(M))$ the Lie algebra of vector fields on M. Then (L, R) is a Lie-Rinehart algebra, where the action $\rho :\; L = \mathcal{D}er_{\mathbb{R}}(R) \to \mathcal{D}er_{\mathbb{R}}(R)$ is the identity map.

EXAMPLE 5.4. Let $R = C^\infty(M)$ and (L, R) be a Lie-Rinehart algebra such that L is a finitely generated projective R-module. Then it follows from Swan's theorem that $L = C^\infty(E)$, i.e., it is the space of smooth sections of a vector bundle over M. Since $\rho : C^\infty(E) \to C^\infty(TM)$ is R-linear, it is induced by a bundle map $\rho : E \to TM$. In this way we recover Lie algebroids as a particular example of Lie-Rinehart algebras.

Next we recall the definition of the homology of a Lie-Rinehart algebra [33]. This homology theory is a simultaneous generalization of Lie algebra homology and de Rham homology. Let (L, R) be a Lie-Rinehart algebra. A module over (L, R) is a left R-module M and a left Lie L-module $\varphi : L \to End_k(M)$, denoted by $\varphi(X)(m) = X(m)$ such that for all $X \in L$, $a \in R$ and $m \in M$,

$$X(am) = aX(m) + X(a)m$$
$$(aX)(m) = a(X(m)).$$

Alternatively, we can say an (L, R)-module is an R-module endowed with a *flat connection* defined by $\nabla_X(m) = X(m),\;\; X \in L,\; m \in M$.

Let $C_n = C_n(L, R; M) = M \otimes_R \mathcal{A}lt^n_R(L)$, where $\mathcal{A}lt^n_R(L)$ denotes the n-th exterior power of the R-module L over R. Let $d : C_n \to C_{n-1}$ be the differential defined by

$$d(m \otimes X_1 \wedge \ldots \wedge X_n) = \sum_{i=1}^{n} (-1)^{i-1} X_i(m) \otimes X_1 \wedge \ldots \wedge \hat{X}_i \ldots \wedge X_n$$
$$+ \sum_{1 \le i < j \le n} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \ldots \wedge \hat{X}_j \ldots \wedge X_n.$$

It is easy to check that $d^2 = 0$ and thus we have a complex (C_n, d). The homology of this complex is, by definition, the homology of the Lie-Rinehart algebra (L, R) with coefficients in M and we denote this homology by $H_*(R, L; M)$.

To interpret this homology theory as a derived functor, Rinehart in [33] introduced the *universal enveloping algebra* of a Lie-Rinehart algebra (L,R). It is an associative k-algebra, denoted $U(L, R)$, such that the category of (L, R)-modules as defined above is equivalent to the category of $U(L, R)$-modules. It is defined as follows. One can see easily that the following bracket defines a k-Lie algebra structure on $R \oplus L$:

$$[r+X, s+Y] = [X,Y] + X(s) - Y(r) \text{ for } r,s \in R \text{ and } X,Y \in L.$$

Let $\tilde{U} = U(R \oplus L)$ be the universal enveloping algebra of the Lie algebra $R \oplus L$, and let $\tilde{U}^+$ be the linear span of all monomials generated by the image of the canonical embedding $\imath : R \oplus L \to \tilde{U}$. (In other words, $\tilde{U}^+$ is the augmentation ideal—the kernel of the counit map—of the Hopf algebra $\tilde{U}$.) Then $U(L,R) = \tilde{U}^+/I$, where I is the two-sided ideal generated by the set $\{\imath(rZ) - \imath(r)\imath(Z) \mid r \in R \text{ and } Z \in R \oplus L\}$. In [33] Rinehart showed that if L is a projective R-module, then

$$H_*(L,R;M) \cong Tor_*^{U(L,R)}(R,M).$$

EXAMPLE 5.5. The universal enveloping algebra $U(L,R)$ of a Lie-Rinehart algebra (L,R) is an extended Hopf algebra over the algebra R. For $X \in L$ and $r \in R$, we define

$$\begin{array}{cc} \Delta(X) = X \otimes_R 1 + 1 \otimes_R X, & \Delta(r) = r \otimes_R 1, \\ \epsilon(X) = 0, & \epsilon(r) = r, \\ S(X) = -, X & S(r) = r. \end{array}$$

Using the Poincaré-Birkhoff-Witt theorem of Rinehart [33], we extend Δ to be a multiplicative map, S to be an anti-multiplicative map and ϵ by $\epsilon(rX_1 \dots X_n) = 0$ for $n \geq 1$. The source and target maps are the natural embeddings $\alpha = \beta : R \hookrightarrow U(L,R)$ and $\widetilde{S} = S$. We define the section $\gamma : U(L,R) \otimes_R U(L,R) \to U(L,R) \otimes U(L,R)$ by $\gamma(rX_1 \dots X_n \otimes_R sY_1 \dots Y_m) = rsX_1 \dots X_n \otimes Y_1 \dots Y_m$. One can check that γ is well defined and $U(L,R)$ is an extended Hopf algebra.

Next we compute the cyclic cohomology groups of the extended Hopf algebra $U(L,R)$ of a Lie-Rinehart algebra (L,R). Let $S(L)$ be the symmetric algebra of the R-module L. It is an extended Hopf algebra over R. In fact it is the enveloping algebra of the pair (L,R) where L is an abelian Lie algebra acting by zero derivations on R. Let $\wedge(L)$ be the exterior algebra of the R-module L. The following lemma computes the Hochschild cohomology of the cocyclic module $S(L)_\natural$.

LEMMA 5.1. *Let R be a commutative k-algebra and let L be a flat R-module. Then*

$$H^*(S(L)_\natural) \cong \wedge^*(L).$$

The following proposition computes the periodic cyclic cohomology of the extended Hopf algebra $U(L,R)$ associated to a Lie-Rinehart algebra (L,R) in terms of its Rinehart homology. It extends a similar result for the enveloping algebra of Lie algebras from [11].

PROPOSITION 5.2 ([22]). *If L is a projective R-module, then we have*

$$HP^n(U(L,R)) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H_i(L,R;R),$$

where HP^* means periodic cyclic cohomology.

Lie-Rinehart algebras interpolate between Lie algebras and commutative algebras, exactly in the same way that groupoids interpolate between groups and spaces. In fact Lie-Rinehart algebras can be considered as the infinitesimal analogue of groupoids. For more information on Lie-Rinehart algebras one can see [2, 19, 33].

COROLLARY 5.1. *Let M be a smooth closed manifold and $\mathcal{D}$ be the algebra of differential operators on M. It is an extended Hopf algebra and its periodic cyclic homology is given by*

$$HP_n(\mathcal{D}) = \bigoplus_{i=n \pmod 2} H^i_{dR}(M).$$

Proof. We have $\mathcal{D} = U(L, R)$, where $L = C^\infty(TM)$ and $R = C^\infty(M)$. Dualizing the above proposition, we obtain

$$HP_n(\mathcal{D}) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(L, R) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i_{dR}(M). \blacksquare$$

DEFINITION 5.4 (Haar system for bialgebroids). Let (H, R) be a bialgebroid. Let $\tau : H \to R$ be a right R-module map. We call τ a *left Haar system* for H if

$$\alpha(\tau(h^{(1)}))h^{(2)} = \alpha(\tau(h))1_H$$

and $\alpha\tau = \beta\tau$. We call τ a *normal* left Haar system if $\tau(1_H) = 1_R$.

We give a few examples of Haar systems. Let H be the Hopf algebroid of a groupoid with finite base. Then it is easy to see that $\tau : H \to R$ defined by $\tau(id_x) = id_x$ for all $x \in Obj(\mathcal{G})$ and 0 otherwise is a normal Haar system for H. In a related example, one can directly check that the map $\tau : A_\theta \to \mathbb{C}[U, U^{-1}]$ defined by

$$\tau(U^n V^m) = \delta_{m,0} U^n$$

is a normal Haar system for the noncommutative torus A_θ. It is shown in [22] that A_θ is an extended Hopf algebra over $\mathbb{C}[U, U^{-1}]$.

PROPOSITION 5.3. *Let H be an extended Hopf algebra that admits a normal left Haar system. Then $HC^{2i+1}(H) = 0$ and $HC^{2i}(H) = \ker\{\alpha - \beta\}$ for all $i \geq 0$.*

Finally in this section we compute the periodic Hopf cyclic cohomology of commutative Hopf algebroids in terms of Hochschild cohomology. Given an extended Hopf algebra (H, R), we denote the Hochschild cohomology of the cocyclic module $H_\natural$ by $H^i(H, R)$. It is the cohomology of the complex

$$R \xrightarrow{d_0} H \xrightarrow{d_1} H \otimes_R H \xrightarrow{d_2} H \otimes_R H \otimes_R H \xrightarrow{d_3} \dots$$

Here the first differential is $d_0 = \alpha - \beta$ and d_n is given by

$$d_n(h_1 \otimes_R \dots \otimes_R h_n) = 1_H \otimes_R h_1 \otimes_R \dots \otimes_R h_n + \sum_{i=1}^n (-1)^i h_1 \otimes_R \dots \otimes_R \Delta(h_i) \otimes_R \dots \otimes_R h_n + (-1)^{n+1} h_1 \otimes_R \dots \otimes_R h_n \otimes_R 1_H.$$

By a commutative extended Hopf algebra we mean an extended Hopf algebra (H, R) where both H and R are commutative algebras. In [23], it is shown that the periodic cyclic cohomology, in the sense of Connes-Moscovici, of a commutative Hopf algebra admits a simple description. In fact, if $\mathcal{H}$ is a commutative Hopf algebra then we have ([23], Theorem 4.2):

$$HC^n_{(\epsilon,1)}(\mathcal{H}) \cong \bigoplus_{i \geq 0} H^{n-2i}(\mathcal{H}, k), \tag{4}$$

where the cohomologies on the right hand side are Hochschild cohomology of the coalgebra $\mathcal{H}$ with trivial coefficients. Since the cocyclic module of Theorem 5.1 reduces to Connes-Moscovici cocyclic module if H happens to be a Hopf algebra, it is natural to expect that the analogue of isomorphism (4) holds true for commutative extended Hopf algebras. Furthermore, the analogues of Propositions 4.2 and 4.3 in [23], which are crucial in establishing the above isomorphism (4), are true for extended Hopf algebras with similar proofs [22]. This leads us to the following conjecture.

CONJECTURE 5.1. *Let (H, R) be a commutative extended Hopf algebra. Then its cyclic cohomology is given by*

$$HC^n(H) \cong \bigoplus_{i \geq 0} H^{n-2i}(H, R).$$

6. Cohomology of smash products. A celebrated problem in cyclic homology theory is to compute the cyclic homology of the crossed product algebra $A \ltimes G$, where the group G acts on the algebra A by automorphisms. If G is a discrete group, there is a spectral sequence, due to Feigin and Tsygan [16], which converges to the cyclic homology of the crossed product algebra. This result generalizes Burghelea's calculation of the cyclic homology of a group algebra [26]. In [17] Getzler and Jones gave a new proof of this spectral sequence using their Eilenberg-Zilber theorem for cylindrical modules. In [1], this spectral sequence has been extended to all Hopf algebras with invertible antipode. In this section we recall this result.

Let $\mathcal{H}$ be a Hopf algebra and A an $\mathcal{H}$-module algebra. We define a bicomplex, in fact a cylindrical module $A \natural \mathcal{H}$ as follows: Let

$$(A \natural H)p, q = \mathcal{H}^{\otimes(p+1)} \otimes A^{\otimes(q+1)}, \qquad p, q \geq 0.$$

The vertical and horizontal operators, $\tau^{p,q}$, $\delta^{p,q}$, $\sigma^{p,q}$ and $t^{p,q}$, $d^{p,q}$, $s^{p,q}$ are defined by

$$\begin{aligned}
\tau^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0^{(1)}, \ldots, g_p^{(1)} \mid S^{-1}(g_0^{(0)} g_1^{(0)} \ldots g_p^{(0)}) \cdot a_q, a_0, \ldots, a_{q-1}),\\
\delta_i^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_p \mid a_0, \ldots, a_i a_{i+1}, \ldots, a_q), \quad 0 \leq i < q,\\
\delta_q^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0^{(1)}, \ldots, g_p^{(1)} \mid (S^{-1}(g_0^{(0)} g_1^{(0)} \ldots g_p^{(0)}) \cdot a_q) a_0, \ldots, a_{q-1}),\\
\sigma_i^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_p \mid a_0, \ldots, a_i, 1, a_{i+1}, \ldots, a_q), \quad 0 \leq i \leq q,\\
t^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_p^{(q+1)}, g_0, \ldots, g_{p-1} \mid g_p^{(0)} \cdot a_0, \ldots, g_p^{(q)} \cdot a_q),\\
d_i^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_i g_{i+1}, \ldots, g_p \mid a_0, \ldots, a_q), \quad 0 \leq i < q,\\
d_q^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_p^{(q+1)} g_0, g_1, \ldots, g_{p-1} \mid g_p^{(0)} \cdot a_0, \ldots, g_p^{(q)} \cdot a_q),\\
s_i^{p,q}(g_0, \ldots, g_p \mid a_0, \ldots, a_q) &= (g_0, \ldots, g_i, 1, g_{i+1}, \ldots, g_p \mid a_0, \ldots, a_q), \quad 0 \leq i \leq q.
\end{aligned}$$

REMARK. The cylindrical module $A \natural \mathcal{H}$ in [1] is defined for all Hopf algebras. For applications, however, one has to assume that S is invertible. The above formulas are essentially isomorphic to those in [1], when S is invertible.

THEOREM 6.1 ([1]). *Endowed with the above operations, $A \natural \mathcal{H}$ is a cylindrical module.*

COROLLARY 6.1. *The diagonal $d(A \natural \mathcal{H})$ is a cyclic module.*

Our next task is to identify the diagonal $d(A\natural\mathcal{H})$ with the cyclic module of the smash product $(A\,\#\,\mathcal{H})_\natural$. Define a map $\phi : (A\,\#\,\mathcal{H})_\natural \to d(A\natural\mathcal{H})$ by

$$\begin{aligned}&\phi(a_0\otimes g_0,\dots,a_n\otimes g_n) = \\ &(g_0^{(1)},g_1^{(2)},\dots,g_n^{(n+1)} \mid S^{-1}(g_0^{(0)}g_1^{(1)}\dots g_n^{(n)})\cdot a_0, S^{-1}(g_1^{(0)}g_2^{(1)}\dots g_n^{(n-1)})\cdot a_1,\dots,\\ &\qquad S^{-1}(g_{n-1}^{(0)}g_n^{(1)})\cdot a_{n-1}, S^{-1}(g_n^{(0)})\cdot a_n)\end{aligned}$$

By a long computation one shows that ϕ is a morphism of cyclic modules [1].

THEOREM 6.2 ([1]). *We have an isomorphism of cyclic modules $d(A\natural\mathcal{H}) \cong (A\#\mathcal{H})_\natural$.*

Proof. Define a map $\psi : d(A\natural\mathcal{H}) \to (A\,\#\,\mathcal{H})_\natural$ by

$$\begin{aligned}&\psi(g_0,\dots,g_n \mid a_0,\dots,a_n)\\ &\quad = ((g_0^{(0)}g_1^{(0)}\dots g_n^{(0)})\cdot a_0\otimes g_0^{(1)}, (g_1^{(1)}\dots g_n^{(1)})\cdot a_1\otimes g_1^{(2)},\dots, g_n^{(n)}\cdot a_n\otimes g_n^{(n+1)}).\end{aligned}$$

Then one can check that $\phi\circ\psi = \psi\circ\phi = id$. ■

Now we are ready to give an spectral sequence to compute the cyclic homology of the smash product $A\#\mathcal{H}$. By using the Eilenberg-Zilber theorem for cylindrical modules, we have:

THEOREM 6.3. *There is a quasi-isomorphism of mixed complexes*

$$Tot((A\natural\mathcal{H})) \cong d(A\natural\mathcal{H}) \cong (A\,\#\,\mathcal{H})^\natural,$$

and therefore an isomorphism of cyclic homology groups,

$$HC_\bullet(Tot(A\natural\mathcal{H})) \cong HC_\bullet(A\,\#\,H).$$

Next, we show that one can identify the E^2-term of the spectral sequence obtained from the column filtration. To this end, we define an action of $\mathcal{H}$ on the first row of $A\natural\mathcal{H}$, denoted by $A^\natural_{\mathcal{H}} = \{\mathcal{H}\otimes A^{\otimes(n+1)}\}_{n\geq 0}$ by

$$h\cdot(g\mid a_0,\dots,a_n) = (h^{(n+1)}\cdot g \mid h^{(0)}\cdot a_0,\dots,h^{(n)}\cdot a_n)$$

where $h^{(n+1)}\cdot g = h^{(n+1)}g\, S^{-1}(h^{(n+2)})$ is an action of $\mathcal{H}$ on itself. We let $C_\bullet^{\mathcal{H}}(A)$ be the space of invariants of $\mathcal{H}\otimes A^{\otimes(n+1)}$ under the above action. So in $C_\bullet^{\mathcal{H}}(A)$, we have

$$h\cdot(g\mid a_0,\dots,a_n) = \epsilon(h)(g\mid a_0,\dots,a_n).$$

We define the following operators on $C_\bullet^{\mathcal{H}}(A)$:

$$\begin{aligned}\tau_n(g\mid a_0,\dots,a_n) &= (g^{(1)}\mid (S^{-1}(g^{(0)})\cdot a_n), a_0,\dots,a_{n-1})\\ \delta_i(g\mid a_0,\dots,a_n) &= (g\mid a_0,\dots,a_ia_{i+1},\dots,a_n)\\ \delta_n(g\mid a_0,\dots,a_n) &= (g^{(1)}\mid (S^{-1}(g^{(0)})\cdot a_n)a_0, a_1,\dots,a_{n-1})\\ \sigma_i(g\mid a_0,\dots,a_n) &= (g\mid a_0,\dots,a_i,1,a_{i+1},\dots,a_n)\end{aligned}$$

PROPOSITION 6.1 ([1]). *$C_\bullet^{\mathcal{H}}(A)$ with the operators defined above is a cyclic module.*

Let M be a left $\mathcal{H}$-module. Then M is an $\mathcal{H}$-bimodule if we let $\mathcal{H}$ act on the right on M via the counit map: $m.h = \varepsilon(h)m$. We denote the resulting Hochschild homology groups by $H_\bullet(\mathcal{H},M)$. Explicitly they are computed from the complex $C_p(\mathcal{H},M) = \mathcal{H}^{\otimes p}\otimes M$,

$p \geq 0$, with the differential $\delta : C_p(\mathcal{H}, M) \to C_{p-1}(\mathcal{H}, M)$ defined by

$$\delta(g_1, g_2, \dots, g_p, m) = \epsilon(g_1)(g_2, \dots, g_p, m) + \sum_{i=1}^{p-1}(-1)^i(g_1, \dots, g_i g_{i+1}, \dots, g_p, m) + (-1)^p(g_1, \dots, g_{p-1}, g_p \cdot m).$$

Let $C_q(A^\natural_\mathcal{H}) = \mathcal{H}^{\otimes q} \otimes A^\natural_\mathcal{H}$ and let $\mathcal{H}$ act on it by $h \cdot (g_1, \dots, g_p \mid m) = (g_1, \dots, g_p \mid h \cdot m)$, where the action of $\mathcal{H}$ on $A^\natural_\mathcal{H}$ is given by conjugation. Thus we can construct $H_p(\mathcal{H}, C_q(A^\natural_\mathcal{H}))$.

Now we can show that our original cylindrical complex $(A \natural \mathcal{H}, (\delta, \sigma, \tau), (d, s, t))$ can be identified with the cylindrical complex $(\mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H}), (\mathfrak{d}, \mathfrak{s}, \mathfrak{t}), (\bar{\mathfrak{d}}, \bar{\mathfrak{s}}, \bar{\mathfrak{t}}))$ under the transformations $\beta : (A \natural \mathcal{H})_{p,q} \to \mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H}))$ and $\gamma : \mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H})) \to (A \natural \mathcal{H})_{p,q}$ defined by

$$\beta(g_0, \dots, g_p \mid a_0, \dots, a_q) = (g_1^{(0)}, \dots, g_p^{(0)} \mid g_0 g_1^{(1)} \dots g_p^{(1)} \mid a_0, \dots, a_q),$$
$$\gamma(g_1, \dots, g_p \mid g \mid a_0, \dots, a_q) = (g S^{-1}(g_1^{(1)} \dots g_p^{(1)}), g_1^{(0)}, \dots, g_p^{(0)} \mid a_0, \dots, a_q).$$

One checks that $\beta\gamma = \gamma\beta = id$. To compute the homologies of the mixed complex $(Tot(C(A \natural \mathcal{H}), b + \bar{b} + \mathbf{u}(B + \bar{B}))$ we filter it by the subcomplexes (column filtration)

$$\mathsf{F}^i_{pq} = \sum_{q \leq i}(\mathcal{H}^{\otimes(p+1)} \otimes A^{\otimes(q+1)}).$$

THEOREM 6.4 ([1]). *The E^0-term of the spectral sequence is isomorphic to the complex*

$$\mathsf{E}^0_{pq} = (\mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H})), \delta)$$

and the E^1-term is

$$\mathsf{E}^1_{pq} = (H_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H})), \mathfrak{b} + \mathbf{u}\mathfrak{B})).$$

The E^2-term of the spectral sequence is

$$\mathsf{E}^2_{pq} = HC_q(H_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H}))),$$

the cyclic homologies of the cyclic module $H_p(\mathcal{H}, \mathsf{C}_q(A^\natural_\mathcal{H}))$.

7. From invariant cyclic homology to Hopf-cyclic homology with coefficients. The daunting task of verifying the axioms of cyclic modules, specially the cyclicity axiom $t_n^{n+1} = id$, for the Connes-Moscovici cyclic module of Hopf algebras or its dual cyclic module, prompts one to search for a conceptual foundation for the whole theory.

As we already indicated in the introduction, the close relationship between cyclic homology of enveloping algebras and group algebras as Hopf algebras on the one hand, and Lie algebra homology and group homology of the corresponding Lie algebras and groups on the other hand, suggest that there should be an approach based on invariant theoretic considerations. The strongest hint that such an approach might in fact work came from two sources: (1) The work of Chevalley and Eilenberg [3] where they define an invariant de Rham cohomology theory for any triple (M, G, V) consisting of a smooth manifold M, a Lie group G acting smoothly on M and a G-module V. They further showed that for $G = M$ acting on itself via left translations, the invariant de Rham complex is isomorphic with the Lie algebra cohomology complex. (2) The fact that cyclic cohomology is the noncommutative analogue of de Rham cohomology.

These two points taken together suggested that perhaps there exists a noncommutative analogue of invariant de Rham cohomology for a (co)algebra endowed with a (co)action of a Hopf algebra, and perhaps cyclic (co)homology of Hopf algebras are just invariant cyclic cohomology with regard to the natural translation (co)action of the Hopf algebra on itself. This was shown to be the case in Khalkhali-Rangipour paper [21]. A byproduct of this conceptualization, among other things, was a much simpler proof of the cyclicity axiom $t_n^{n+1} = id$ in a much broader context.

One of the questions that remained unsettled in our paper cited above [21] was the issue of coefficients. Inspired by the commutative case of triples (M, G, V) mentioned above, in [21] we introduced triples $(A, \mathcal{H}, M)$, called Hopf triples, where M is an $\mathcal{H}$-module and A is an $\mathcal{H}$-comodule algebra. Due to non-(co)commutativity of $\mathcal{H}$, to form the complex of invariant chains on A, we had to further assume that there is a coaction of $\mathcal{H}$ on M and the action and coaction are compatible (see below for precise definitions). Such pairs $(\mathcal{H}, M)$ are called matched pairs in involution in [21].

Since the module M plays the role of coefficients for invariant cyclic homology and in particular is a noncommutative analogue of coefficients of Lie algebra and group homology theories, it is of utmost importance to understand what is the most general type of coefficients allowable in invariant cyclic homology theory beyond matched and comatched pairs. This problem is completely solved in Hajac-Khalkhali-Rangipour-Sommerhäuser paper [18] by introducing the class of stable anti-Yetter-Drinfeld modules over a Hopf algebra. Matched and comatched pairs are special cases of stable anti-Yetter-Drinfeld modules.

In this section we first recall, very briefly, the Chevalley-Eilenberg definition of invariant de Rham cohomology from [3]. We then recall the notion of a stable anti-Yetter-Drinfeld module over a Hopf algebra from [18] as the ultimate generalization of the notion of Hopf triple and cotriple. One can think of invariant cyclic homology as the noncommutative analogue of invariant de Rham cohomology as defined by Chevalley and Eilenberg [3]. We indicate that cyclic homology of Hopf algebras is an example of invariant cyclic homology. We also present our Morita invariance theorem for invariant cyclic homology. Note that the result could not be formulated for cyclic homology of Hopf algebras since the algebra of $n \times n$ matrices over a Hopf algebra is not a Hopf algebra. One can find the details of this section in [21].

Let G be a Lie group acting smoothly on a manifold M and let V be a G-module. Then G acts (diagonally) on the complex $\Omega^* M \otimes V$ of differential forms on M with coefficients in V. This action preserves the differential of this complex and thus we obtain a well defined complex $(\Omega^* M \otimes V)^G$ of G-invariant differential forms on M with values in V. The invariant de Rham cohomology of (M, G, V) is, by definition, the cohomology of the latter complex [3].

There are at least two advantages in defining invariant de Rham cohomology that were the main reasons for their introduction by Chevalley and Eilenberg. Firstly, if G is compact and connected then it is not difficult to see that the natural inclusion of invariant forms into forms is a quasi-isomorphism of complexes. This result has many applications to topology of Lie groups, but unfortunately has so far found no generalization to the noncommutative realm. Secondly, for $G = M$ acting on itself via left translations, the complex of G-invariant forms on G with values in V is obviously isomorphic to the exterior algebra of the dual of the Lie algebra of G tensored with V. Moreover, using Cartan's formula for the exterior derivative, one can check that the de Rham differential exactly coincides with the Chevalley-Eilenberg differential in Lie algebra cohomology. In this way, invariant de Rham cohomology of G is seen to be isomorphic with the Lie algebra cohomology of the Lie algebra of G.

This second point, as we will see below, admits a full generalization to noncommutative geometry. We note that the process is reversed here: while classically cohomology of Lie algebras was derived from invariant de Rham cohomology, in noncommutative geometry cyclic (co)homology of Hopf algebras was discovered first and only after that invariant cyclic (co)homology was defined.

Definition 7.1. Let H be a Hopf algebra with a bijective antipode S, and M a module and comodule over H. We call M an *anti-Yetter-Drinfeld module* if the action and coaction are compatible in the following sense:

$$
\begin{aligned}
{}_M\Delta(hm) &= h^{(1)}m^{(-1)}S^{-1}(h^{(3)}) \otimes h^{(2)}m^{(0)} && \text{if } M \text{ is a left module and a left comodule;}\\
\Delta_M(hm) &= h^{(2)}m^{(0)} \otimes h^{(3)}m^{(1)}S(h^{(1)}) && \text{if } M \text{ is a left module and a right comodule;}\\
{}_M\Delta(mh) &= S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)} && \text{if } M \text{ is a right module and a left comodule;}\\
\Delta_M(mh) &= m^{(0)}h^{(2)} \otimes S^{-1}(h^{(1)})m^{(1)}h^{(3)} && \text{if } M \text{ is a right module and a right comodule.}
\end{aligned}
$$

In the first case we say M is *stable* if $m^{(-1)}m^{(0)} = m$ for all $m \in M$ (similar definitions apply in other cases).

Example 7.1. (i) (Modular pairs in involution). Let $M = k$ be the commutative ground ring. It is easily checked that stable anti-Yetter-Drinfeld module structures on M are in one to one correspondence with modular pairs in involution (δ, σ), where the action (resp. coaction) of $\mathcal{H}$ is defined via the character δ (resp. the group like element σ). We denote this module by ${}_\sigma^k\delta$...

(ii) (Matched and comatched pairs in involution). An intermediate step between modular pairs in involution and stable anti-Yetter-Drinfeld modules are the matched and comatched pairs in involution of [21]. Thus stable anti-Yetter-Drinfeld modules where the coaction (resp. action) of $\mathcal{H}$ is defined via a group like element σ (resp. via a character δ) are called matched pairs (rep. comatched pairs) in involution in [21]. For more general examples of stable anti-Yetter-Drinfeld modules we refer the reader to [18].

Now let us recall the definition of invariant cyclic (co)homology from [18] and [21]. We start with the definition of invariant cyclic cohomology of coalgebras. Let C be an $\mathcal{H}$-module colagebra and M be a left $\mathcal{H}$-comodule. Let $C^n(C, M) := M \otimes C^{\otimes(n+1)}$, $n \in \mathbb{N}$. It is easily checked that the following operators define a paracocyclic module structure on $\{C^n(C, M)\}_n$.

$$\delta_i(m \otimes c_0 \otimes \ldots \otimes c_{n-1}) = m \otimes c_0 \otimes \ldots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_{n-1}, \qquad 0 \le i < n, \tag{5}$$

$$\delta_n(m \otimes c_0 \otimes \ldots \otimes c_{n-1}) = m^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \ldots \otimes c_{n-1} \otimes m^{(-1)}c_0^{(1)}, \tag{6}$$

(7) $\sigma_i(m \otimes c_0 \otimes \ldots \otimes c_{n+1}) = m \otimes c_0 \otimes \ldots \otimes \varepsilon(c_{i+1}) \otimes \ldots \otimes c_{n+1}, \qquad 0 \leq i \leq n$

(8) $\tau_n(m \otimes c_0 \otimes \ldots \otimes c_n) = m^{(0)} \otimes c_1 \otimes \ldots \otimes c_n \otimes m^{(-1)} c_0.$

Now let us assume that M is also a right $\mathcal{H}$-module. We can treat $C^{\otimes(n+1)}$ as a left $\mathcal{H}$-module via the diagonal action (i.e., $h(c_0 \otimes \ldots \otimes c_n) = h^{(1)} c_0 \otimes \ldots \otimes h^{(n+1)} c_n$) and define the quotient (invariant) complex $C^n_{\mathcal{H}}(C, M) := M \otimes_{\mathcal{H}} C^{\otimes(n+1)}$, $n \in \mathbb{N}$. Except for τ_n and δ_n it is clear that the aforementioned morphisms are well defined on the quotient complex. The key result of [18] is that τ_n is well defined for any module coalgebra C if and only if M is an anti-Yetter-Drinfeld module. More precisely, we have:

THEOREM 7.1 ([18]). *Let M be a left $\mathcal{H}$-comodule and a right $\mathcal{H}$-module. Then the map τ_n given by the formula (8) is well defined on $M \otimes_{\mathcal{H}} C^{\otimes(n+1)}$ for any $n \in \mathbb{N}$ and any $\mathcal{H}$-module coalgebra C if and only if M is an anti-Yetter-Drinfeld module. If furthermore M is stable, then $\{C^n_H(C, M)\}_{n \in \mathbb{N}}$ is a cocyclic module.*

There is a similar definition for invariant cyclic homology of comodule algebras [18]. The following result shows that the Connes-Moscovici cyclic cohomology for Hopf algebras is an special case of invariant cyclic cohomology of colagebras. A similar result holds for cyclic homology of Hopf algebras.

PROPOSITION 7.1. *The cyclic modules $\{\mathcal{H}^n_{(\delta,\sigma)}\}_{n \in \mathbb{N}}$ and $\{C^n_{\mathcal{H}}(\mathcal{H}, \delta^k_\sigma)\}_{n \in \mathbb{N}}$ are isomorphic.*

Let A be an $\mathcal{H}$-comodule algebra. One can easily see that $M_n(A)$ is also an $\mathcal{H}$-comodule algebra, where the coaction of $\mathcal{H}$ on $M_n(A)$ is induced by the coaction of A, i.e., for all $a \otimes u \in A \otimes M_n(k) = M_n(A)$, $\rho(a \otimes u) = a^{(-1)} \otimes a^{(0)} \otimes u$. The following theorem was first proved in [21] for matched pairs in involution. But the same proof carries over to the more general stable anti-Yetter-Drinfeld modules.

THEOREM 7.2 (Morita invariance, [21]). *For any stable anti-Yetter-Drinfeld module M and any $k \geq 1$ one has*

$$HC^{\mathcal{H}}_n(A, M) \cong HC^{\mathcal{H}}_n(M_k(A), M), \qquad n \geq 0.$$

It is shown in [18] that stable anti-Yetter-Drinfeld modules are the most general possible coefficients that one can use in invariant cyclic (co)homology. We believe that this notion will play an important role in the development of the subject as is already evident from the recent work [20], where it is shown that the relative cyclic homology of Hopf-Galois extensions is isomorphic to a variant of Hopf-cyclic homology with coefficients. The cyclic module introduced in [20], though different, seems to be closely related to those considered in [18]. The exact relationship between these theories will be studied elsewhere.

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